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AUTHORITY

USARO ltr, 15 Mar 1971

ARO-D Report 68-2

PROCEEDINGS OF THE THIRTEENTH CONFERENCE ON THE DESIGN OF EXPERIMENTS IN ARMY RESEARCH DEVELOPMENT AND TESTING

Sponsored by

[illegible] Information Steering Committee

[illegible]

U.S. Army Research Office - Durham
Report No. 68-2
November 1968

PROCEEDINGS OF THE THIRTEENTH CONFERENCE ON THE DESIGN OF EXPERIMENTS IN ARMY RESEARCH, DEVELOPMENT AND TESTING

Sponsored by the Army Mathematics Steering Committee

Host

The U.S. Army Mobility Equipment and Development Center

and

The U.S. Army Engineer Topographic Laboratories

1-3 November 1967

U.S. Army Research Office - Durham
Box CM, Duke Station
Durham, North Carolina

FOREWORD

The theme of the Thirteenth Conference on the Design of Experiments in Army Research, Development and Testing, as suggested by Dr. Walter Foster, was "Design and Analysis for Engineering Experimentation". This was a very appropriate theme in the light of the recent activities of the two hosts for the meeting. This conference was held at Fort Belvoir, Virginia, on 1-3 November 1967, and the U.S. Army Mobility Equipment Research and Development Center served together with the U.S. Army Engineer Topographic Laboratories as joint hosts. The Army Mathematics Steering Committee, sponsors of these meetings on behalf of the Office of the Chief of Research and Development, would like to thank these two agencies for so ably serving the conference in this capacity. A large number of persons at Fort Belvoir helped with the various details needed to run a meeting of this size. We would like to express the thanks of the attendees for the many courtesies shown them. In particular, their thanks are due to Mr. James B. Duff, Chairman on Local Arrangements, for his excellent execution of the many details needed to make the symposium run smoothly.

The invited speakers for the conference featured five nationally known scientists. Their names and the titles of their addresses are noted below:

Regression Analysis
Professor Francis J. Anscombe, Yale University

Some Comments on Matching
Professor K.A. Brownlee, University of Chicago

Some Statistical Methods in Machine Intelligence Research
Professor I.J. Good, Virginia Polytechnic Institute

Maximum Likelihood Estimation of Reliability
Dr. Frank Proschan, Boeing Company

Data Analysis
Dr. M.B. Wilk, Bell Telephone Laboratories

In addition to these talks, there were 29 contributed papers which covered a wide range of design and statistical problems. Following the banquet, which was held at the Officers' Club, it was my pleasure to present the Third Wilks Memorial Medal to Professor William G. Cochran of Harvard University. We are pleased to be able to include in these Proceedings Dr. Cochran's acceptance speech.

This volume of the Proceedings contains 26 of the papers which were presented at this meeting. The Army Mathematics Steering Committee has asked that these articles on modern principles on the design of experiments, together with the application of these ideas, be made available in the form of this technical manual. Members of this committee take this opportunity to express their thanks to the many speakers and other research workers who participated in the conference.

The conference had an attendance of 173 scientists; and 71 organizations were represented. Speakers and panelists came from Yale University, University of Chicago, North Carolina State University at Raleigh, the National Institutes of Health, Harvard Computing Center, University of North Carolina at Chapel Hill, University of Georgia, Cornell University, University of Wisconsin, Boeing Scientific Research Laboratories, Stanford University, Duke University, Virginia Polytechnic Institute, Stanford Research Institute, the National Aeronautics and Space Administration, Bell Telephone Laboratories, and fourteen Army facilities.

Lieutenant Colonel John H. Cain, Deputy Commander of the U.S. Army Mobility Equipment Research and Development Center, and Lieutenant Colonel William R. Cordova, Deputy Commander of the U.S. Army Engineer Topographic Laboratories, both welcomed members of the conference to Fort Belvoir. Their comments to those in attendance contained many interesting statements about the work being performed by the host installation. Their remarks are printed here for the benefit of those who did not have an opportunity to attend the symposium.

The Chairman wishes to take this opportunity to thank members of his Advisory Committee (Cuthbert Daniel, F.G. Dressel, Walter D. Foster, Fred Frishman, Lawrence Gambino, Bernard Greenberg, Bernard Harris, Boyd Harshbarger, J.S. Hunter, William Kruskal, H.L. Lucas, Jr., Clifford Maloney, and Frank Robertson) for their assistance in formulating the agenda and their help in selecting the invited speakers.

Frank E. Grubbs
Conference Chairman

WELCOME REMARKS

LTC John H. Cain

Good morning, ladies and gentlemen!

I am Colonel Cain, deputy commander of the U.S. Army Mobility Equipment Research and Development Center, your co-host for this three-day meeting. Speaking on behalf of Colonel O'Donnell, the Center's commanding officer, I am happy to welcome you here today.

I want to thank the Army Mathematics Steering Committee for sponsoring its 13th annual Conference on the Design of Experiments at the R&D Center. It gives so many more of our people an opportunity to become acquainted with the latest in statistical and mathematical methods for application in their scientific and engineering work.

Since the R&D Center is both co-host and participant I would like to take a few minutes to acquaint you briefly with its mission and facilities.

