Derivation of the Distribution Function for the Tampered Brownian Motion Process Model[1]

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Abstract
The tampered Brownian motion process (BMP) arises in the context of partial step-stress accelerated life testing when the underlying system fatigue accumulated over time is modeled by two constituent BMPs, one governing up to the predetermined time point at which the stress level is elevated and the other afterwards. A conditioning argument obtains the probability distribution function (pdf) of the corresponding time-to-failure random variable. This result has been reported and studied in the literature, but its derivation has not been published.

Key Words: Brownian motion process, step-stress testing, distribution function, first hitting time

1. Introduction

The tampered BMP (Bhattacharyya 1987, Lu & Storer 2001) arises in the context of partial step-stress accelerated life testing when the underlying system fatigue accumulated over time t, $B(t)$, is modeled by two separate BMPs, one applicable before the stress level is elevated at a predetermined fixed time point τ and the other afterwards (assuming that an item under test has not failed by time τ). Specifically, let

$$B(t) = \begin{cases} B_1(t), & t \leq \tau \\ B_1(\tau) + B_2(t - \tau), & t > \tau, \end{cases} \tag{1}$$

where $B_i(t) = B_i(t;\eta_i,\delta)$, i = 1, 2, are independent BMPs with positive drifts η_i and a common diffusion parameter δ^2, and the system fails when $B(t)$ first attains a critical threshold value ξ. The ordering $\eta_2 > \eta_1$ ensures that fatigue accrues relatively faster at the higher stress value.

A primary impetus for prescribing the representation (1) is its plausible physical basis. Additionally, the corresponding single stress setting problem is known to yield the prominent inverse Gaussian (IG) distribution for the first passage time of the BMP with respect to a critical boundary (Shrödinger 1915, Smoluchowski 1915, Tweedie 1945). The IG pdf accommodates a spectrum of shapes, adheres to the structure of an exponential family, and supports well-developed statistical inference procedures (Folks & Chhikara 1978). It has been applied extensively in the modeling of reliability, fatigue life, and long-tailed phenomena (Chhikara & Folks 1977, Bhattacharyya & Fries 1982b, Seshadri 1999). The IG pdf and cdf take the forms:

$$g(t) = g(t;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi t^3}}\, exp\left[\frac{-\lambda(t-\mu)^2}{2\mu^2 t}\right], \tag{2}$$

$$G(t) = G(t;\mu,\lambda) = \Phi\left(\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}-1\right)\right) + exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}+1\right)\right),$$

denoting the mean as μ, the shape parameter as λ, and the standard N(0,1) cdf as $\Phi(\cdot)$.

Doksum & Hóyland (1992) examine variable accelerated life testing experiments for which the time-to-failure distribution is expressed in terms of linear time-transformed IG distribution functions, a construct that is incompatible with (1). Both Lu & Storer (2001) and Doksum & Hóyland (1992) employ a common characterization of the failure time: $T = \inf\{t: B(t) > \xi\}$. Different representations of $B(t)$, however, lead to distinct pdfs.

Based on (1), Lu & Storer (2001) report the pdf for their tampered BMP model to be:

$$f(t) = \begin{cases} \sqrt{\frac{\lambda}{2\pi t^3}}\, exp\left\{-\frac{\lambda t c^2(\mu_1,t)}{2}\right\} & t \le \tau, \\ \sqrt{\frac{\lambda}{2\pi t^3}}\, exp\left\{-\frac{\lambda t}{2}\left[c^2(\mu_1,t) + \frac{t}{\mu_2^2}c(\tau,t)\right]\right\} s(t), & t > \tau, \end{cases} \tag{3}$$

specifying $\lambda = \xi^2/\delta^2$, $\mu_i = \xi/\eta_i$, for i = 1, 2, $c(a,b) = (1/a - 1/b)$ for a, $b \neq 0$, $\Delta = \tau\cdot c(\mu_2,\mu_1)$, $s(t) = q(t,\Delta+1,\lambda) - q(t,\Delta-1,\lambda)$, and $q(t,a,\lambda) = a\cdot exp(½a^2\lambda c(\tau,t))\cdot\Phi(a(\lambda c(\tau,t))^{½})$. On the interval $(0,\tau]$, $f(t)$ matches $g(t)$, the IG pdf given in (2), with parameters μ_1 and λ. For larger values of t, $f(t)$ incorporates μ_2 from $B_2(t)$ and takes on an unwieldy form. Lu & Storer (2001) establish numerous properties of (3): $f(t)$ is continuous and may be either unimodal or bimodal; all positive integer moments exist; and maximum likelihood estimators are unique with probability tending to 1, are strongly consistent, and are asymptotically normally distributed.

