A Comprehensive Set of Transformations

A. Glen, S. Butler, N. Mankovich

Partially Funded by ARO Grant 67906-MA

Outline

1. Introduction
   - Transformation Theorem
   - The Research
2. Demonstration
   - Maple and Latex Output
3. Some Results
   - Database with Results
   - Unique Distributions and Interesting Properties
   - Catalog of Results: A Distribution Matrix

Outline

1. Introduction
   - Transformation Theorem
   - The Research
2. Demonstration
   - Maple and Latex Output
3. Some Results
   - Database with Results
   - Unique Distributions and Interesting Properties
   - Catalog of Results: A Distribution Matrix

Transformation Theorem

- $Y = g(X)$

Transformation Theorem

- $Y = g(X)$
- $f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|$

Outline

1. Introduction
   - Transformation Theorem
   - The Research
2. Demonstration
   - Maple and Latex Output
3. Some Results
   - Database with Results
   - Unique Distributions and Interesting Properties
   - Catalog of Results: A Distribution Matrix

The Research

- 23 distributions $\times$ 21 transformations

The Research

- 23 distributions $\times$ 21 transformations
- Generate the general PDF of the new distribution

The Research

- 23 distributions $\times$ 21 transformations
- Generate the general PDF of the new distribution
- Generate a representative instance of the new distributions and calculate pertinent characteristics

The Research

- 23 distributions $\times$ 21 transformations
- Generate the general PDF of the new distribution
- Generate a representative instance of the new distributions and calculate pertinent characteristics
- Catalogue and organize the information from these (number) distributions

Name	Distribution
ArcSine	$\frac{1}{\pi\sqrt{x(1-x)}}$
ArcTangent	$\frac{a}{(\arctan(a\,b)+\pi/2)\left(1+a^2(x-b)^2\right)}$
Beta	$\frac{\Gamma(a+b)x^{a-1}(1-x)^{b-1}}{\Gamma(a)\Gamma(b)}$
Chi	$\frac{x^{a-1}\mathrm{e}^{-1/2\,x^2}}{2^{a/2-1}\Gamma(a/2)}$
Chi Squared	$\frac{x^{a/2-1}\mathrm{e}^{-x/2}}{2^{a/2}\Gamma(a/2)}$
Exponential	$\mathrm{e}^{-a\,x}$
Exponential Power	$\mathrm{e}^{1-\mathrm{e}^{a\,x^b}}\,\mathrm{e}^{a\,x^b}\,a\,b\,x^{b-1}$
FRV	$\frac{\Gamma(a/2+b/2)x^{a/2-1}}{\Gamma(a/2)\Gamma(b/2)}\left(\frac{a}{b}\right)^{a/2}\left(\left(\frac{a\,x}{b}+1\right)^{a/2+b/2}\right)^{-1}$
Gamma	$a\,\left(a\,x\right)^{b-1}\mathrm{e}^{-a\,x}\Gamma\left(b\right)$
Pareto	$\left(a+\frac{c}{x+b}\right)\left(1+\frac{x}{b}\right)^{-c}\mathrm{e}^{-a\,x}$
Gompertz	$a\,b^x\mathrm{e}^{-\frac{a\left(b^x-1\right)}{\ln(b)}}$
Hyperexponential	$\mathrm{e}^{-3\,x}+2\,\mathrm{e}^{-4\,x}$