The Center [see the first of the following figures] was for 20 years, until two months ago today to be exact, the Engineer Research and Development Laboratories. The change in name, however, in no way changed its location in the Defense chain shown here. Now, as then, the Center is THE R&D agency of the Mobility Equipment Command in St. Louis, a major sub-command of the Army Materiel Command.

Our mission [second figure] remains the same, and in each aspect, from research thru engineering for procurement, the goal remains the ultimate in mobility equipment for the Army.

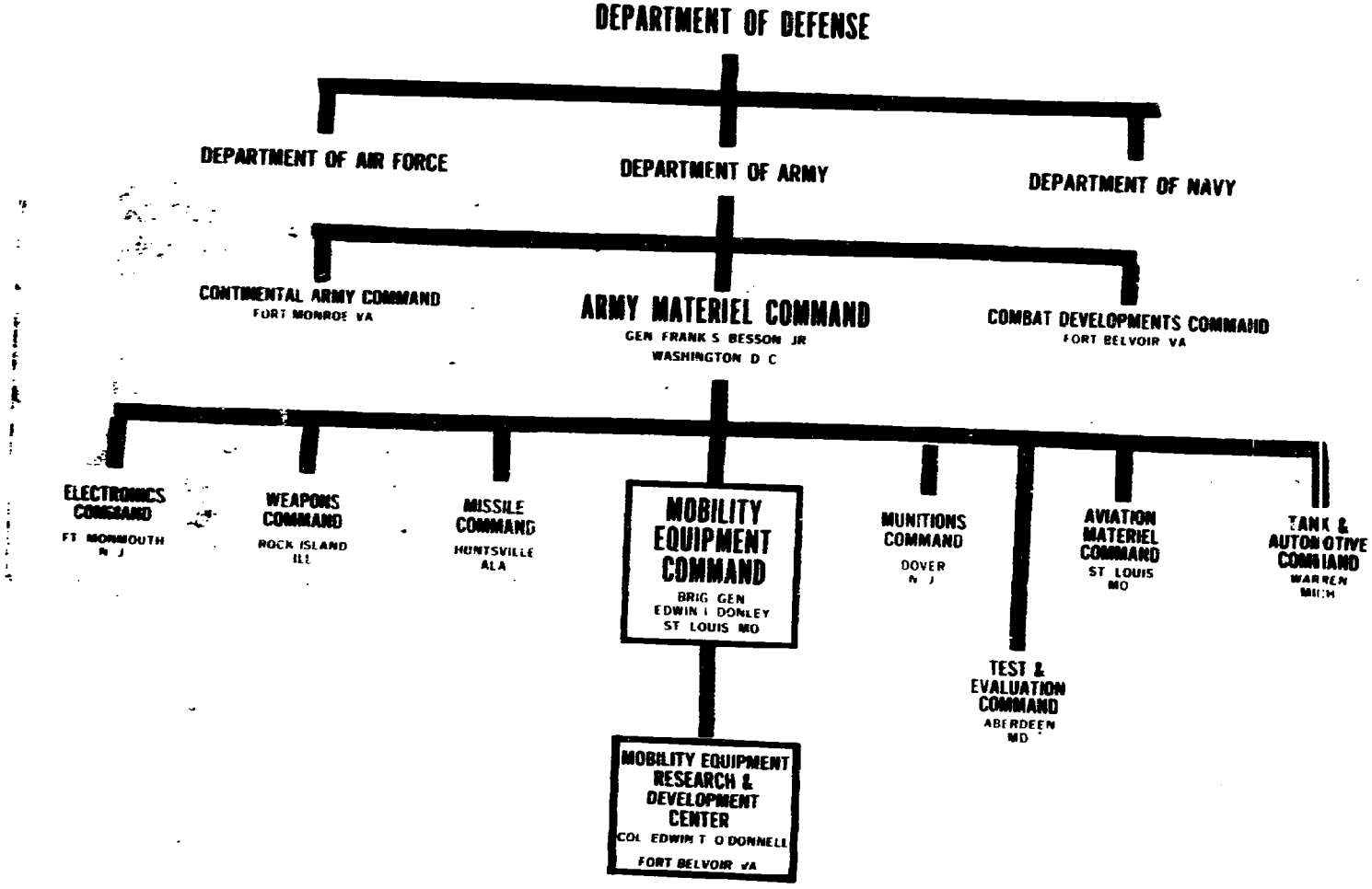
To achieve this goal [third figure] the Center engages in R&D in some 13 areas. You can see from the diversity of these fields of endeavor, that a wide range is offered for design of experiments.

Our organization, as shown here [fourth figure] features four R&D Laboratories: Military, Electro, and Mechanical Technology, and Intrusion Detection and Sensor. The Engineering Laboratory prepares technical data packages which give industry the specifications, drawings and other information it needs to build quality mobility equipment in quantity.

Some 1400 scientific, engineering and support personnel are employed at the Center [last figure]. The main physical plant is just down the road a piece. Approximately 30 permanent structures on a 240-acre site house of the best R&D facilities in the country. Additional test facilities are supplied at the 900 acre annex on Belvoir's North Area.

The Center, you will note has several tenants. One of these, the U.S. Army Engineer Topographic Laboratories, is co-host for this conference.

To give ETL a chance to add its welcome, I will close now with best wishes to Chairman Dr. Frank E. Grubbs and to all of you for a lucky 13th.


