Computational geometry for multivariate statistics

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Washington, DC, USA

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Outline

ARO Multidisciplinary University Research Initiative (MURI)

5 year grant on Multivariate Heavy Tailed Phenomena

Richard Davis - Columbia University (Statistics)
Weibo Gong - UMass Amherst (ECE)
John Nolan - American University (Math)
Sidney Resnick - Cornell University (ORIE)
Gennady Samorodnitsky - Cornell University (ORIE)
Ness Shroff - Ohio State University
R. Srikant - University of Illinois at Urbana-Champaign (ECE)
Don Towsley - UMass Amherst (CS)
Zhi-Li Zhang - University of Minnesota

Some topics the MURI group has worked on

- Growth of social networks - preferential attachment model leads to joint heavy tailed model for (in-degree, out-degree)
- Random walks on large graphs
- Analysis of communication networks with heavy tailed traffic
- Resource allocation in cloud computing
- Reducing power consumption on mobile devices
- Multivariate extreme value distributions - calculations and estimation
- Threshold selection in heavy tailed inference
- Dimensionality reduction for heavy tailed data - robust PCA and ICA

Outline

There is a need for non-traditional models for multivariate data. Working in dimension $d > 2$ requires new tools.

- grids and meshes on non-rectangular shapes
- numerical integration over surfaces
- simulate from a shape

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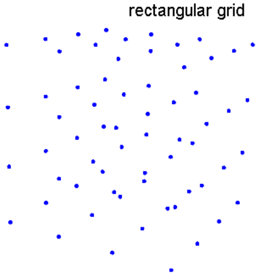
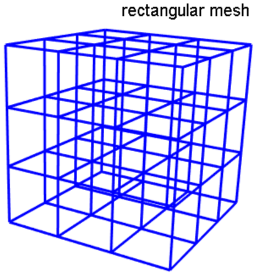
R software packages

- mvmesh - MultiVariate Meshes (CRAN)
- SphericalCubature (CRAN)
- Simplicial Cubature (CRAN)
- gensphere (manuscript submitted)
- mvevd - MultiVariate Extreme Value Distributions (in progress)

mvmesh

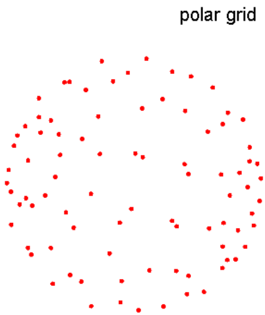
Rectangular grids are straightforward in any dimensions

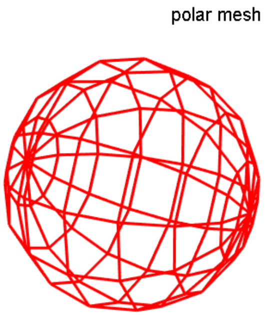




Grid points evenly spaced, easy to determine which cell a point is in.

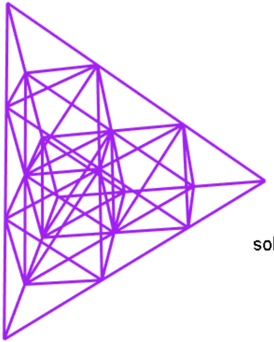
Other shapes are not well described by rectangular grids

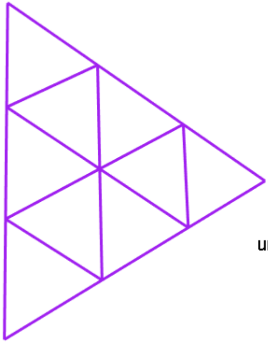




Points are not evenly spread, faces have different numbers of vertices.

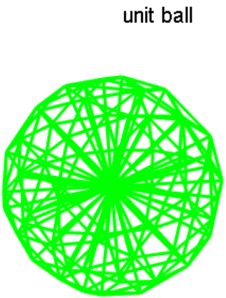
Simplices - equal area subdivisions

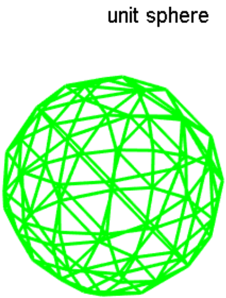




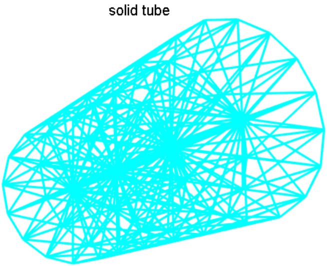
This and following shapes can be generated in any dimension $d \geq 2$.