Lu and Storer (2001) state that (3) was obtained *after* Bhattacharyya (1987) and attribute the derivation to Bhattacharyya – contradicting Bhattacharyya (1987, p. 156): "The distribution … *has been derived* by using a conditioning approach *which led to* a closed form expression for the pdf" *[emphasis added]*. I derived the tampered BMP pdf (Fries 1982) while awaiting my PhD defense. Gouri Bhattachatyya, my advisor, posed the problem to me (Bhattacharyya 1982) and shortly thereafter crafted a skeleton of a draft manuscript (Bhattacharyya & Fries, 1982a) streamlining portions of my exposition and introducing the exact parameterization (3). Section 2 below details the approach taken in the derivation.

2. Pdf Derivation

Two lemmas support the development of (3). Both were obtained from first principles in 1982, but at present it suffices to cite published sources. Lemma 1 establishes the probability that a BMP in the future will attain a particular value, given that it earlier had reached a specified point at some prescribed instance in time – a fundamental probability arising naturally in the context of conditioning arguments. Lemma 2 simplifies certain integral expressions involving exponential functions.

Lemma 1 (Wang & Pötzelberger 1997, Eq. (2)). Let $B^*(t) = B^*(t;\eta,\delta)$ be a BMP with positive drift η and variance δ^2, and let τ, a, and b be positive constants. Then, independent of η,

$$P\left[\sup_{s\,\epsilon\,[0,\tau]} B^*(s) \geq a \mid B^*(\tau) = b\right] = \begin{cases} exp\left(\dfrac{-2a(a-b)}{\delta^2\tau}\right) & if\ b\ < a, \\ 1 & if\ b\ \geq a. \end{cases}$$

Lemma 2 (Gradshteyn & Ryzhik 2007, pp. 365 & 1030). Let $\phi(\cdot) = \Phi'(\cdot)$ denote the standard normal pdf. For $\alpha > 0$,

$$I(\alpha,\beta) \equiv \int_0^\infty \vartheta \cdot exp\left(-(\alpha\vartheta^2 + \beta\vartheta)\right) \mathrm{d}\vartheta = \frac{1}{2\alpha}\left(1 - \left(\frac{\beta}{\sqrt{2\alpha}}\right)\frac{\Phi\left(-\frac{\beta}{\sqrt{2\alpha}}\right)}{\phi\left(-\frac{\beta}{\sqrt{2\alpha}}\right)}\right).$$

Derivation of (3). On the time interval $(0,\tau]$, it is straightforward to determine the associated component of the cdf $F(t) = \mathrm{P}[T \leq t]$. For a fixed $t \leq \tau$, $B(t) = B_1(t)$ and

$$P[T \leq t] = P\left[\sup_{s\,\epsilon\,[0,t]} B(s) > \xi\right] = P\left[\sup_{s\,\epsilon\,[0,t]} B_1(s) > \xi\right] = G(t;\mu_1,\lambda).$$

For the non-trivial case, $t > \tau$, the derivation proceeds by conditioning on $B_1(\tau)$ and invoking the independence of $B_1(\cdot)$ and $B_2(\cdot)$:

$P[T \leq t] = P[T \leq \tau] + P[\tau < T \leq t]$, where

$$P[\tau < T \leq t] = P\left[\left[\sup_{s\,\epsilon\,[0,\tau]} B(s) < \xi\right] \cap \left[\sup_{s\,\epsilon\,(\tau,t]} B(s) \geq \xi\right]\right]$$

$$= \int_{-\infty}^{\infty} P\left[\left(\left[\sup_{s\,\epsilon\,[0,\tau]} B(s) < \xi\right] \cap \left[\sup_{s\,\epsilon\,(\tau,t]} B(s) \geq \xi\right]\right) \mid B_1(\tau) = b\right] f_{B_1(\tau)}(b)\mathrm{d}b \qquad (4)$$

$$= \int_{-\infty}^{\infty} P\left[\left(\left[\sup_{s\,\epsilon\,[0,\tau]} B_1(s) < \xi\right] \cap \left[\sup_{s\,\epsilon\,(0,t-\tau]} B_2(s) \geq \xi - b\right]\right) \mid B_1(\tau) = b\right] f_{B_1(\tau)}(b)\mathrm{d}b$$

$$= \int_{-\infty}^{\infty} P\left[\sup_{s\,\epsilon\,[0,\tau]} B_1(s) < \xi \mid B_1(\tau) = b\right] \cdot P[(\xi - b)] f_{B_1(\tau)}(b)\mathrm{d}b.$$