DEPARTMENT OF DEFENSE
DEPARTMENT OF AIR FORCE
DEPARTMENT OF ARMY
DEPARTMENT OF NAVY
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FORT MONROE VA
ARMY MATERIEL COMMAND
GEN FRANK S BESSON JR
WASHINGTON D C
COMBAT DEVELOPMENTS COMMAND
FORT BELVOIR VA
ELECTRONICS COMMAND
FT MONMOUTH N J
WEAPONS COMMAND
ROCK ISLAND ILL
MISSILE COMMAND
HUNTSVILLE ALA
MOBILITY EQUIPMENT COMMAND
BRIG GEN EDWIN I DONLEY
ST LOUIS MO
MUNITIONS COMMAND
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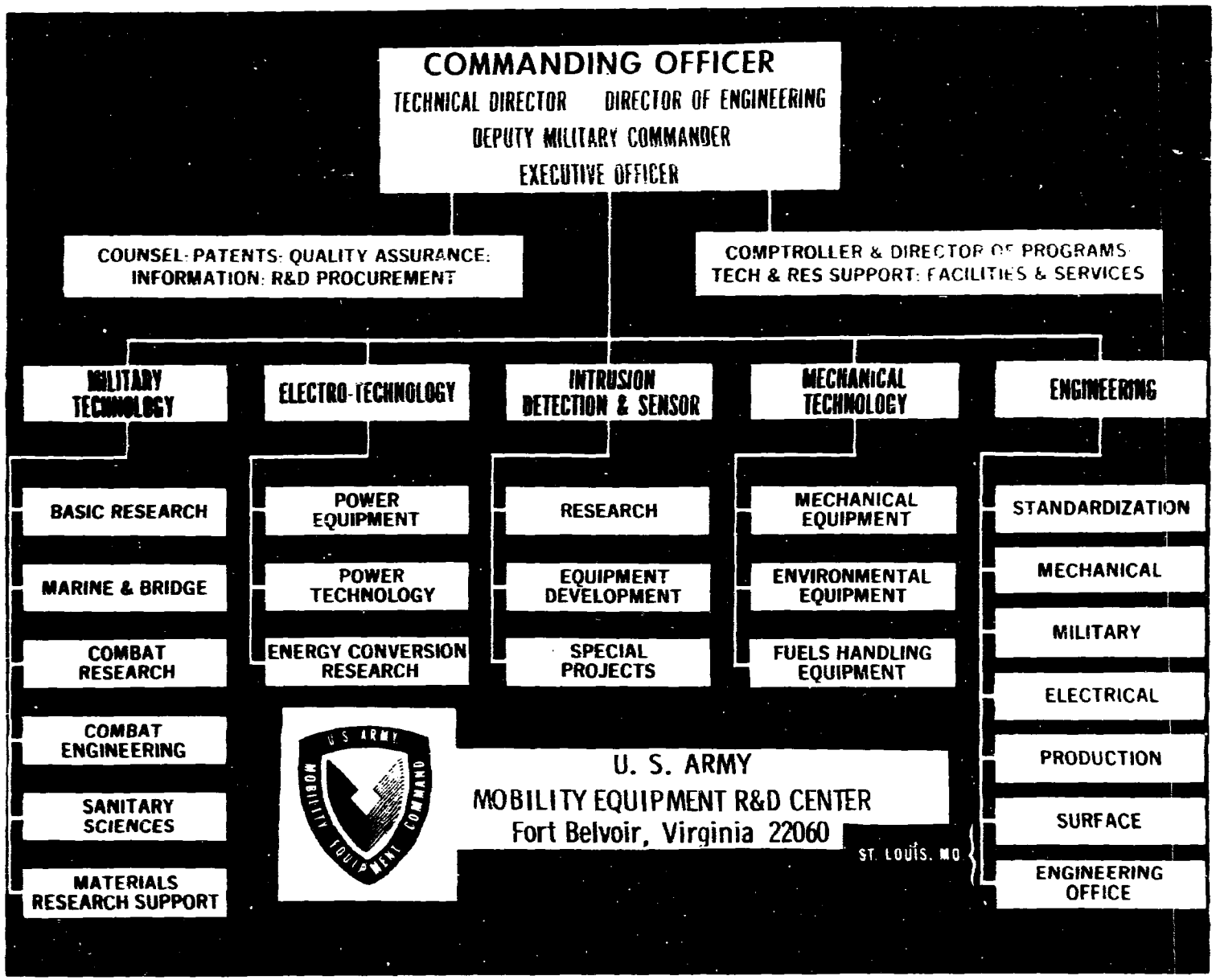
TEST & EVALUATION COMMAND
ABERDEEN MD
AVIATION MATERIEL COMMAND
ST LOUIS MO
TANK & AUTOMOTIVE COMMAND
WARREN MICH
MOBILITY EQUIPMENT RESEARCH & DEVELOPMENT CENTER
COL EDWIN T O'DONNELL
FORT BELVOIR VA


