A new procedure for sensitivity testing with two stress factors

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- Sensitivity testing : problem formulation.
- Review of the 3pod (3-phase optimal design) procedure with *one* stress factor.
- A new procedure for two stress factors, partly inspired by 3pod; not a trivial extension.
- An illustration.
- Comments and further work.

(joint work with Dianpeng Wang, Beijing Inst. of Technology)

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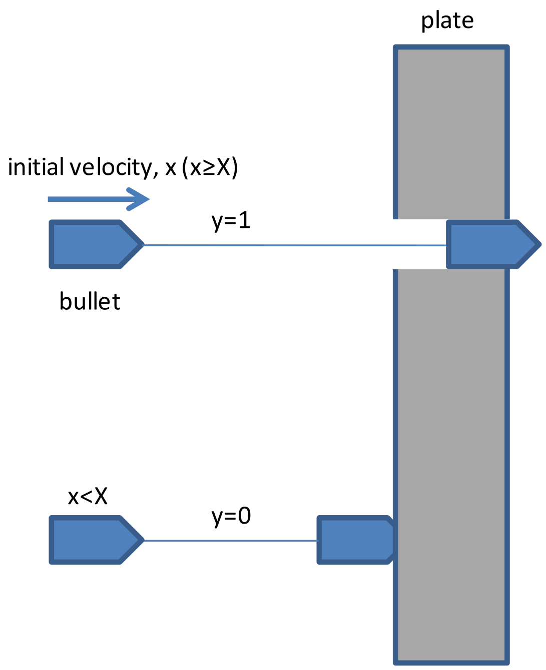
Sensitivity testing

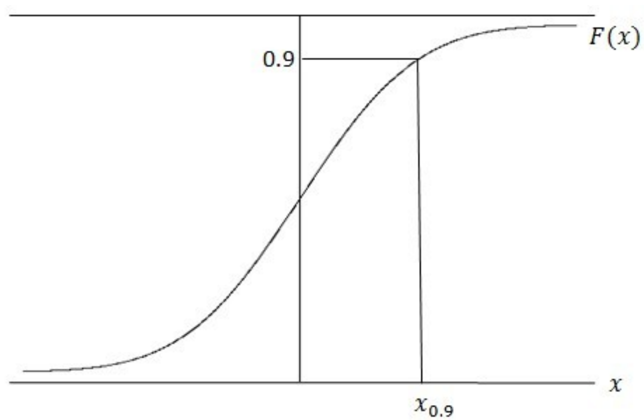
- Stress/stimulus level x: launching velocity, drop height
- Response/nonresponse y = 1 or 0: penetrate, explode



X=unknown critical level (a random quantity)

Quantal response curve

- Quantal response curve $F(x)$ = prob ($y = 1 | x$)*;* interested in estimating the p-th quantile x_p with $F(x_p) = p$, p typically high, e.g., p = 0.9, 0.99, 0.999. Useful for certification or quantification of test items. Common in military and heavy industry applications



- Choice of F: probit, logit, or skewed version
- Problem/challenge: find a sequential design procedure to estimate x_p efficiently and for small samples

Three-phase optimal design

- A trilogy of search-estimate-approximate:
  I. (*search*) to generate $y = 1$ and $y = 0$, to "close-in" on region of interest and to obtain *overlapping* data pattern
  II. (*estimate*) use *D*-optimality criterion to generate design points; *spread out* design points
  III. (*approximate*) Taking $\hat{\mu} + F^{-1}(p)\hat{\sigma}$, where $\hat{\mu}, \hat{\sigma}$ are MLE of μ,σ based on data in I-II, as the starting value, use the Robbins-Monro-Joseph (RMJ) procedure to generate design points
- 3-phase optimal design, dubbed as 3pod (for its steady performance ☺)

  Wu-Tian (2014, JSPI)

Phase I of 3pod

- It has three stages I1, I2, I3
- I1. (quickly obtain y = 1 and y = 0). Choose ($\mu_{\min}$, $\mu_{\max}$) for location parameter μ and σ_g as guessed value of scale parameter σ and $\mu_{\max}$- $\mu_{\min} \geq 6\sigma_g$. Take y_1 and y_2 at $x_1 = \frac{3}{4}\mu_{min} + \frac{1}{4}\mu_{max}$, $x_2 = \frac{1}{4}\mu_{min} + \frac{3}{4}\mu_{max}$.
  Four cases result:
  (i) $y_1 = y_2 = 0$ ⟶ x_1, x_2 to the left of μ; take $x_3 = \mu_{\max} + 1.5\sigma_g$. If y_3= 1, move to I2. If y_3= 0, take $x_4 = \mu_{\max} + 3\sigma_g$. If y_4= 1, move to I2. If y_4= 0, range not large; increase x by $1.5\sigma_g$ until y=1.

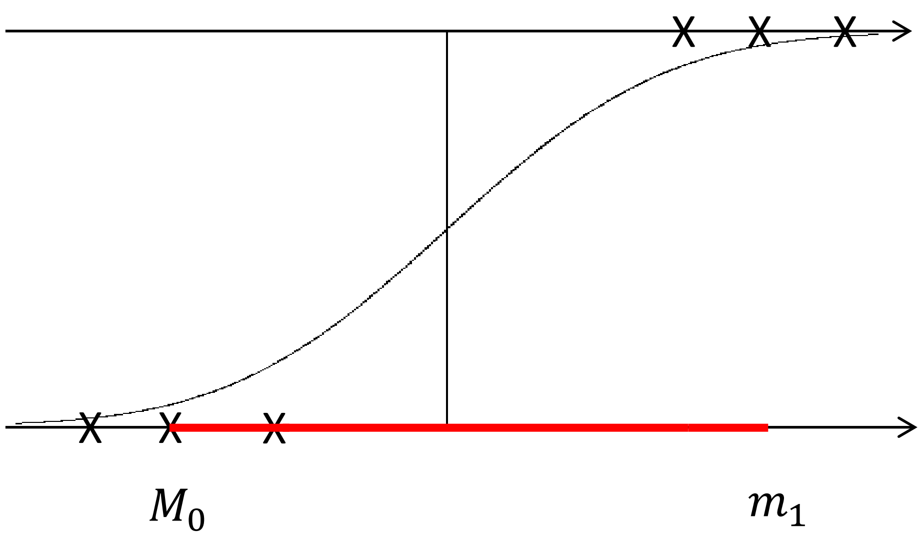
Phase I of 3pod (continued)

(ii) $y_1 = y_2 = 1$, do the mirror image of (i)
(iii) $y_1 = 0$, $y_2 = 1$: good! Move to I2
(iv) $y_1 = 1$, $y_2 = 0$: range too narrow around μ,
expand it by taking $x_3 = \mu_{\text{min}} - 3\sigma_g$,
$x_4 = \mu_{\text{max}} + 3\sigma_g$; move to I2

- Note: I1 is like "dose ranging" in dose-response studies

Trapped in separation?

- Let M_0 = largest x value with y = 0, m_1= smallest x value with y = 1. Overlapping iff $M_0 > m_{1;}$ separation iff $M_0 \leq m_1$
- Running test within the separation interval $[M_0, m_1]$ will forever be *trapped in separation* ☹. ➡ When the interval is small, get out to avoid logjam



I2: stage 2 of phase I

- If overlapping in data from I1, move to I3. Otherwise, take next level at $\hat{\mu}$ (=MLE assuming probit and σ_g); if overlapping, move to I3. If no overlapping, update M_0, m_1, $\hat{\mu}$, take next level at $\hat{\mu}$ until m_1-M_0<1.5 σ_g. Then choose x levels outside the separation interval [M_0, m_1]. See next.
- Take next run at $m_1+0.3\sigma_g$; if $y = 0$, overlapping, move to I3. If $y = 1$, next run at $M_0-0.3\sigma_g$; if $y = 1$, overlapping, move to I3. Otherwise it suggests σ_g is too large, *reduce* it to $\frac{2}{3}\sigma_g$, repeat I2 until seeing overlapping.