Name	Distribution
Hypoexponential	$\frac{b\,a\,c\left(\mathrm{e}^{-c\,z}a-\mathrm{e}^{-c\,z}b+\mathrm{e}^{-a\,z}b-\mathrm{e}^{-a\,z}c-\mathrm{e}^{-b\,z}a+\mathrm{e}^{-b\,z}c\right)}{(a-b)(a-c)(b-c)}$
Inverse Gaussian	$1/2\,\sqrt{2}\sqrt{\frac{a}{\pi\,x^3}}\mathrm{e}^{-1/2\,\frac{a\,(x-b)^2}{b^2x}}$
Inverted Gamma	$\frac{x^{-a-1}}{\Gamma(a)b^a}\mathrm{e}^{-\frac{1}{xb}}$
Log Logistic	$\frac{a\,b\,(a\,x)^{b-1}}{\left(1+(a\,x)^b\right)^2}$
Log Normal	$(1/2\,)\frac{\sqrt{2}}{\sqrt{\pi}xb}\mathrm{e}^{-1/2\,\frac{(\ln(x)-a)^2}{b^2}}$
Lomax	$b\,a\,\left(b\,x+1\right)^{-a-1}$
Makeham	$\left(a+b\,c^x\right)\mathrm{e}^{-a\,x-\frac{b\left(c^x-1\right)}{\ln(c)}}$
Muth	$\left(\mathrm{e}^{a\,x}-a\right)\mathrm{e}^{-\frac{\mathrm{e}^{a\,x}}{a}+a\,x+a^{-1}}$
Rayleigh	$2\,a^2x\mathrm{e}^{-a^2x^2}$
Weibull	$b\,a^b x^{b-1}\mathrm{e}^{-(a\,x)^b}$

Transformations

Transformation Table g(x)=	
x^2	$1/\tanh(x+1)$
$\sqrt{x}$	$1/\sinh(x+1)$
$1/x$	$\text{arccsch}(x+1)$
$\arctan(x)$	$1/\text{arcsinh}(x+1)$
e^x	$1/\text{csch}(x)+1$
$\tanh(1/x)$	$\text{csch}(1/x)$
$\ln(x)$	$\text{arcsinh}(x)$
$\text{arccsch}(1/x)$	$\text{csch}(x+1)$
e^{-x}	
$-\ln(x)$	
$\ln(x+1)$	
$1/(\ln(x+2))$	
$\tanh(x)$	
$\sinh(x)$	

Outline

1. Introduction
   - Transformation Theorem
   - The Research
2. Demonstration
   - Maple and Latex Output
3. Some Results
   - Database with Results
   - Unique Distributions and Interesting Properties
   - Catalog of Results: A Distribution Matrix

Outline

1. Introduction
   - Transformation Theorem
   - The Research
2. Demonstration
   - Maple and Latex Output
3. Some Results
   - Database with Results
   - Unique Distributions and Interesting Properties
   - Catalog of Results: A Distribution Matrix