COMMANDING OFFICER
TECHNICAL DIRECTOR
DIRECTOR OF ENGINEERING
DEPUTY MILITARY COMMANDER
EXECUTIVE OFFICER
COUNSEL: PATENTS: QUALITY ASSURANCE: INFORMATION: R&D PROCUREMENT
COMPTROLLER & DIRECTOR OF PROGRAMS: TECH & RES SUPPORT: FACILITIES & SERVICES
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ELECTRO-TECHNOLOGY
INTRUSION DETECTION & SENSOR
MECHANICAL TECHNOLOGY
ENGINEERING
BASIC RESEARCH
MARINE & BRIDGE
COMBAT RESEARCH
COMBAT ENGINEERING
SANITARY SCIENCES
MATERIALS RESEARCH SUPPORT
POWER EQUIPMENT
POWER TECHNOLOGY
ENERGY CONVERSION RESEARCH
RESEARCH
EQUIPMENT DEVELOPMENT
SPECIAL PROJECTS
MECHANICAL EQUIPMENT
ENVIRONMENTAL EQUIPMENT
FUELS HANDLING EQUIPMENT
STANDARDIZATION
MECHANICAL
MILITARY
ELECTRICAL
PRODUCTION
SURFACE
ENGINEERING OFFICE
U. S. ARMY
MOBILITY EQUIPMENT R&D CENTER
Fort Belvoir, Virginia 22060
ST. LOUIS, MO

U. S. ARMY
MOBILITY EQUIPMENT R&D CENTER
Fort Belvoir, Virginia 22060

R & D. FIELDS OF ENDEAVOR

BRIDGES AND MARINE CRAFT

SANITARY ENGINEERING

CAMOUFLAGE AND DECEPTION

FORTIFICATIONS AND BARRIERS

ELECTRIC POWER GENERATION

ELECTRIC PROPULSION

INDUSTRIAL ENGINES AND TURBINES

DETECTORS AND SENSORS

ENVIRONMENTAL CONTROL

CONSTRUCTION EQUIPMENT

MATERIALS AND CARGO HANDLING

FUELS HANDLING

NUCLEAR WEAPONS EFFECTS

U.S. ARMY MOBILITY EQUIPMENT RESEARCH & DEVELOPMENT CENTER, FORT BELVOIR, VA.

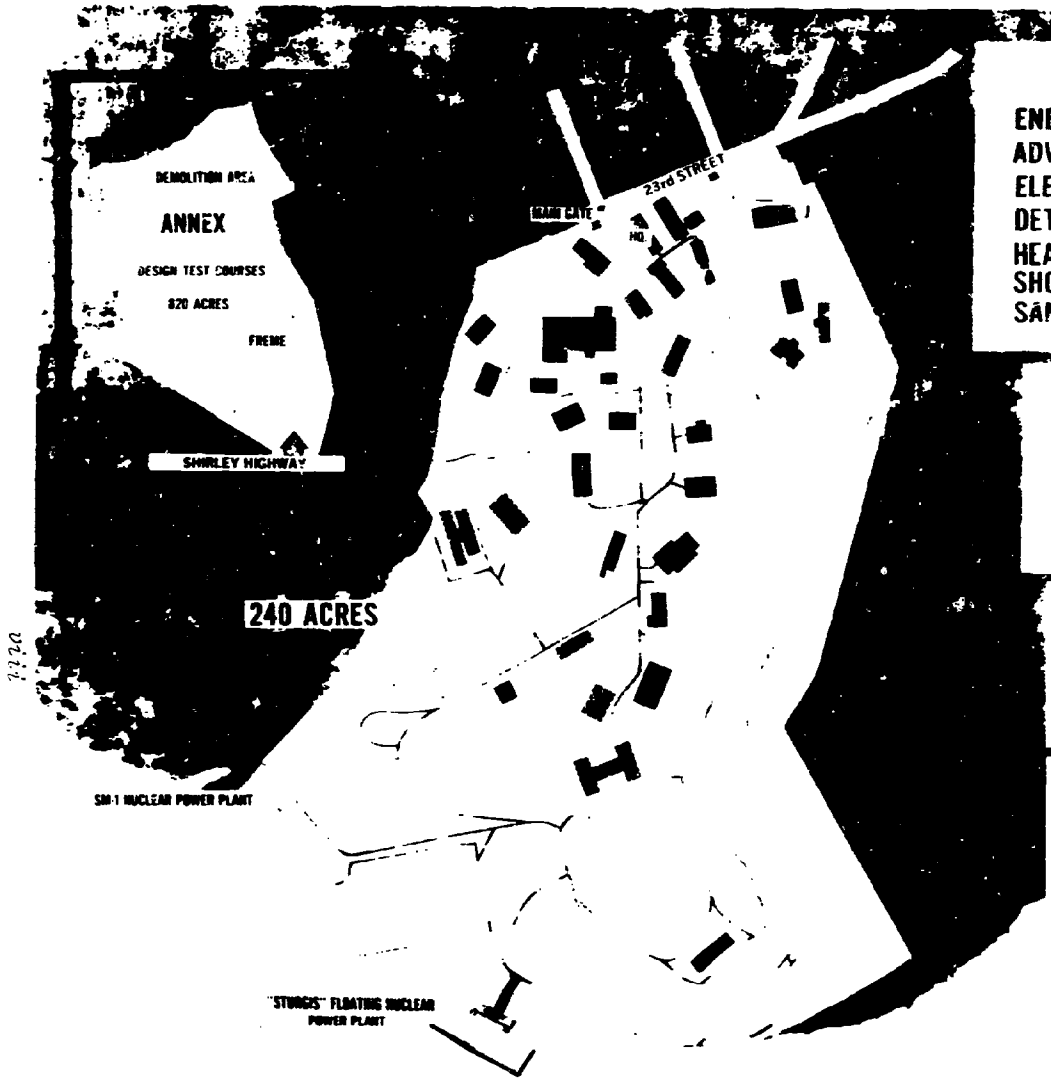
RESEARCH AND DEVELOPMENTAL ENGINEERING OF EQUIPMENT....