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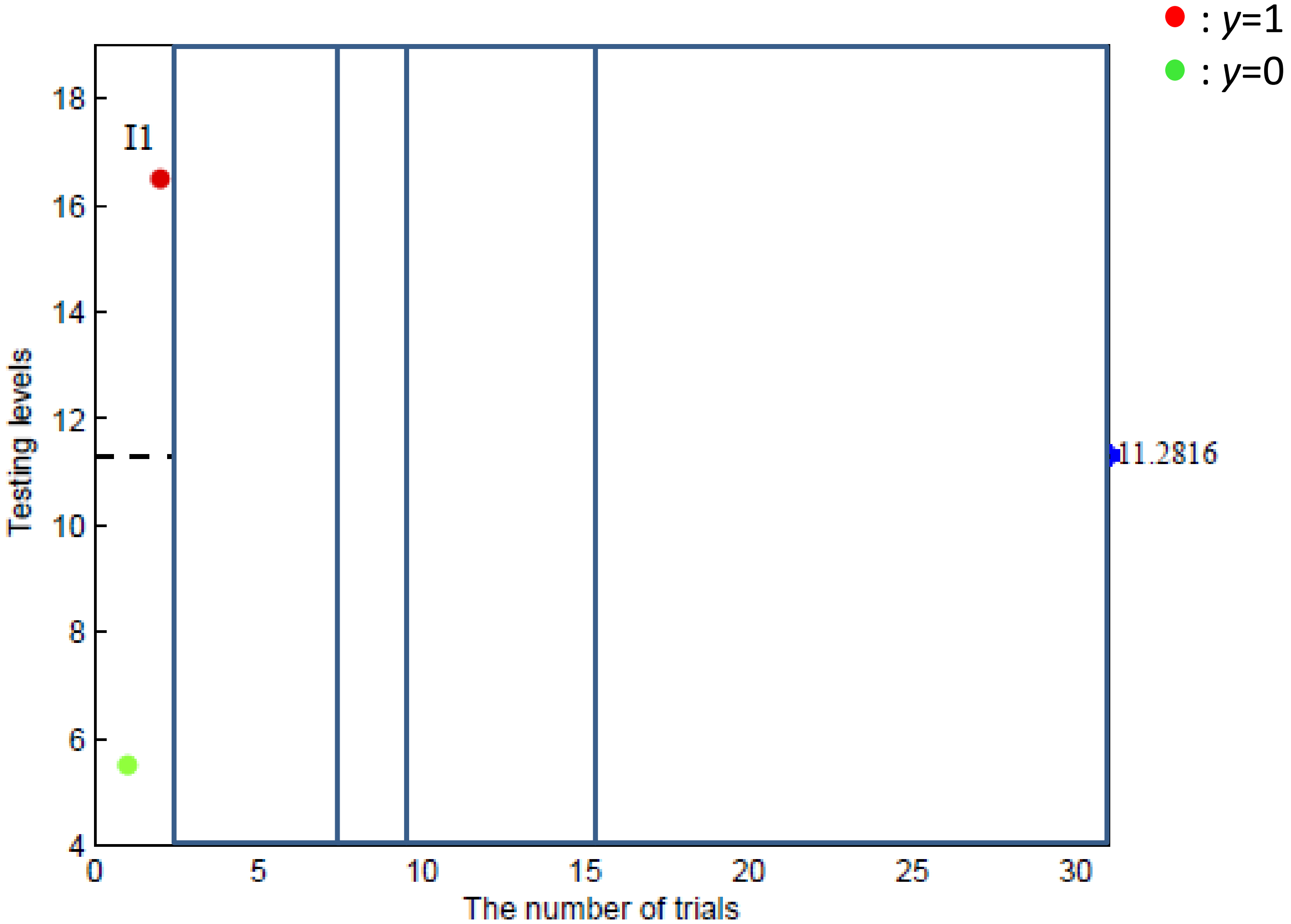
Illustrative Example

(0,22), probit, μ=10, σ=1, σ_g=3, $x_{0.99}$=11.2816



Problem formulation with two stress factors

- Two stress factors, $x = (x_1, x_2)$, which are not independent. Example: temperature and voltage in detonation of ammunition.
- The outcome is binary data, $y = 1$ or $y = 0$.
- $P(y = 1) = G[f(x)]$, where $G(\cdot)$ is a location-scale distribution function with μ, σ, and $f(x)$ is an unknown latent function of x. G is a standard choice like logit, probit, or a skewed version.
- Each quantile is a curve, $\zeta_p = \{x = (x_1, x_2) \in \mathbb{R}^2 | f(x) = G^{-1}(p)\}$.

Two-step procedure for approximating the quantile curve

- The proposed two-step procedure is inspired by some ideas in the 3pod procedure.
- The clue comes from the ideas in phase I, whose goal is to achieve an *overlapping* pattern.
- Note that step I(1) of 3pod is to determine the region of interest by using a modified binary search and step I(2) is to break the separation pattern.

Two-step procedure: further details

- The new procedure has two steps:
  - (I) search for an overlapping pattern,
  - (II) approximate the quantile curves of interest.
- For two dimensions, an overlapping pattern means that the levels with $y = 1$ and the levels with $y = 0$ cannot be separated by a straight line.

Step I: search for overlapping pattern

- Assume that the investigators can make a guess of the region, $[x_{1L}, x_{1U}] \times [x_{2L}, x_{2U}]$, in which both outcomes can occur with high probability.
- Run tests at the four corners of the rectangles, $(x_{1L}, x_{2L}), (x_{1L}, x_{2U}), (x_{1U}, x_{2L}), (x_{1U}, x_{2U})$.
- There are three situations according to the types of the outcomes.