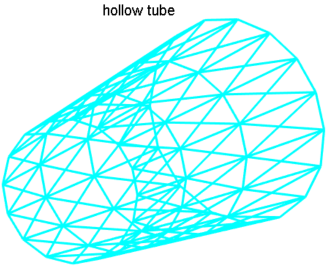
Balls/spheres - approximate equal area subdivisions



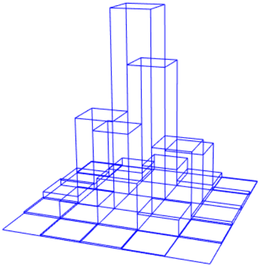


Tubes- approximate equal area subdivisions





Rectangular histograms



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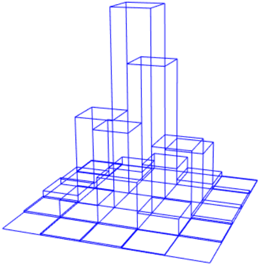
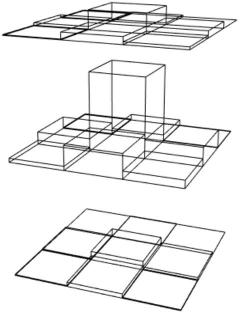
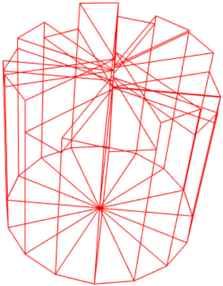




Histograms of non-rectangular regions



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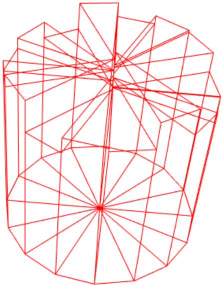
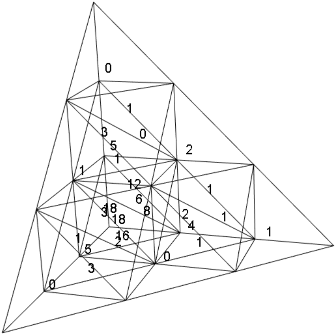
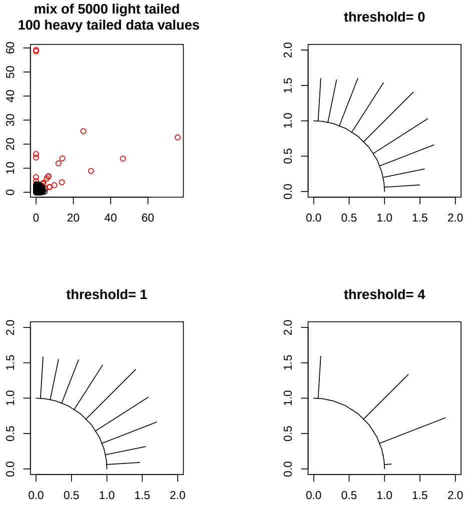


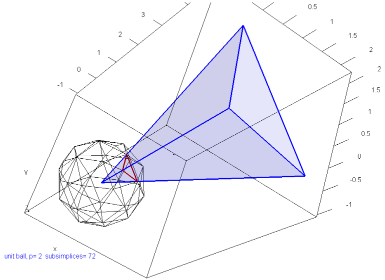


Directional histogram $d = 2$ - count how many in each "direction"

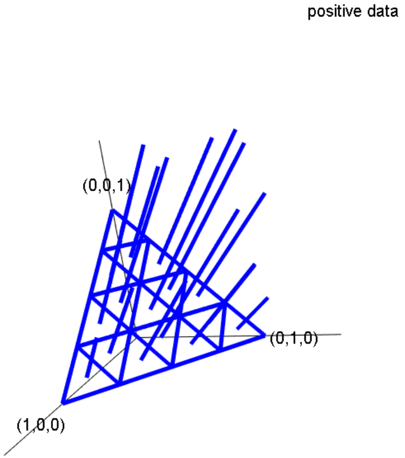


Generalize to $d \geq 3$?