ADAPT COMMERCIAL EQUIPMENT TO MILITARY REQUIREMENTS....

DEVELOP TECHNIQUES AND APPLICATIONS....

ENGINEER FOR QUANTITY PROCUREMENT....

A WIDE RANGE OF EQUIPMENT IS BEING DEVELOPED TO ACCELERATE THE MOBILITY OF OUR FORCES, AND IMPEDE THE MOVEMENT OF THE ENEMY ON ANY BATTLEFIELD.



● laboratories

ENERGY CONVERSION	BASIC RESEARCH
ADVANCED POWER EQUIP	MATERIALS
ELECTRIC PROPULSION	CLIMATIC
DETECTION & SENSOR	POWER PLANT
HEATING & AIR COND.	METROLOGY
SHOCK & VIBRATION	PICTORIAL
SANITARY SCIENCES	COMPUTER

● future labs

ELECTRIC POWER

INDUSTRIAL ENGINEERING

● shops

MODEL, PAINT & PLASTICS:
MACHINE; WOODWORKING;
MODIFICATION & REPAIR

● test areas

MECHANICAL
PETROLEUM EQUIPMENT
MINE DETECTION LANES
BRIDGE & MARINE

tenants

- NIGHT VISION LAB & PROJ. MGR.
- NUCLEAR POWER FIELD OFFICE
  ARMY ENGINEER TOPO. LABS
- ELECTRIC POWER PROJECT MGR.

U S ARMY
MOBILITY EQUIPMENT RESEARCH AND DEVELOPMENT CENTER

WELCOME REMARKS

LTC William R. Cordova

Thank you Colonel Cain.

On behalf of Colonel Anderson, the Commanding Officer of the U.S. Army Engineer Topographic Laboratories, I take pleasure in welcoming you this morning.

I might begin by saying that we too have had a recent name change, prior to 28 July 1967 being known as the U.S. Army Engineer, Geodesy, Intelligence and Mapping Research and Development Agency, the acronym being GIMRADA.

The mission of USAETL is as follows:

The U.S. Army Engineer Topographic Laboratories (ETL) is a Class II activity under the Chief of Engineers. It is the principal field activity of the Corps of Engineers for the accomplishment of research and development of equipment, procedures and techniques in the specific field of geodesy, military geography and mapping for application both to troop and to base plant operations. The Chief of Engineers may assign work to these Laboratories under research and development projects utilizing either RDT&E funds or other appropriate funds.

Our research and development program in Mapping and Geodesy includes activities within the entire spectrum from basic research through exploratory development, advanced systems development and finally engineering development, where a particular system or item is engineered for production and service use.

Our primary goals are as follows:

a. Develop the capability to provide current and adequate "Terrain Data" when and where needed for military purposes.

b. Minimize the geodesy and gravity portion of the error budget of weapons and missiles systems.

c. Maintain superiority in technology to be able to project the state of art and to provide meaningful forecasts to customers.

The USAETL organization is comprised of two major technical operating elements:

a. The Research Institute, which conducts basic and applied research and individually oriented exploratory development involving the disciplines related to mapping and geodetic sciences, is located in GSA rental space in Alexandria, Virginia. I believe a number of you know Mr. Larry Gambino of the Research Institute who will be presenting

a paper at this session.

b. Our Mapping and Geographic Sciences Laboratory, which conducts feasibility studies, design, development and tests and evaluation of systems, equipment and techniques in the specific fields of mapping, geodesy and geographic sciences, is located within the MERDC area along with the Headquarters and support offices. We currently have 28 trailers. The Chief of Engineers has approved a site on the North Fort Belvoir Post for our new building which we hope to get approved in the FY 69 budget.

I hope this very brief presentation has given you a general feel for our mission, goals and work. If I can be of assistance to any of you, please stop by my office.

Thank you very much. I hope you have an enjoyable and fruitful conference.

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* This paper was presented at the conference. It does not appear in the Proceedings.

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THIRTEENTH CONFERENCE ON THE DESIGN OF EXPERIMENTS

IN ARMY RESEARCH, DEVELOPMENT AND TESTING

1-3 November 1967

Wednesday, 1 November

0800-0900 REGISTRATION - Main Lobby of Humphrey's Hall

0900-0915 OPENING OF THE CONFERENCE - Auditorium of Humphrey's Hall

James B. Duff, Chairman on Local Arrangements, U.S. Army Mobility Equipment Research & Development Center

WELCOME

Colonel Edwin T. O'Donnell, CE, Commanding Officer
U.S. Army Mobility Equipment Research & Development Center
and
Colonel Edward G. Anderson, CE, Commanding Officer
U.S. Army Engineer Topographical Laboratories

0915-1130 GENERAL SESSION 1 - Auditorium*

Chairman: Dr. Walter D. Foster, Biomathematics Division, U.S. Army Biological Laboratories, Fort Detrick, Frederick, Maryland

REGRESSION ANALYSIS

Professor Francis J. Anscombe, Department of Statistics, Yale University, New Haven, Connecticut