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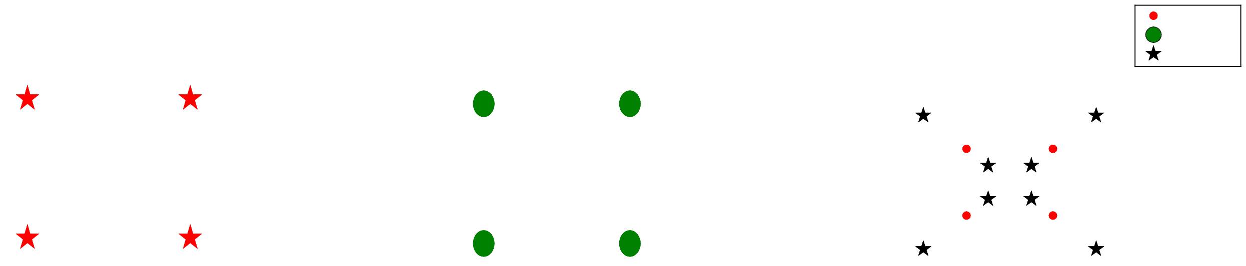
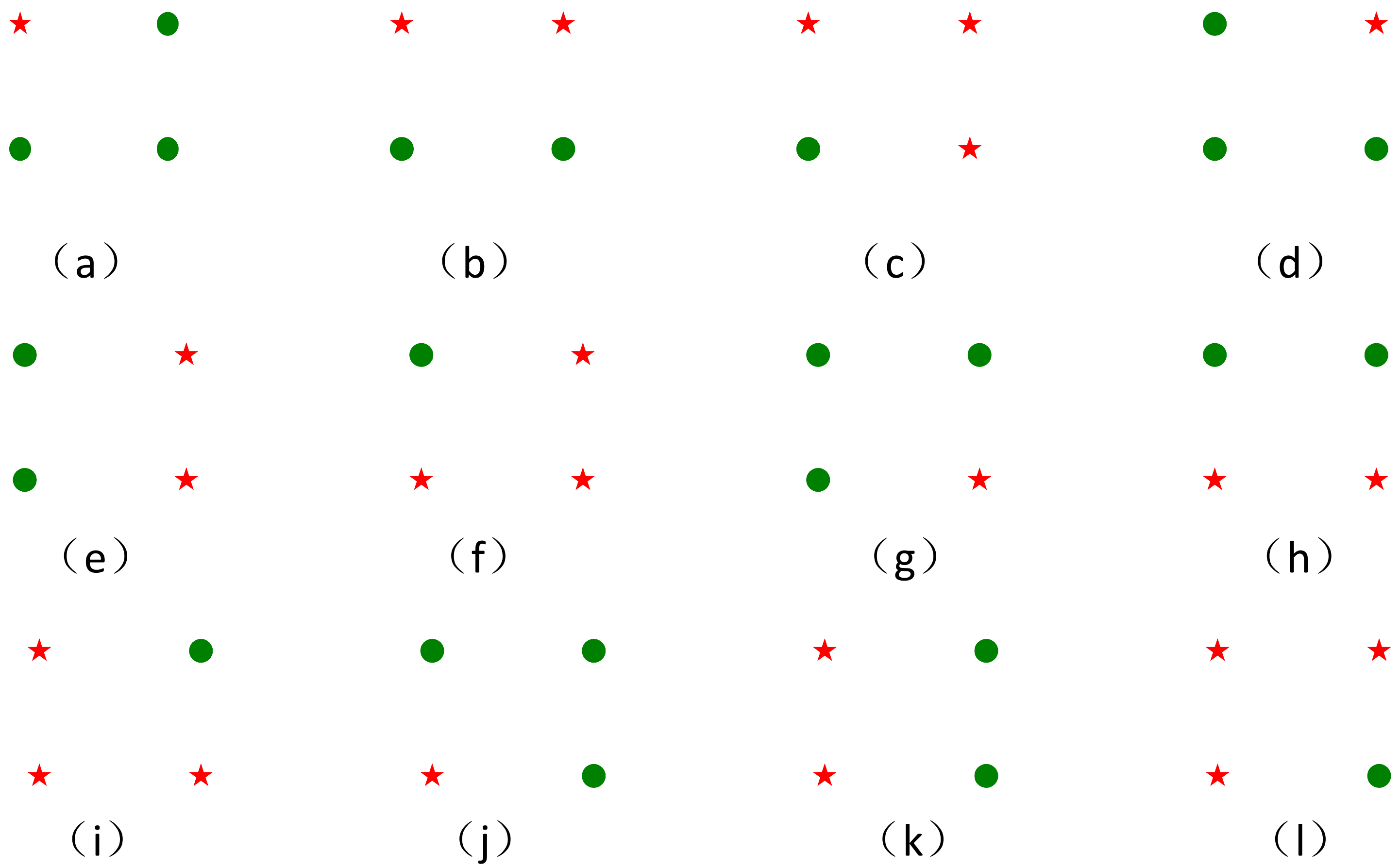
Only one type of outcome (i.e., y = 0 only, or y = 1 only) is observed

- This implies that the region of rectangle is too small or too large.
- Run additional tests both inside and outside the rectangle, until both $y = 1$ and $y = 0$ are observed.
- To choose additional tests outside the rectangle, we double the sides of the rectangle and keep the same center.
- To choose the tests inside the rectangle, we halve the sides of the rectangle and keep the same center.

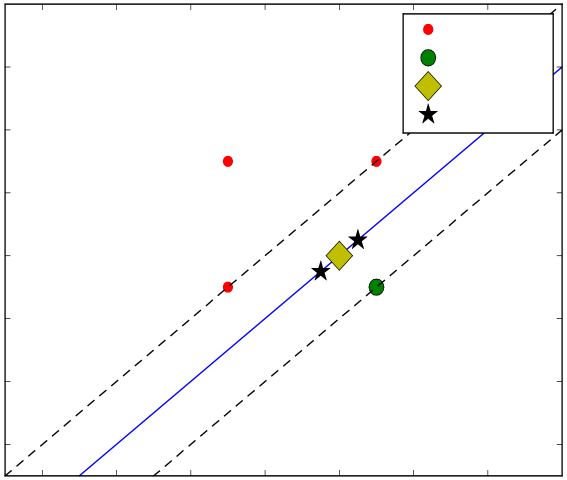


Both y = 0 and y = 1 are observed but can be separated by a straight line



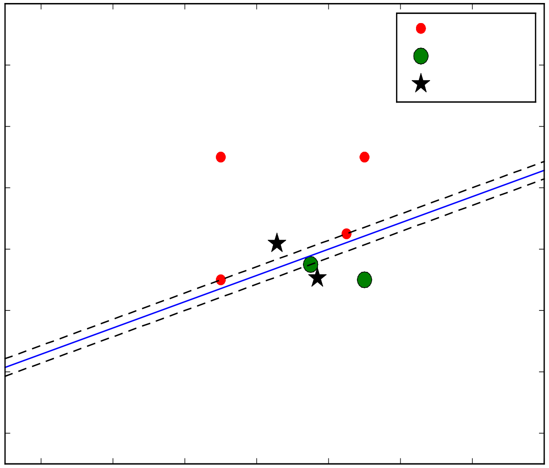
SVM (Support Vector Machine) is used to exploring the middle region

- Denote C_0 as the mean of support vectors with $y = 0$, C_1 the mean of support vectors with $y = 1$. Denote k_0 as the number of tests with $y = 0$, k_1 the number of tests with $y = 1$.
- If $D_{margin} > D_g$ and $k_0 > k_1$, choose two tests on the separator with the projection of C_1 as its center.
- If $D_{margin} > D_g$ and $k_0 \leq k_1$, choose two tests on the separator with the projection of C_0 as its center.
- The distance between these two tests is $D_{margin}/2$.



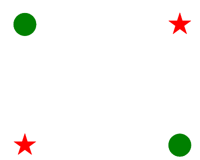
Exploring the middle region (continued)

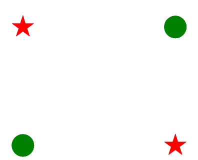
- If $D_{margin} \leq D_g$, it implies that the margin is too narrow.
- Choose tests outside the margin to avoid being trapped in a wrong region.
- Choose one point at each side of the margin. The projection of the test, which is chosen from the side with $y = 1$ (and resp. $y = 0$), on the separator is C_1(and resp. C_0).
- The distances between the new tests and the separator are both D_{margin}.
- Continue the SVM steps until the overlapping pattern is obtained.

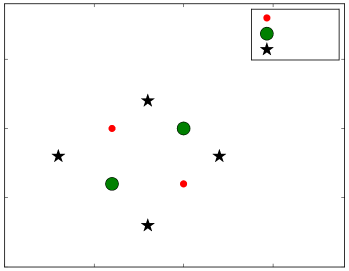


Overlapping pattern is achieved

- This indicates two red dots on the diagonal and two green dots on the off-diagonal (or vice versa), which usually suggests that the initial guess of the region is too narrow.
- Add four tests outside the rectangle to get more information.
- Keep the center (i.e., the mean) of the new tests as before, and set the length of the sides to be 1.5 times the length of the initial sides and *rotate* it 45 degrees.







Step II: approximating the curve of interest

- $X = \{x^1, x^2, \cdots, x^n\}$, $Y = \{y^1, y^2, \cdots, y^n\}$ and $f = \{f^1, f^2, \cdots, f^n\}$, where $f^i = f(x_i)$. Recall $f(x)$ is a latent function in $P(y = 1) = G[f(x)]$, which connects binary y with continuous x.
- We employ a binary Gaussian process: $f \sim GP(0, K(x, x'))$, where covariance function $K(x, x') = \sigma^2 \exp\{-\| x - x' \|^2 / 2l\}$.
- Let $\theta = (\sigma, l)$. The posterior distribution of f: $p(f|X, Y, \theta) = \frac{N(f;0,K|X,\theta)}{p(Y|X,\theta)} \prod_{i=1}^{n} G(f^i)$.