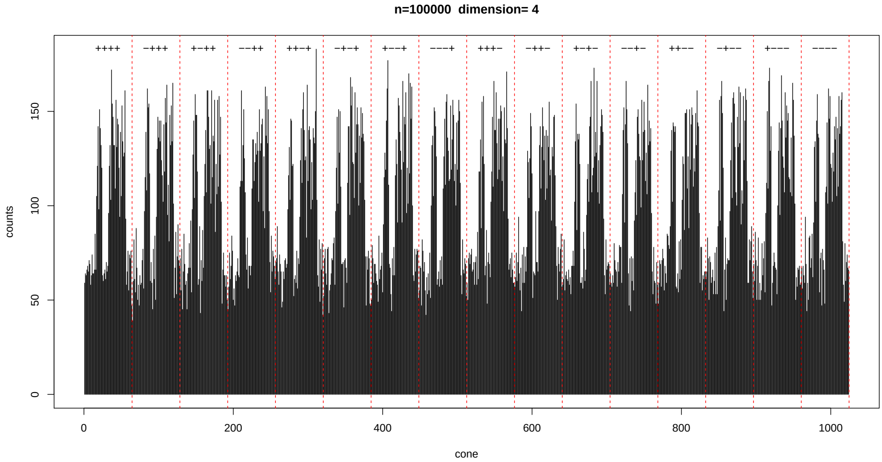
- triangulate/tessellate the sphere
- each simplex on sphere determines a cone starting at the origin
- loop through data points, seeing which cone each falls in
- plot histogram



Directional histogram $d = 3$, positive data

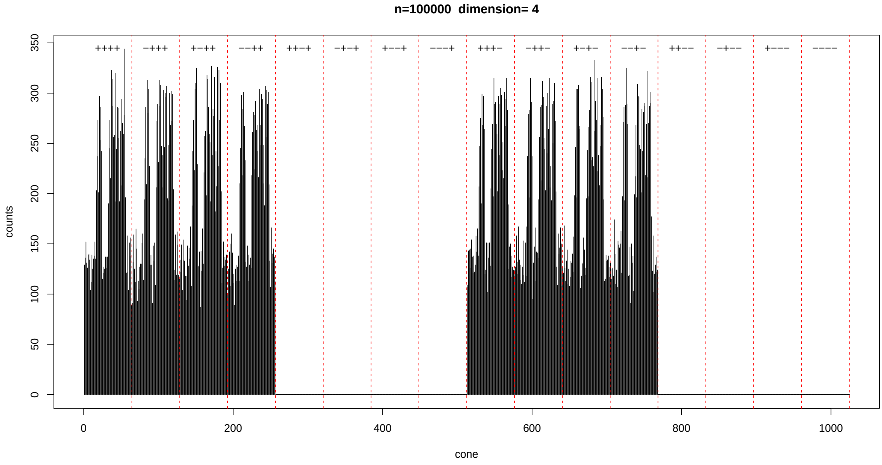


Directional histogram $d = 4$



Radially symmetric data in $\mathbb{R}^4$

Directional histogram $d = 4$



All octants where 3^{rd} component is negative are empty.

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Generalized spherical distributions

Distributions with level sets that are all scaled versions of a star shaped region. Flexible scheme for building up nonstandard star shaped contours.