SOME COMMENTS ON MATCHING

Professor K.A. Brownlee, Statistics Research Center, University of Chicago, Chicago, Illinois

1130-1300 LUNCH

1300-1500 CLINICAL SESSION A - Auditorium

Chairman: Joseph Weinstein, Electronics Components Laboratory, U.S. Army Electronics Command, Fort Monmouth, New Jersey

Panelists:
Francis J. Anscombe, Yale University
Cuthbert Daniel, Private Consultant, New York, N.Y.
Frank E. Grubbs, U.S. Army Ballistic Research Laboratories
William Kruskal, University of Chicago
H.L. Lucas, Jr., North Carolina State University
Clifford J. Maloney, National Institutes of Health

*All sessions of the conference will be held in Humphrey's Hall. Lunches will be served in Mackenzie Hall.

Wednesday (Continued)

ON METHODS OF OPTIMIZATION OF A MULTIOBJECTIVE SURVEY

John C. Atkinson, Harvard Computing Center, Medical Branch, Boston, Massachusetts

COMPONENTS OF VARIANCE OF A LINEAR FUNCTION IN REPEATED TRIALS

Walter D. Foster, Biomathematics Division, U.S. Army Biological Laboratories, Fort Detrick, Frederick, Maryland

1300-1500 TECHNICAL SESSION I - Room 2E

Chairman: Cyrus Martin, Quality Assurance Group, U.S. Army Engineer Topographic Laboratories, Fort Belvoir, Virginia

THE DERIVATION OF THE OPERATING CHARACTERISTIC CURVE OF A SKIP LOT SAMPLING PLAN

Allen C. Endres, U.S. Army Ammunition Procurement and Supply Agency, Joliet, Illinois

A MODEL FOR THE FORMULATION OF QUALITY INCENTIVE CLAUSES FOR ITEMS PROCURED ACCORDING TO ACCEPTANCE CRITERIA INVOLVING SINGLE SAMPLING PLANS BY ATTRIBUTES

Roger R. Rymer and Eugene Dutoit, Picatinny Arsenal, Dover, New Jersey

OPTIMUM SAMPLING PLANS FOR GRADING BINOMIAL POPULATIONS

Paul B. Nickens, Surveillance and Reliability Laboratory, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland

1300-1500 TECHNICAL SESSION II - Room 2F

Chairman: Frank Robertson, U.S. Army Mobility Equipment Research and Development Center, Fort Belvoir, Virginia

METHODOLOGY OF ASSESSMENT OF BIOCELLULAR REACTIONS TO ABSORBED ENERGY

George I. Lavin, Terminal Ballistic Laboratory, U.S. Army Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland

THE EFFECT OF INVENTORY FORECASTING UPON SUPPLY EFFECTIVENESS

Patsy Courtney, U.S. Army Aviation Material Command, St. Louis, Missouri

Wednesday (Continued)

THE ABBA SEQUENCE: A SEQUENTIAL PROCEDURE FOR COMPARISON TESTING

Arthur Pillersdorf, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland

1500-1530 BREAK

1530-1700 TECHNICAL SESSION III - Auditorium

Chairman: Gideon A. Culpepper, Missile Test and Evaluation Control Division, White Sands Missile Range, New Mexico

ON EXPECTED PROBABILITIES OF MISCLASSIFICATION IN DISCRIMINANT ANALYSIS

P.A. Lachenbruch, School of Public Health, Department of Biostatistics, University of North Carolina, Chapel Hill, North Carolina

INTRA-PROFILE VARIANCE

Claude F. Bridges, Institutional Research Division, Office of Research, U.S. Military Academy, West Point, N.Y.

1530-1700 TECHNICAL SESSION IV - Room 2E

Chairman: Henry Ellner, Quality Assurance Directorate, U.S. Army Materiel Command, Washington, D.C.

A STATISTICAL TEST OF TWO HYPOTHETICAL RELIABILITY GROWTH CURVES OF THE LOGISTIC FORM IN THE DISCRETE CASE

William P. Henke, Research Analysis Corporation, McLean, Virginia

ON FITTING OF THE WEIBULL DISTRIBUTION WITH NON-ZERO LOCATION PARAMETERS AND SOME APPLICATIONS

Oskar M. Essenwanger, Physical Sciences Laboratory, Research and Development Division, Redstone Arsenal, Alabama

1730-1830 SOCIAL HOUR - Mackenzie Hall (Officer's Club)*

1830- Banquet - (As above)

Presentation of the Samuel S. Wilks Memorial Award

*Attendees will not be able to return to motels unless they have their own transportation.