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An alternative: curve approximation by GLM

- An alternative to the GP model is the use of GLM, i.e., logit or probit regression.
- Let $f^i = \left(x_1^i, x_2^i, x_1^{i,2}, x_2^{i,2}, x_1^i x_2^i\right)\beta$, where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)^T$.
- $p\left(y^i = 1 \middle| x^i\right) = G(f^i)$, where G is a logit or probit.
- $\hat{\beta}$ can be obtained based on the observations.
- Given a new point x^*, f^* can be predicted by using $\left(x_1^*, x_2^*, x_1^{*,2}, x_2^{*,2}, x_1^* x_2^*\right)\hat{\beta}$.

Step II (continued)

- Given a new point x^*, the posterior distribution of f^* can be predicted by using the density $p(f^*|x^*, X, Y, \theta) = \int p(f^*|f, x^*, X, \theta) p(f|X, Y, \theta) df$.
- Choose two new tests at
$$x^{c_1} = x^c + a_1(1,0)', x^{c_2} = x^c + a_2(0,1)'.$$
- a_1 and a_2 are chosen such that $E\left(f(x^{c_i})\right) = G^{-1}(p), i = 1, 2.$
- Let the new center point be $x^c = (x^{c_1} + x^{c_2})/2.$
- In the beginning, choose $x^c = x^s$, whose latent value $f(x^s) = G^{-1}(0.5)$.
- If x^{c_1} and x^{c_2} are very close, choose the new x^s from $\mathbb{R}\backslash U$, where U is a given neighborhood of the previous starting point.
- After N samples are completed, update the estimation of θ and using the GP or logit model to approximate the curves.

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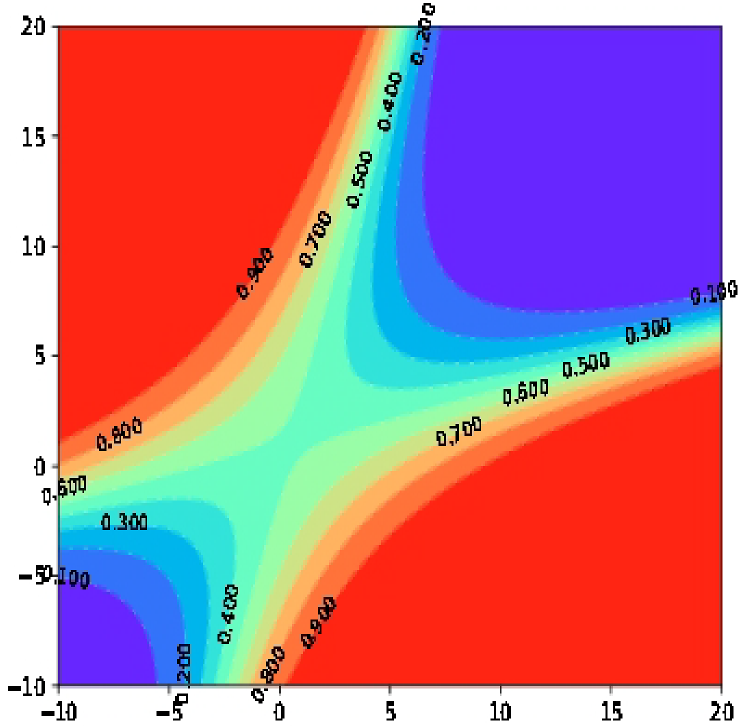
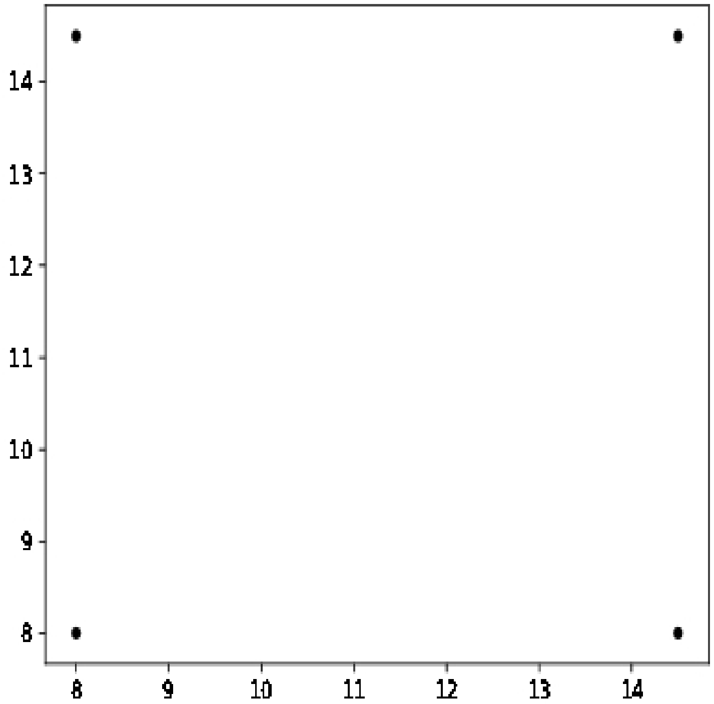
An Illustrative example



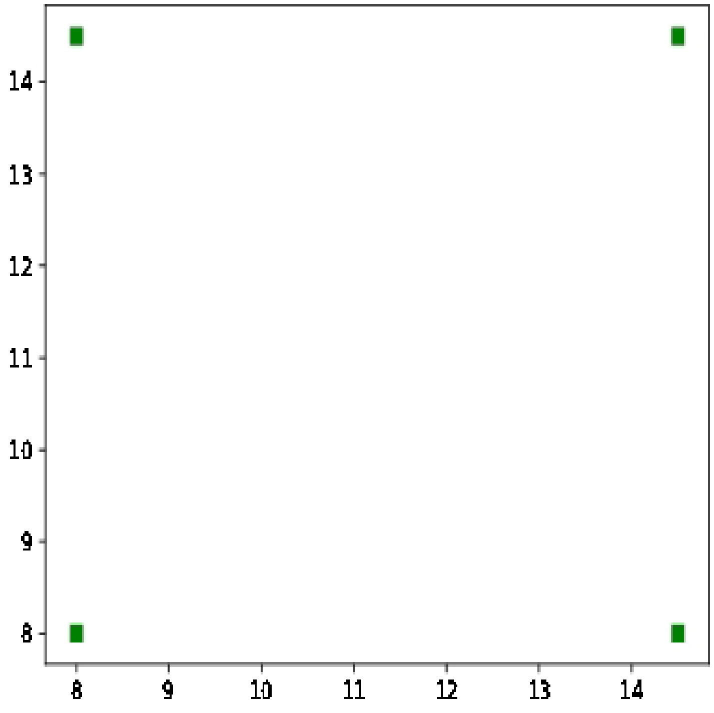
- $prob(y = 1|x) = G\big(f(x)\big)$.
- $G(z) = \frac{1}{1+\exp\{-z\}}$.
- $f(x) = \frac{1}{50}(x_1^2 + x_2^2 - 4x_1x_2 + 3x_1)$.
- The true contour of $p(y = 1)$ is given in the left figure.