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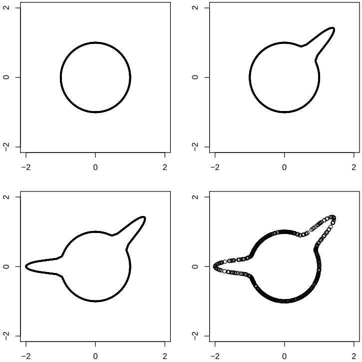
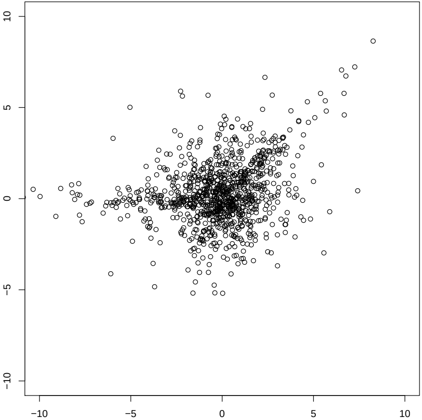
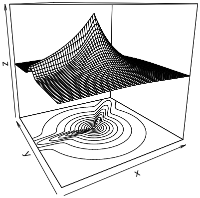


A tessellation based on the added 'bumps' is automatically generated and used in simulating from the contour. Process requires the arclength/ surface area of the contour.

Add a radial component to get a distribution: $\mathbf{X} = R\mathbf{Z}$, where $\mathbf{Z}$ is uniform w.r.t. $(d-1)$-dimensional surface area on contour. Here $R \sim \Gamma(2,1)$