Thursday, 2 November

0830-1000 CLINICAL SESSION B - Auditorium

Chairman: A.C. Cohen, Department of Statistics, University of Georgia, Athens, Georgia

Panelists:
Robert Bechhofer, Cornell University
Cuthbert Daniel, Private Consultant
Bernard Harris, Mathematics Research Center, U.S. Army
Henry Mann, Mathematics Research Center, U.S. Army
Frank Proschan, Boeing Scientific Research Laboratories
Herbert Solomon, Stanford University

DETERMINATION OF TBO BY WEIBULL PROBABILITY PARAMETERS FOR REPAIRABLE COMPONENTS

John L. Mundy, U.S. Army Aviation Materiel Command, St. Louis, Missouri

0830-1000 TECHNICAL SESSION V - Room 2E

Chairman: Raymond Schnell, U.S. Army Chemical Corps, Edgewood Arsenal, Maryland

A TECHNIQUE FOR INTERPRETING HIGH ORDER INTERACTIONS

Melvin O. Braaten and John Tonzetich, Duke University, Representing Shaw Air Force Base, South Carolina, and the North Carolina Operations Analysis Standby Unit, University of North Carolina, Chapel Hill, North Carolina

A SIMPLIFIED METHOD FOR FINDING OPTIMUM EXPERIMENTAL DESIGNS

Melvin O. Braaten, Duke University; Ray L. Miller, Jr., Shaw Air Force Base, South Carolina; Fred W. Judge, Wood-Ivey Systems Corporation, Winter Park, Florida. Representing Shaw Air Force Base, S.C., and the North Carolina Operations Analysis Standby Unit, University of North Carolina, Chapel Hill, North Carolina

0830-1000 TECHNICAL SESSION VI - Room 2F

Chairman: Erwin Biser, U.S. Army Electronics Command, Fort Monmouth, New Jersey

DEFINITIVE CALIBRATION OF AN AERIAL CAMERA IN ITS OPERATING ENVIRONMENT

Lawrence A. Gambino, U.S. Army Topographic Laboratories, Fort Belvoir, Virginia

DESIGN AND ANALYSIS OF A STATISTICAL EXPERIMENT ON HIGH VOLTAGE BREAKDOWN IN VACUUM

M.M. Chrepta, G.W. Taylor, and M.H. Zinn, U.S. Army Electronics Command, Fort Monmouth, New Jersey

1000-1030 BREAK

1030-1130 TECHNICAL SESSION VII - Auditorium

Chairman: Henry Dihm, Advanced Systems Laboratory, Directorate of Research and Development, U.S. Army Missile Command, Redstone Arsenal, Alabama

A MODERATELY DISTRIBUTION FREE TECHNIQUE FOR SMALL SAMPLE RELIABILITY ESTIMATION

Michael G. Billings, U.S. Army Chemical Corps, Dugway Proving Ground, Utah

1030-1130 TECHNICAL SESSION VIII - Room 2E

Chairman: Agatha Wolman, U.S. Army Strategy and Tactics Group, Bethesda, Maryland

USE OF REFERENCE COMPONENT MIXTURE DESIGNS IN A CALIBRATION APPLICATION

Raymond H. Myers, Department of Statistics, Virginia Polytechnic Institute, Blacksburg, Virginia, and
Bernard J. Alley, U.S. Army Missile Command, Redstone Arsenal, Alabama

1030-1130 TECHNICAL SESSION IX - Room 2F

Chairman: Joseph Mandelson, Quality Evaluation Division, Quality Assurance Directorate, U.S. Army Edgewood Arsenal, Maryland

DEVELOPMENT OF AN IMPROVED MODEL FOR ACOUSTIC SOUND RANGING

Robert P. Lee, Atmospheric Sciences Office, U.S. Army Electronics Command, White Sands Missile Range, New Mexico

AN EXPERIMENT ON THE METEOROLOGICAL EFFECTS ON SOUND RANGING

William H. Hatch, Atmospheric Sciences Office, U.S. Army Electronics Command, White Sands Missile Range, New Mexico

1130-1300 LUNCH

Thursday (Continued)

1300-1520 CLINICAL SESSION C - Auditorium

Chairman: Harold Fassberg, Research Analysis Corporation, McLean, Virginia

Panelists:
Robert Bechhofer, Cornell University
O.P. Bruno, U.S. Army Ballistic Research Laboratories
A.C. Cohen, University of Georgia
Walter D. Foster, U.S. Army Biological Laboratories
Boyd Harshbarger, Virginia Polytechnic Institute
H.L. Lucas, Jr., North Carolina State University
Herbert Solomon, Stanford University

PARAMETERS IN R&D IN RELATION TO COST/ACCURACY INVESTIGATION

Robert G. Conard, Systems Evaluation Branch, Advanced Systems Laboratory, Research & Development Directorate, U.S. Army Missile Command, Redstone Arsenal, Alabama

ON EXPERIMENTS CONCERNED WITH THE SAMPLING DISTRIBUTION OF LANCHESTER'S PARAMETERS

David R. Howes, U.S. Army Strategy and Tactics Analysis Group, Bethesda, Maryland

1300-1520 TECHNICAL SESSION X - Room 2F

Chairman: William W. Wolman, Traffic Systems Division, Office of Research and Development, Bureau of Public Roads, Washington, D.C.

ESTIMATES OF P(Y < X) AND THEIR APPLICATION TO RELIABILITY PROBLEMS FOR BOTH CONTINUOUS AND QUANTAL RESPONSE DATA

Bernard Harris and J.D. Church, Mathematics Research Center, U.S. Army, University of Wisconsin, Madison, Wisconsin

NUMBERS NEEDED FOR DETECTING IMPORTANT DIFFERENCES IN CHI-SQUARE TESTS

C.J. Maloney, Division of Biologics Standards, National Institutes of Health, Bethesda, Maryland, and F.M. Wadley, Consultant, U.S. Army Biological Laboratories, Fort Detrick, Frederick, Md.