Initial guess about the experimental region



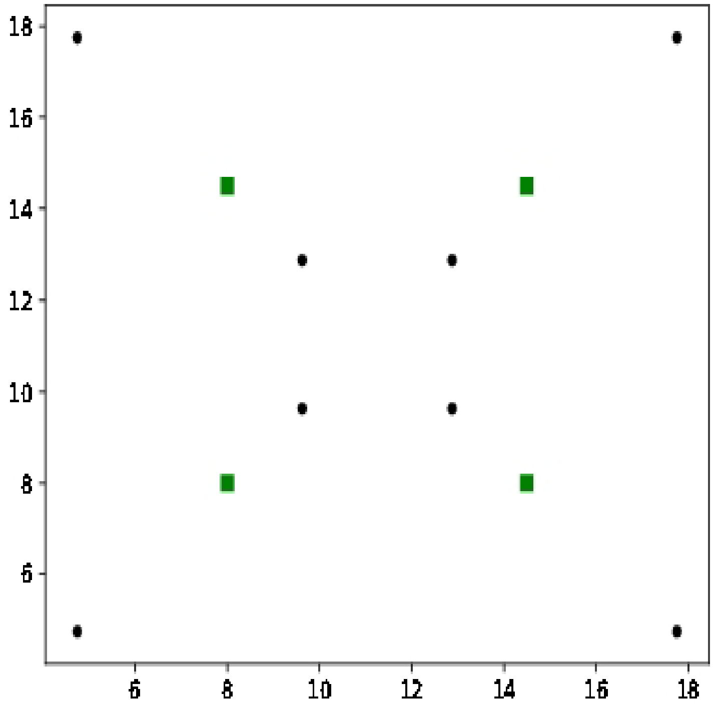
- The investigators can make a guess of the region of interest, $[8, 14.5] \times [8,14.5]$.
- Run tests at the four corners of the rectangles, $(8, 8), (8, 14.5), (14.5, 8), (14.5, 14.5)$.

Only non-response results are observed



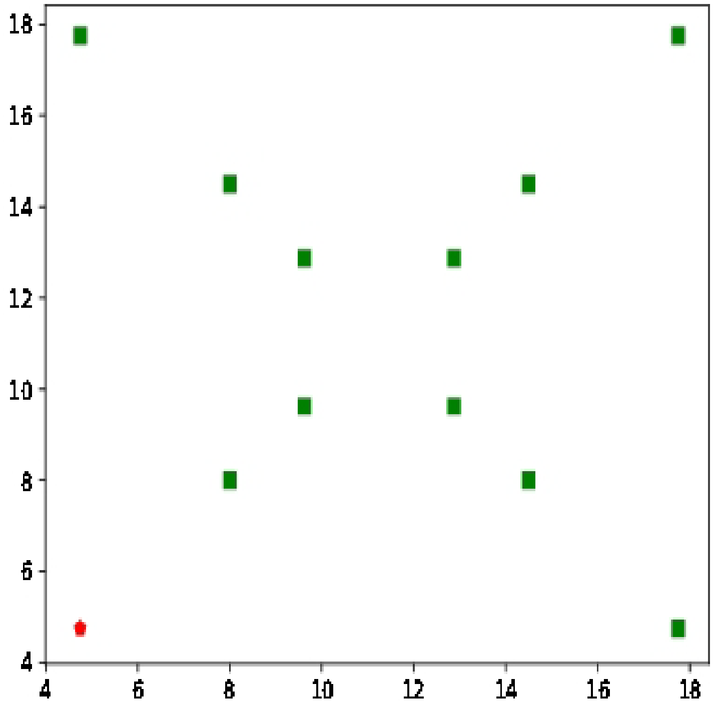
- This implies that the region of rectangle is too small or too large.
- Run additional tests both inside and outside the rectangle, until both $y = 1$ and $y = 0$ are observed.

Only non-response results are observed



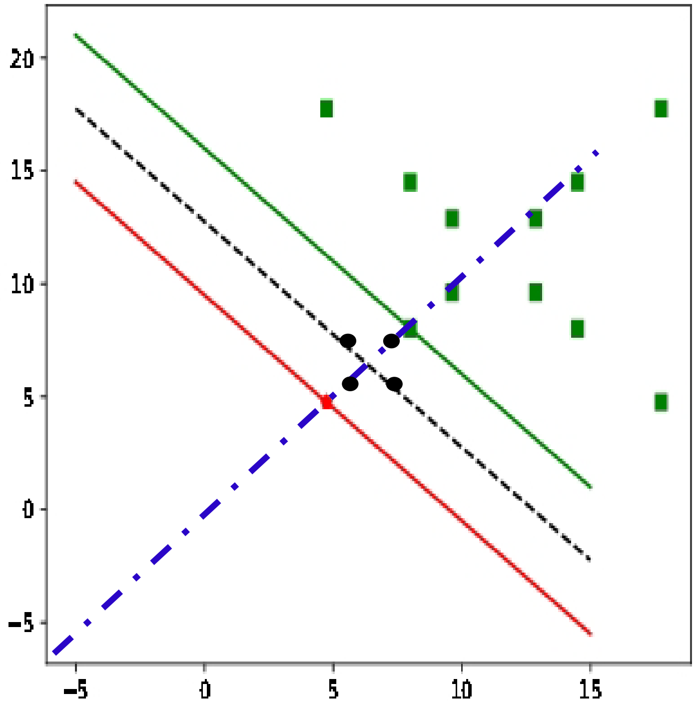
- This implies that the region of rectangle is too small or too large.
- Run additional tests both inside and outside the rectangle, until both $y = 1$ and $y = 0$ are observed.

Both results are observed



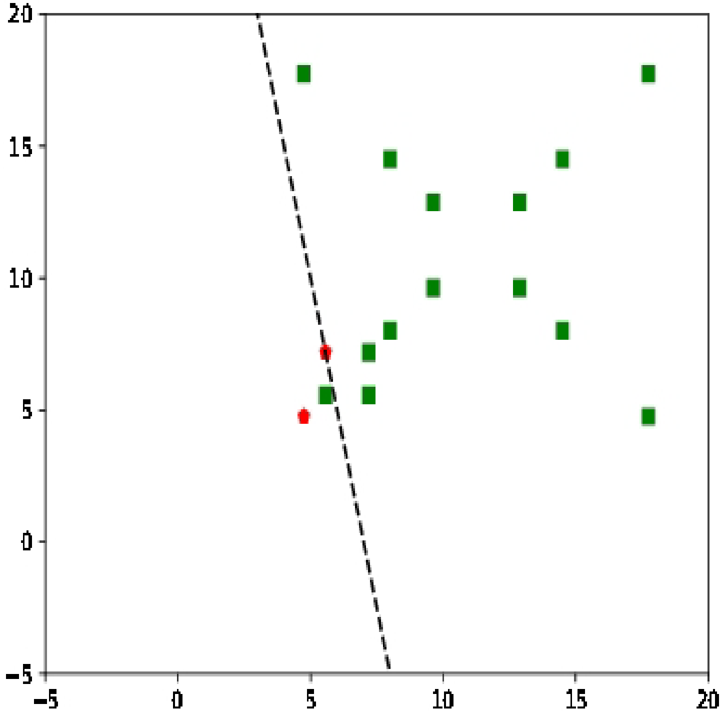
- Both response (y = 1) and non-response (y = 0) are observed but can be separated by a straight line.