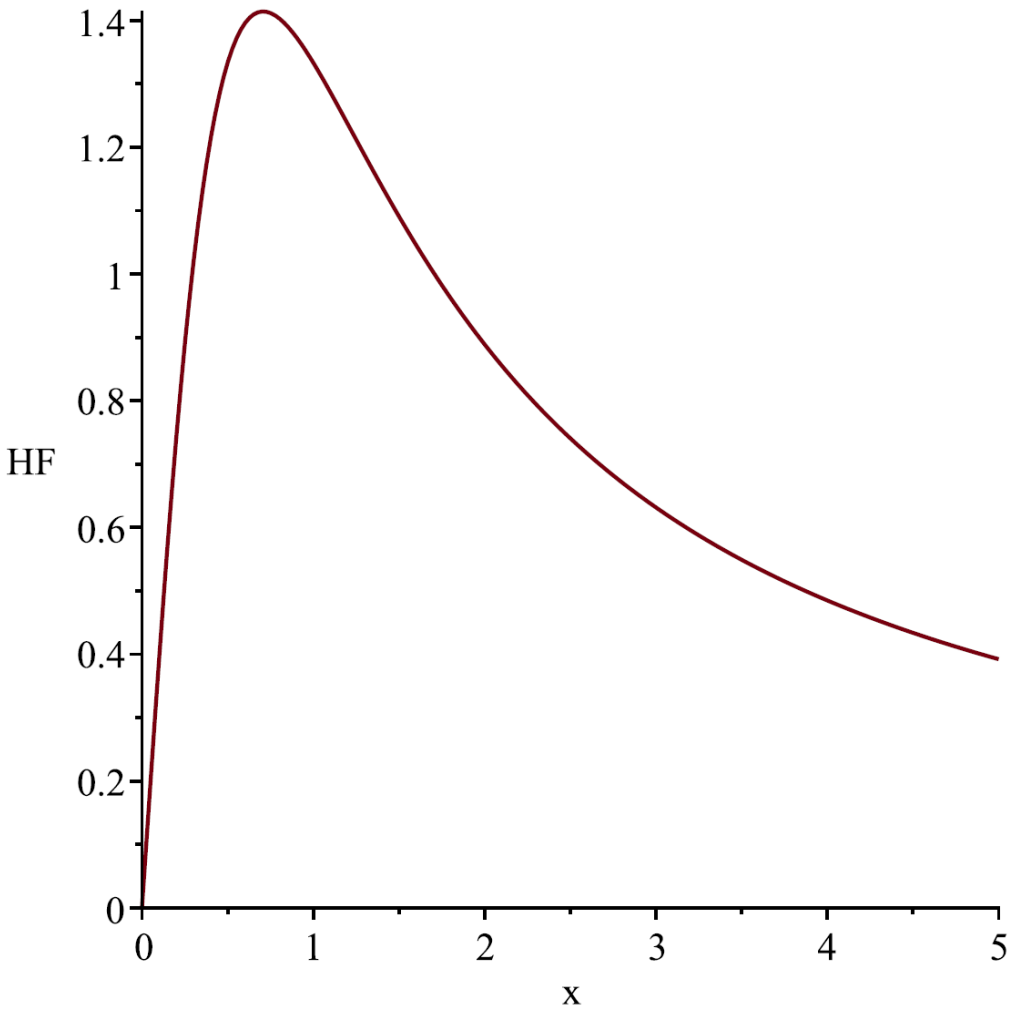
Github Address

https://github.com/nmank/APPLResearch

Outline

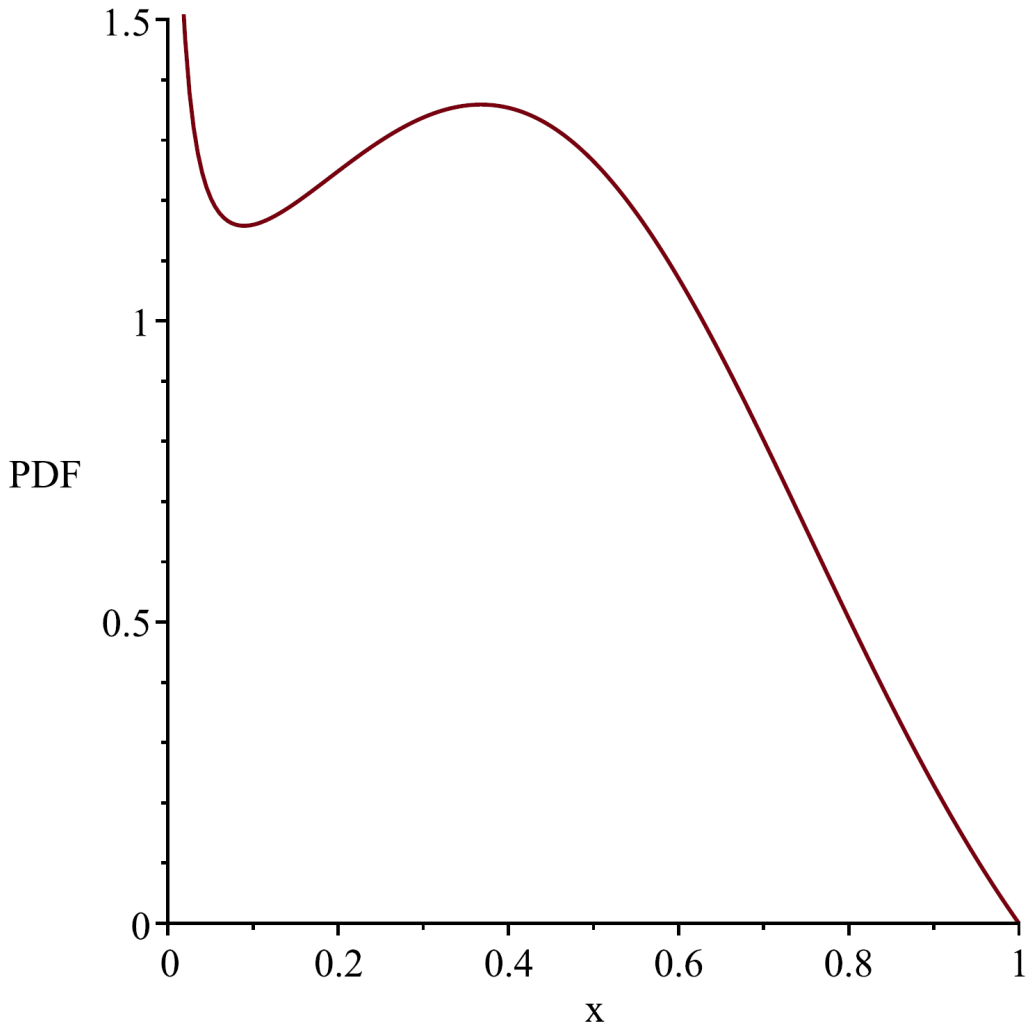
1. Introduction
   - Transformation Theorem
   - The Research
2. Demonstration
   - Maple and Latex Output
3. Some Results
   - Database with Results
   - Unique Distributions and Interesting Properties
   - Catalog of Results: A Distribution Matrix

Upside Down Bathtub Hazard Function



Lomax (1,2), $\rightarrow t^2$

Finite Support PDF



Log Logistic(1,2), $t \to e^{-t}$

Outline

1. Introduction
   - Transformation Theorem
   - The Research
2. Demonstration
   - Maple and Latex Output
3. Some Results
   - Database with Results
   - Unique Distributions and Interesting Properties
   - Catalog of Results: A Distribution Matrix

Sample Distribution Matrix

Log Logistic (1,2)

$$f(x) = \frac{\frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}}{1+(\frac{x}{\alpha})^{\beta}}$$