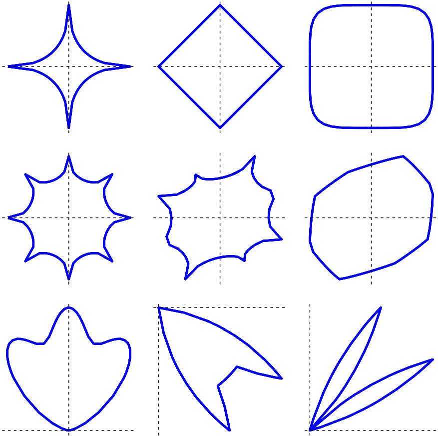


Sample of $\mathbf{X} = R\mathbf{Z}$

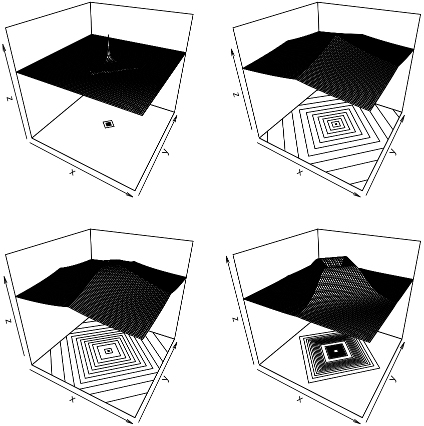


density surface

Many contour shapes possible



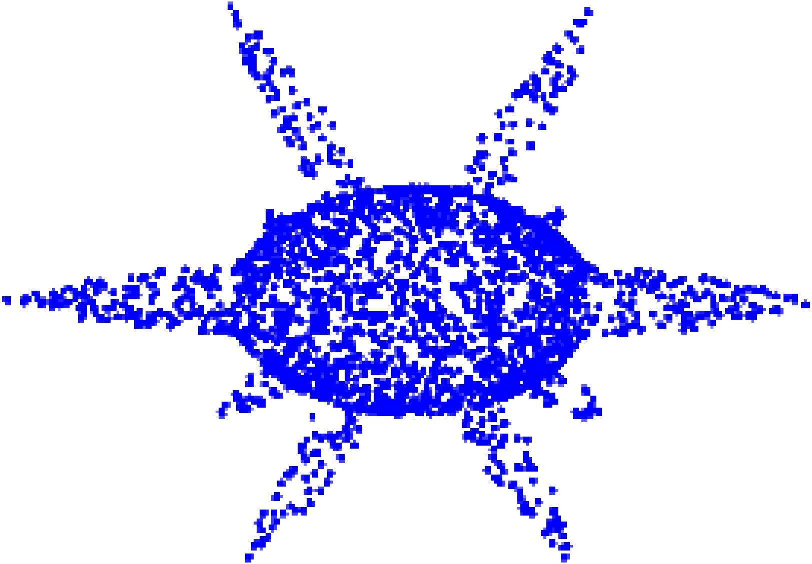
Choice of R determines radial behavior



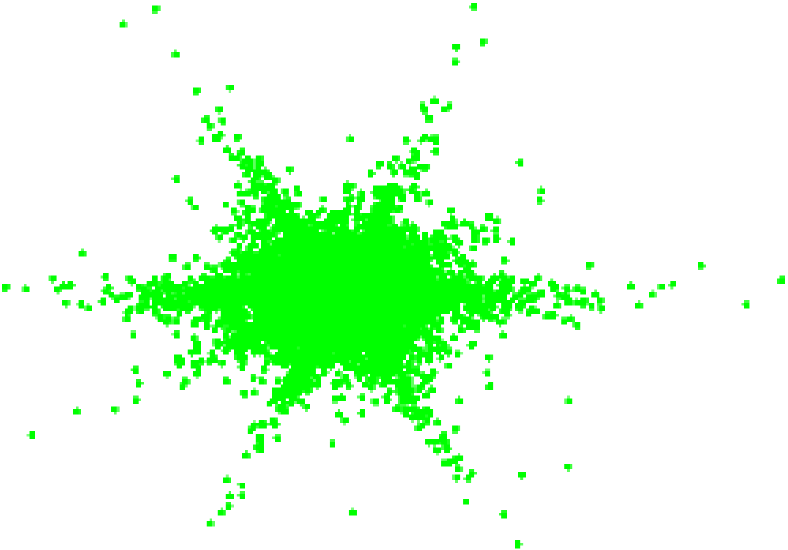
In all cases, contour is a diamond. (a) $R \sim$Uniform(0,1) (b) $R \sim \Gamma(2,1)$
(c) $R = |\mathbf{Y}|$ where $\mathbf{Y}$ is 2D isotropic stable (d) $R \sim \Gamma(5,1)$

3D example - contour

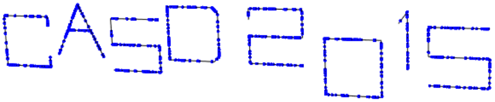
uniform sample from contour



sample from distribution **X** with $R \sim \Gamma(2,1)$

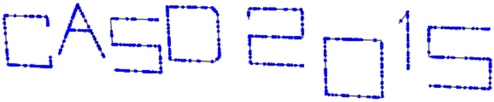


Flexible shapes



Specified some letters in 3D, can sample from this word proportional to arclength

Flexible shapes



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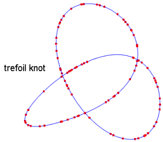

Outline

Multivariate Fréchet Distributions

de Haan and Resnick (1977): **X** max stable, centered with shape index ξ, is characterized by the angular measure H on the unit simplex $\mathbb{W}_+$. The spread of mass by H determines the joint structure. Define the scale function

$$\sigma^{\xi}(\mathbf{u}) = \int_{\mathbb{W}_+} \left(\bigvee_{i=1}^{d} u^{\xi} w_i \right) H(d\mathbf{w}).$$

(If the components of **X** are normalized and $\xi = 1$, then this is the tail dependence function $\ell(\mathbf{u})$.)

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$$G(\mathbf{x}) = P(\mathbf{X} \le \mathbf{x}) = \exp\left(-\sigma^\xi(\mathbf{x}^{-1}) \right).$$

Observation: need to (a) describe different types of measures and (b) integrate over a surface

R package `mvevd`, $d \geq 2$

- Define classes of mvevds: discrete H, generalized logistic, Dirichlet mixture, piecewise constant and linear angular measures (computational geometry)
- Compute scale functions $\sigma(\mathbf{u})$ for above classes (integrate over simplices, computational geometry)
- Fitting mvevd data with any of the above classes (max projections)
- Exact simulation from these classes (Dirichlet mix - Dombry, Engelke & Oesting (EVA 2015), Dieker and Mikosch (2015))
- Compute cdf $G(\mathbf{x}) = P(\mathbf{X} \leq \mathbf{x}) = \exp(-\sigma^{\xi}(\mathbf{x}^{-1}))$, $(\boldsymbol{\mu} = 0, \mathbf{x} \geq 0)$.
- Computation of density $g(\mathbf{x})$ when known (partitions)
- Computation of $H(S)$ for a simplex S to estimate tail probabilities in the direction S. (computational geometry & integrate over simplices)