Both results are observed



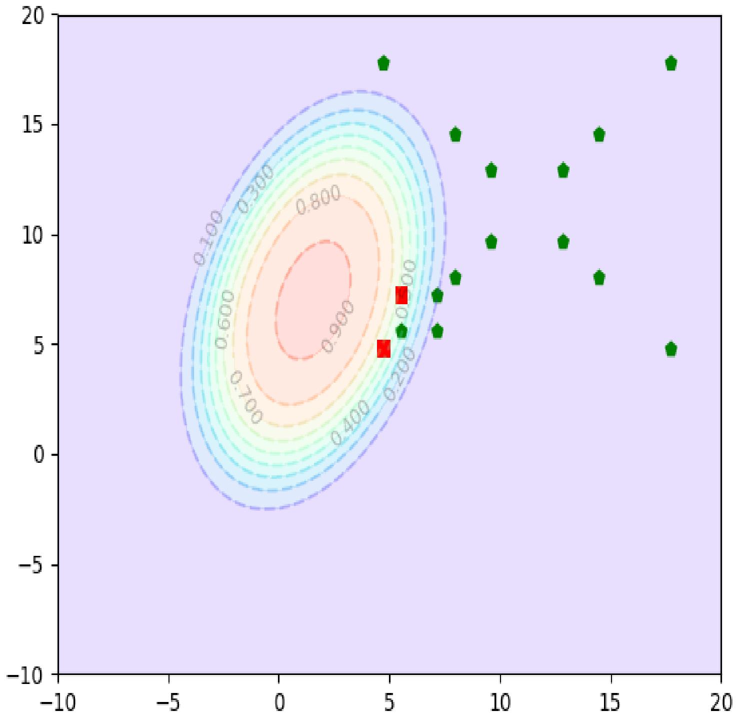
- Both response (y = 1) and non-response (y = 0) are observed but can be separated by a straight line.
- SVM (Support Vector Machine) is used to exploring the middle region.
- Choose additional tests (black points) in the middle region.

Overlapping pattern is achieved



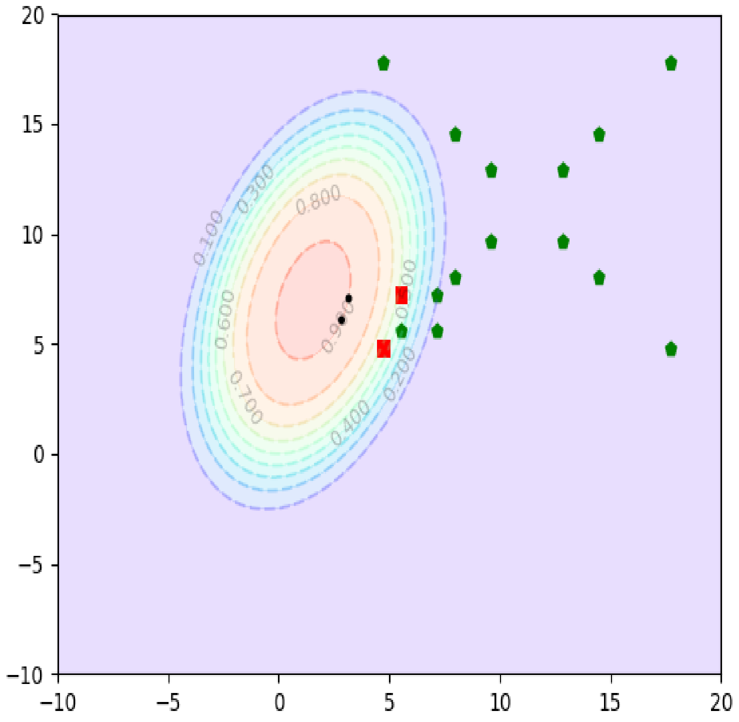
- Runs with (y = 0) and (y = 1) cannot be separated by a straight line.
- The overlapping pattern is achieved.

Fit GLM to approximate curves and select next tests



- Fit GLM based on observed data in step I.

Fit GLM to approximate curves and select next tests



- Fit GLM based on observed data in step I.
- The contours of quantiles based on GLM (color dash curves)
- Choose two new tests (black points).

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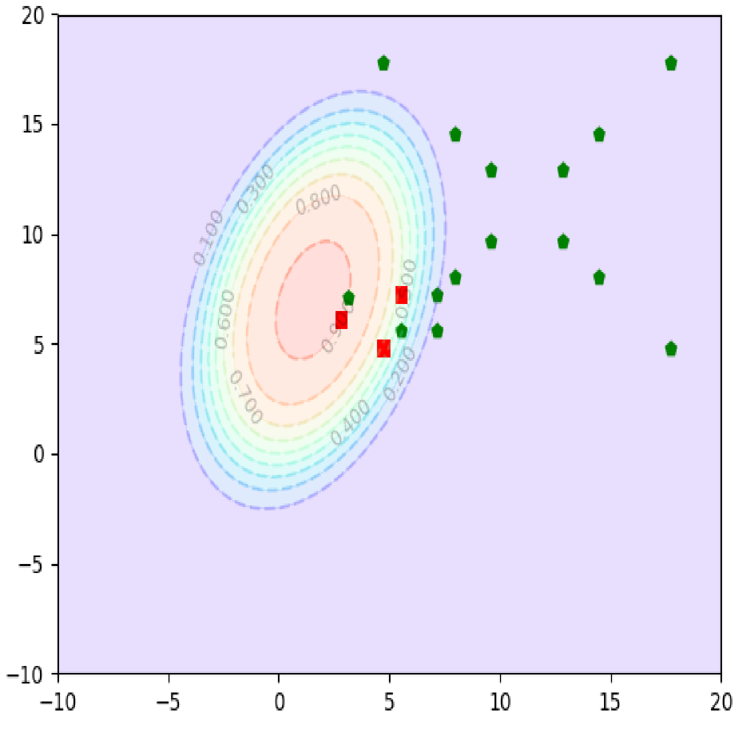
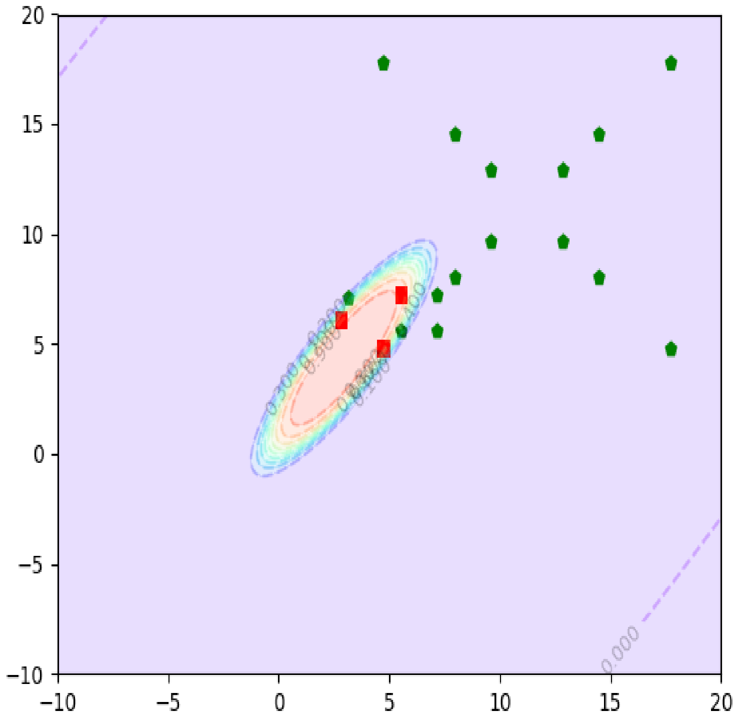
Fit GLM to approximate curves and select next tests