Transformation	General PDF	Example: Log Logistic (1,2) PDF	CDF	HF	IDF	μ	σ^2	MF	MGF	HF Shape	Support	Comment
x^2	✓	✓	✓	✓	✓	∞	∞	✓	✓	DFR	$0, \infty$	
$\sqrt{x}$	✓	✓	✓	✓	✓	✓	✓	✓	✓	UBT	$0, \infty$	
x^{-1}	✓	✓	✓	✓	✓	✓	∞	✓	∂	UBT	$0, \infty$	
$\arctan(x)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	IFR	$0, \frac{\pi}{2}$	
e^x	✓	✓	✓	✓	✓	∞	∞	∞	∂	DFR	$1, \infty$	
$\ln(x)$	✓	✓	✓	✓	✓	✓	✓	∂	∂	IFR	$-\infty, \infty$	
e^{-x}	✓	✓	✓	✓	✓	✓	✓	✓	∂	IFR	$0, 1$	
$-\ln(x)$	✓	✓	✓	✓	✓	✓	✓	∂	∂	IFR	$-\infty, \infty$	
$\ln(x+1)$	✓	✓	✓	✓	✓	✓	✓	∂	∂	UBT	$0, \infty$	
$1/\ln(x+2)$	✓	✓	✓	✓	✓	∂	∂	∂	∂	IFR	$0, \frac{1}{\ln(2)}$	
$\tanh(x)$	✓	✓	✓	✓	✓	∂	∂	∂	∂	IFR	$0, 1$	
$\sinh(x)$	✓	✓	✓	✓	✓	∞	$\varnothing$	∞	∂	UBT	$0, \infty$	
$\text{arcsinh}(x)$	✓	✓	✓	✓	✓	✓	✓	∂	∂	IFR	$0, \infty$	
$\text{csch}(x+1)$	✓	✓	∂	∂		∂	∂	∂	∂		$0, \frac{2}{e-e^{-1}}$	
$\text{arccsch}(x+1)$	✓	✓	✓	✓	✓	✓	✓	∂	∂	IFR	$0, \text{arcsinh}(1)$	
$1/\tanh(x+1)$	✓	✓	✓	✓	✓	∂	∂	∂	∂	IFR	$1, \frac{e+e^{-1}}{e-e^{-1}}$	
$1/\sinh(x+1)$	✓	✓	✓	✓		∂	∂	∂	∂	IFR	$0, \frac{2}{e-e^{-1}}$	
$1/\text{arcsinh}(x+1)$	✓	✓	✓	✓	✓	∂	∂	∂	∂	IFR	$0, \frac{1}{\ln(1+\sqrt{2})}$	
$1/\text{csch}(x)+1$	✓	✓	∂	∂		∞	∞	∞	∂	UBT	$1, \infty$	
$\tanh(x^{-1})$	✓	✓	✓	✓	✓	∂	∂	∂	∂	IFR	$0, 1$	
$\text{csch}(x^{-1})$	✓	✓	∂	∂		∂	∂	∂	∂		$0, \infty$	
$\text{arccsch}(x^{-1})$	✓	✓	✓	✓	✓	✓	✓	∂	∂	IFR	$0, \infty$	

Legend

Symbol	Meaning
✓	Exists, Closed Form
∂	Exists, Not Closed Form
$\emptyset$	Not Possible
	Not Calculated

General PDFs

Log Logistic, $t \to csch(t+1)$

$$f(x) = \frac{b\,a\,\left(b\,\mathrm{arccsch}\left(x\right) - b + 1\right)^{-a-1}}{\sqrt{x^2+1}\,|x|} \qquad 0 < x < 2\left(\mathrm{e} - \mathrm{e}^{-1}\right)^{-1}$$

Arctan, $t \to \sqrt{t}$

$$4\frac{ax}{(2arctan(ab)+\pi)(a^2x^4 - 2a^2bx^2 + a^2b^2 + 1)} \qquad 0 < x < \infty$$

Weibull, $t \to tanh(t^{-1})$

$$f(x) = -\frac{b\,a^b\left(\left(\mathrm{arctanh}\left(x\right)\right)^{-1}\right)^b \mathrm{e}^{-a^b\left(\left(\mathrm{arctanh}(x)\right)^{-1}\right)^b}}{\mathrm{arctanh}\left(x\right)\left(x^2-1\right)} \qquad 0 < x < 1$$

More General PDFs

Rayleigh, $t \rightarrow e^{-t}$

$$f(x) = -2\,\frac{a^2 \ln(x)\, \mathrm{e}^{-a^2(\ln(x))^2}}{x} \qquad 0 < x < 1$$

Chi Squared, $t \rightarrow sinh(t)$

$$f(x) = \frac{(arcsinh(x))^{a/2-1} 2^{-a/2}}{\sqrt{x+\sqrt{x^2+1}}\Gamma(a/2)\sqrt{x^2+1}} \qquad 0 < x < \infty$$

FRV, $t \rightarrow arctan(t)$

$$\frac{a^{a/2} b^{b/2} (tan(x))^{a/2-1} (atan(x)+b)^{-a/2-b/2} (1+tan(x)^2)\Gamma(a/2+b/2)}{\Gamma(a/2)\Gamma(b/2)}$$

$0 < x < \infty$

The End

Questions?