Lemma 1 enables the first term appearing in the final integrand to be evaluated directly, and effectively restricts the upper limit of the integral to be ξ. The second element in the integrand is recognized to be an IG cdf, $G(t-\tau;(\xi-b)/\eta_2,(\xi-b)^2/\delta^2)$. Note that this is the only factor in (4) that involves t. The last component of the integrand can be written as a normal pdf since $B_1(\tau)$ has the distribution $\mathrm{N}(\eta_1\tau,\delta^2\tau)$. Substituting back into (4), rearranging terms, and reparameterizing via the transformation $\upsilon = \xi - b$ yields:

$$F(t) = P[T \leq \tau] + \int_0^\infty G\left(t-\tau;\frac{\upsilon}{\eta_2},\frac{\upsilon^2}{\delta^2}\right)\left(1 - exp\left(-\frac{2\xi\upsilon}{\delta^2\tau}\right)\right)\frac{1}{\delta\sqrt{\tau}}\phi\left(\frac{\upsilon-\xi+\eta_1\tau}{\delta\sqrt{\tau}}\right)\mathrm{d}\upsilon,$$

$$f(t) = F'(t) = \int_0^\infty g\left(t-\tau;\frac{\upsilon}{\eta_2},\frac{\upsilon^2}{\delta^2}\right)\left(1 - exp\left(-\frac{2\xi\upsilon}{\delta^2\tau}\right)\right)\frac{1}{\delta\sqrt{\tau}}\phi\left(\frac{\upsilon-\xi+\eta_1\tau}{\delta\sqrt{\tau}}\right)\mathrm{d}\upsilon.$$

Incorporating (2) and expanding the exponential function terms gives:

$$f(t) = \frac{1}{2\pi\delta^2}\frac{1}{\sqrt{\tau(t-\tau)^3}} \cdot exp\left(-\frac{1}{2\delta^2}\left[\frac{(\eta_1\tau-\xi)^2}{\tau}+\eta_2^2(t-\tau)\right]\right)$$

$$\cdot \int_0^\infty \upsilon \cdot exp\left(-\frac{1}{2\delta^2}\left[\upsilon^2\left(\frac{1}{t-\tau}+\frac{1}{\tau}\right)-2\upsilon\left(\eta_2-\eta_1+\frac{\xi}{\tau}\right)\right]\right)\left(1-exp\left(-\frac{2\xi\upsilon}{\delta^2\tau}\right)\right)\mathrm{d}\upsilon$$

$$= \frac{1}{2\pi\delta^2}\frac{1}{\sqrt{\tau(t-\tau)^3}} \cdot exp\left(-\frac{1}{2\delta^2}\left[\frac{(\eta_1\tau-\xi)^2}{\tau}+\eta_2^2(t-\tau)\right]\right)$$

$$\cdot\left\{I\left(\left[\frac{\frac{1}{t-\tau}+\frac{1}{\tau}}{2\delta^2}\right],-\left[\frac{\eta_2-\eta_1+\frac{\xi}{\tau}}{\delta^2}\right]\right)\right.$$

$$\left.-I\left(\left[\frac{\frac{1}{t-\tau}+\frac{1}{\tau}}{2\delta^2}\right],-\left[\frac{\eta_2-\eta_1-\frac{\xi}{\tau}}{\delta^2}\right]\right)\right\}.$$

The precise form of (3) follows by application of Lemma 2 (observing that the first additive term in that result cancels out due to the difference being taken between the two *I* terms), assimilating the parameter definitions accompanying the initial statement of (3), and routine algebra.

3. Discussion

The original derivation of the pdf for the tampered BMP, over three decades old but hitherto unpublished, has been presented. Extensions to encompass experiments with three or more stress levels conceptually could be developed following analogous conditioning arguments, but cumbersome analytical expressions are encountered, e.g.,

$$\int_0^\infty \Phi(\alpha_1+\beta_1 x)\,\phi(\alpha_2+\beta_2 x)dx.$$

This integral does not seem to be representable in a closed form or even a single series expansion; Fayed & Atiya (2014) establish that a related integral can be written as an infinite series of the normalized incomplete Gamma function and the Hermite polynomial. The identical analytical complexity arises when attempting to integrate the *F(t)* expression under (4) to directly obtain the tampered BMP cdf.

Upon reading an early draft of this paper, Nozer Singpurwalla noted that realizations of an underlying BMP with positive drift are not necessarily monotonically increasing. While such a construct plausibly may model many physical phenomena (e.g., when fatigue or degradation can be partially mitigated by regenerative or restorative processes), it would not realistically portray circumstances for which accumulated levels cannot decrease over time. For these situations, he endorsed modeling based on an underlying Wiener Maximum Process (introduced in Singpurwalla 2006), i.e., the customary $B(t;\eta,\delta)$, a BMP with drift $\eta > 0$ and variance $\delta^2 > 0$, would be replaced by $M(t;\eta,\delta) \equiv sup_{0<s\leq t}\, B(s;\eta,\delta)$. Since the distribution of the first hitting time of a threshold barrier is derived from considerations of the maximum attained value, one obtains the standard IG pdf (2) regardless of whether the phenomenon of interest is modeled by the standard BMP or by its maximum. The derivation of the tampered BMP (3) presented in this paper only considers standard BMPs. It does not account for the prospect that $B_1(t;\eta,\delta)$ and $M_1(t;\eta,\delta)$ are not identical.

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