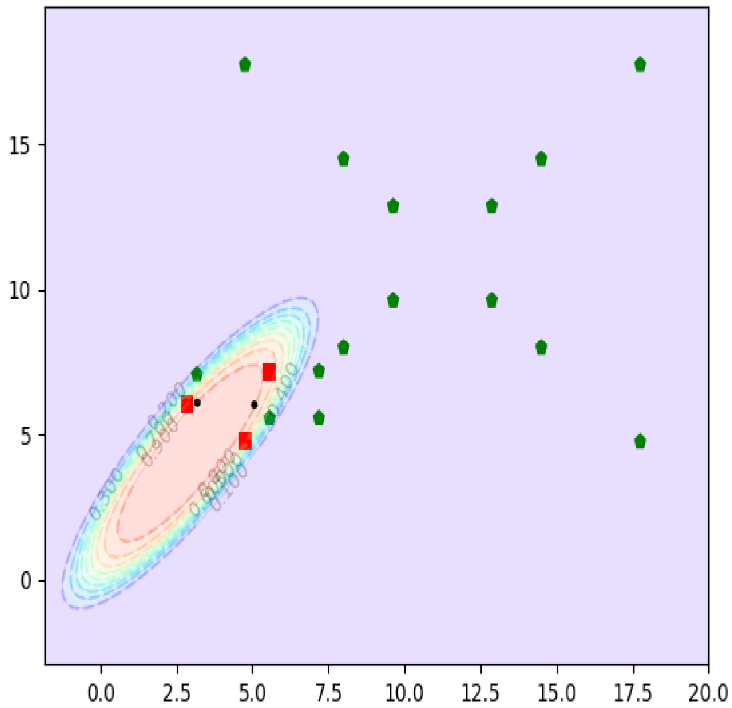
- Fit GLM based on observed data in step I.
- The contours of quantiles by GLM (color dash lines)
- Choose two tests (black points).
- Run tests at the new locations.

Fit GLM to approximate curves and select next tests



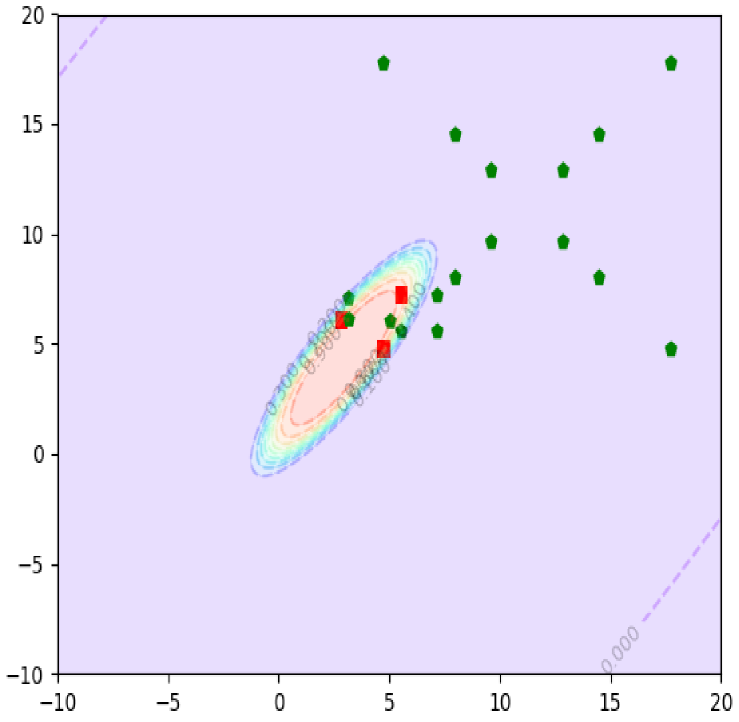
- Re-fit the GLM based on current data.

Fit GLM to approximate curve and select next tests



- Re-fit the GLM based on current data.
- Choose new tests (black).

Fit GLM to approximate curve and select next tests



- Re-fit the GLM based on current data.
- Choose new tests.
- Run tests at the new locations.

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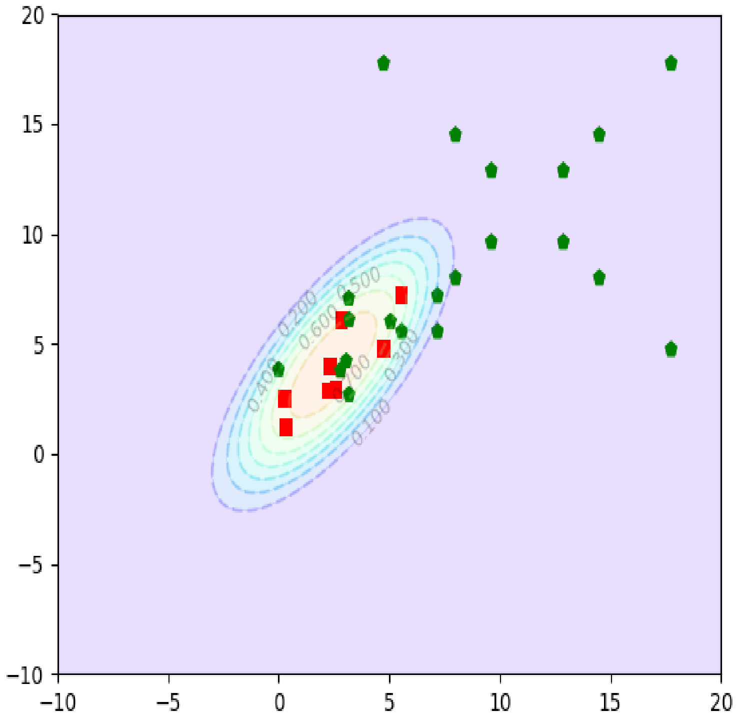
An interim summary



- No information outside current experimental region.
- GLM over-fits observed data.
- Tests trapped in local region.

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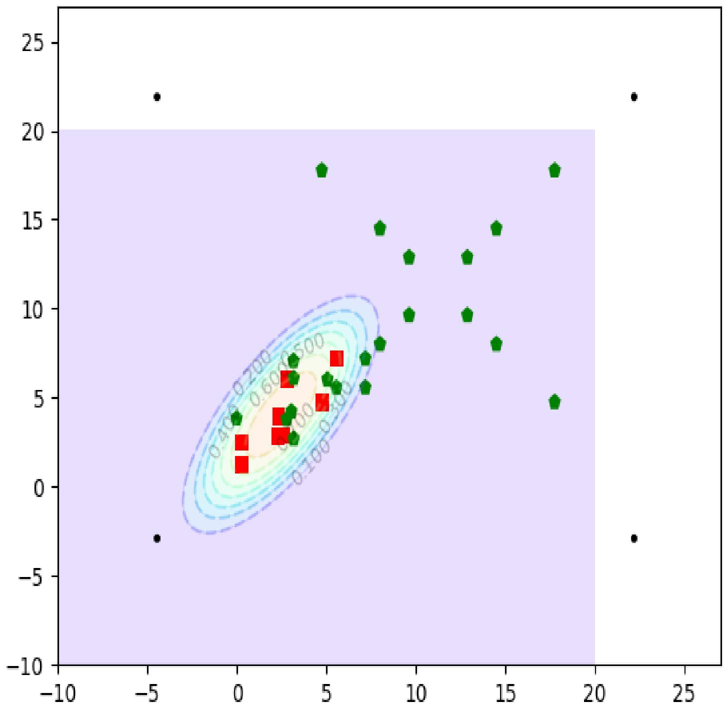
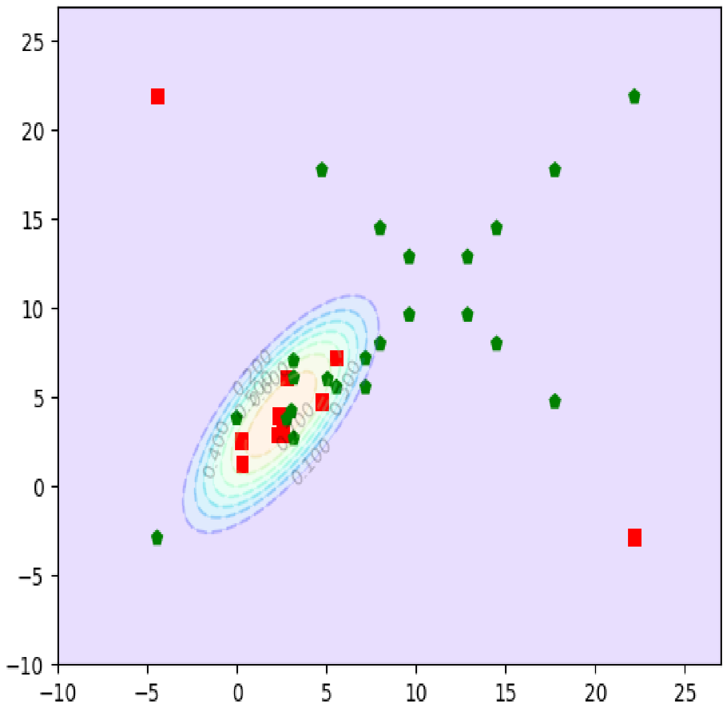
Extend the experimental region



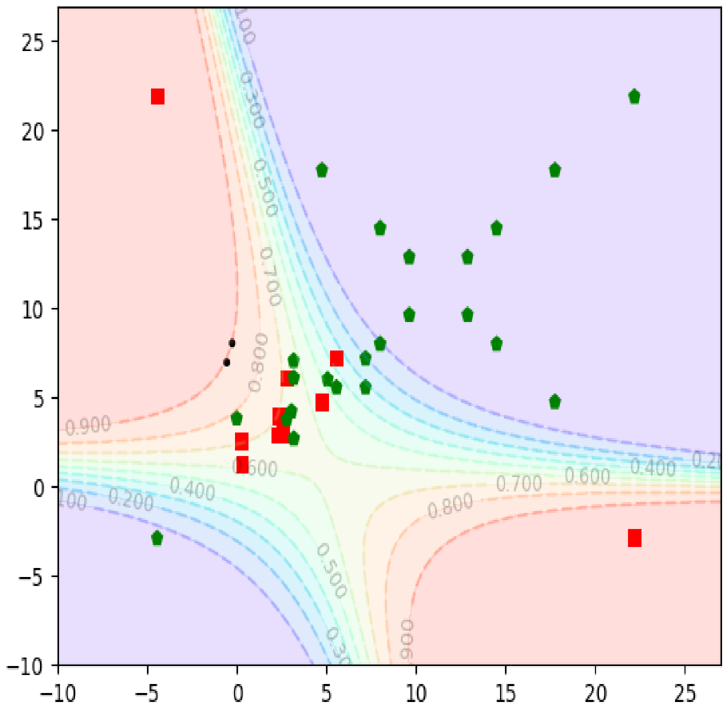
- Choose four tests (black) outside the current region.

Extend the experimental region



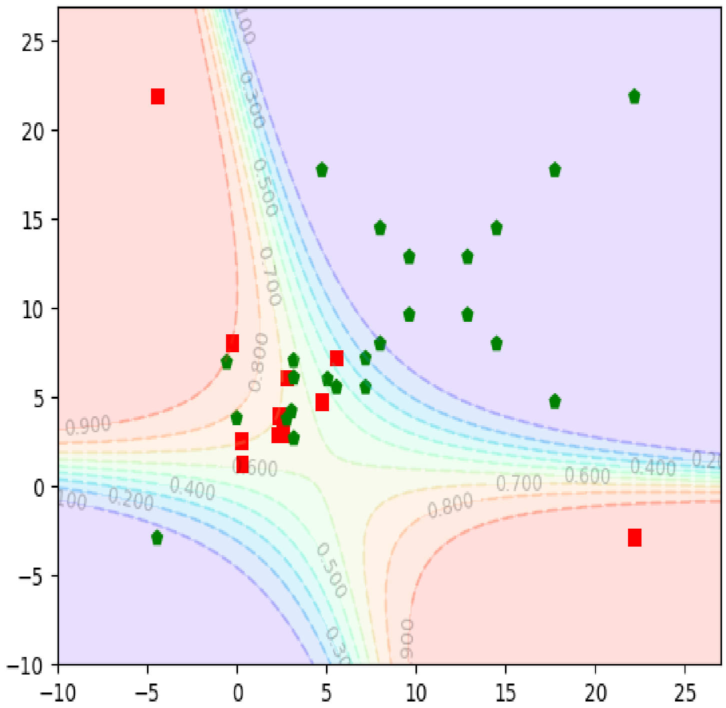
- Choose tests outside the current region.
- Run tests at the new locations.

Extend the experimental region



- Choose tests outside the current region.
- Run tests at the new locations.
- Fit GLM again and choose new tests (black).

Extend the experimental region



- Choose tests outside the current region.
- Run tests at the new locations.
- Fit GLM again and choose new tests.
- Run tests.

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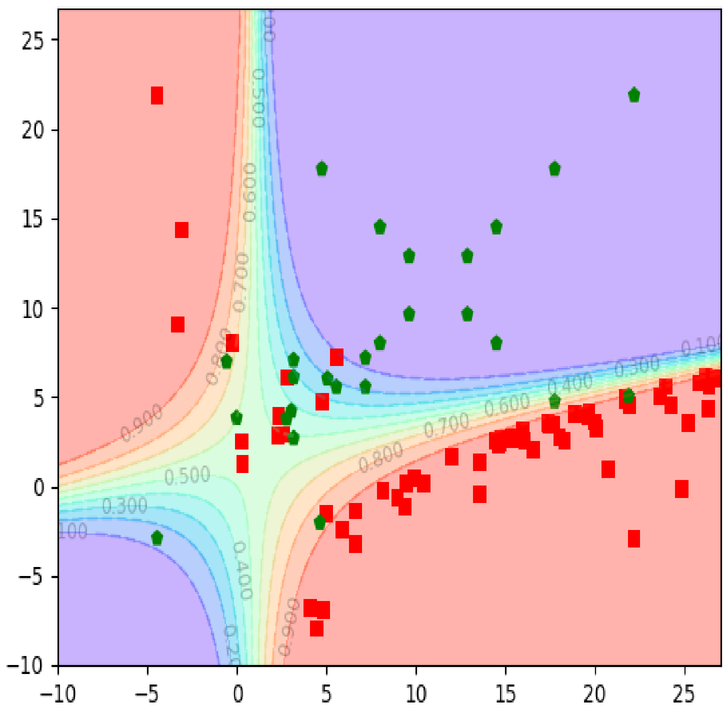
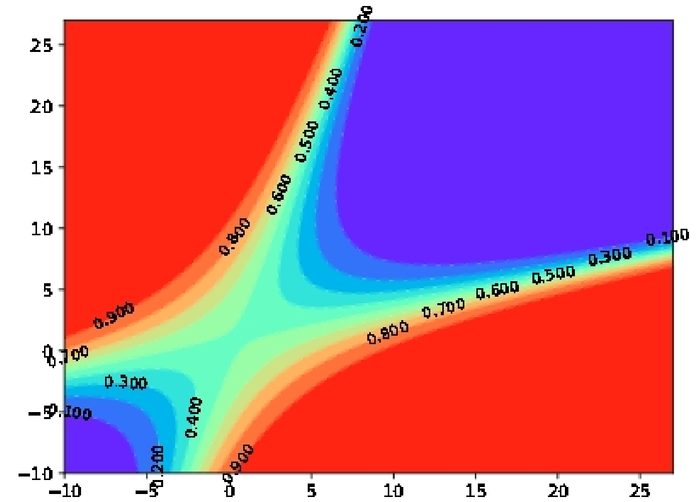
Final fit: curve (left) approximation by GLM (true curve on the right)





Comments and further work

- As far as we know, there is *no* known procedure for sensitivity testing with two or more stress factors. But the problems are encountered in practice. A good procedure is sorely needed!
- The ideas in the proposed procedure are still preliminary, need to be fine-tuned and modified.
- Need a small numerical or simulation study to understand its performance; then do a field test.
- Extension to 3 or more factors.

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