ON A STATISTICALLY CONSISTENT ESTIMATE OF AN AVERAGE CUMULATIVE QUANTAL RESPONSE FUNCTION

George W. Evans II, and Robert C. McCarty, Stanford Research Institute, Menlo Park, California. Representing the U.S. Army Research Office-Durham

1300-1520 TECHNICAL SESSION XI - Room 2E

See next page

Thursday (Continued)

Chairman: Joseph M. Cameron, Statistical Engineering Laboratory, National Bureau of Standards, Gaithersburg, Maryland

DESIGNS OF EXPERIMENTS AS TELESCOPING SEQUENCES OF BLOCKS

Arthur G. Holms, National Aeronautics and Space Administration, Lewis Research Center, Cleveland, Ohio

ON A CLASS OF NONPARAMETRIC TESTS FOR INTERACTIONS IN FACTORIAL EXPERIMENTS

P.K. Sen, School of Public Health, Department of Biostatistics, University of North Carolina, Chapel Hill, North Carolina

ON THE RANK MOD p OF THE DESIGN MATRIX OF A DIFFERENCE SET

Jessie MacWilliams, Bell Telephone Laboratories, Murray Hill, New Jersey, and Henry B. Mann, Mathematics Research Center, U.S. Army, University of Wisconsin, Madison, Wisconsin

1520-1550 BREAK

1550-1700 GENERAL SESSION 2 - Auditorium

Chairman: Professor Boyd Harshbarger, Department of Statistics, Virginia Polytechnic Institute, Blacksburg, Virginia

SOME STATISTICAL METHODS IN MACHINE INTELLIGENCE RESEARCH

Professor I.J. Good, Department of Statistics, Virginia Polytechnic Institute, Blacksburg, Virginia

Friday, 3 November

0830-0915 GENERAL SESSION 3 - Auditorium

OPEN MEETING OF THE AMSC SUBCOMMITTEE ON PROBABILITY AND STATISTICS

Presided over by: Dr. Walter D. Foster, Biometric Division, U.S. Army Biological Laboratories, Fort Detrick, Frederick, Maryland

0925-1200 GENERAL SESSION 4 - Auditorium

Chairman: Dr. Frank E. Grubbs, Chairman of the Conference, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland

MAXIMUM LIKELIHOOD ESTIMATION OF RELIABILITY

Dr. Frank Proschan, Mathematics Research Laboratory, Boeing Scientific Research Laboratories, Headquarters, Offices the Boeing Company, Seattle, Washington

BREAK

DATA ANALYSIS (tentative title)

Dr. M.B. Wilk, Statistics & Data Analysis Research Department,
Bell Telephone Laboratories, Murray Hill, New Jersey

REGRESSION ANALYSIS IN THE COMPUTER AGE

F.J. Anscombe
Department of Statistics
Yale University
New Haven, Connecticut

1. INTRODUCTION. The commonly used methods of statistical analysis took much of their present-day form in the period of rapid development of statistical science between the two world wars. They were conditioned, more than perhaps is generally realized, by the principal computing resource of that period, the desk calculator. They give just about the best return possible for the amount of effort that a human being equipped with a desk calculator could reasonably (or even a little unreasonably) be expected to invest in a statistical analysis.

Now that our computing resources are enormously greater, we need not content ourselves with merely following the procedures suitable for the desk calculator. Almost anything we might ask for can be had at very little cost. What can we make use of? What sorts of calculations and output will give us most understanding, least misunderstanding?

Our extended computing powers can affect statistical methods in two ways. First, we are able to make better use of traditional methods, or of methods closely related thereto. Above all, we can now afford to ask freely for scatter-plots. These are tedious to construct by hand, but trivial with a computer. We can also demand the calculation of residuals, to test agreement of the data with assumptions underlying the method of analysis. We can afford to make transformations of variables and repeat analyses, to see if agreement is improved.

Second, we can consider methods of analysis that are radically different from traditional methods and involve much heavier computation. The great majority of traditional statistical analysis comes under the heading of "least squares" -- regression, analysis of variance, and analogous procedures like the analysis of contingency tables by χ^2. The least squares principle was originally advocated by Laplace and Gauss a century and a half ago because they thought no other method of combining observations would be computationally feasible. Now there are many other possibilities, and these should be explored.

This paper has the modest purpose of illustrating a few features of statistical analysis in the computer age. A set of gunnery readings, to which traditional regression analysis is applicable, is examined.

Section 2 contains a brief digression on computing. In section 3 traditional regression methods are exemplified in their modern guise. In section 4 a non-traditional analysis is briefly reported.

This research was supported by the Army, Navy, Air Force and NASA under a contract administered by the Office of Naval Research.

Wilk's paper [5] further exemplifies the impact of the computer on statistical analysis.

2. STATISTICAL COMPUTING. The computer has not so far had the profound effect on statistics that it has had on some other fields of science and technology. The reason is perhaps that good statistical analysis is done in steps. Methods must be adjusted to fit the data; the adequacy of theoretical descriptions of "models" must be assessed. This requires interaction between the investigator and the computer. Fixed program packages are not altogether satisfactory.

An explosive development of statistical science can be expected once programming can really be done by any interested person, without a large preliminary investment of time in mastering a computer language and without much time spent in actual coding. What makes programming so tedious in FORTRAN and other commonly used languages is the negotiation of arrays. Arithmetical operations are required, not just on individual numbers, but on whole vectors or matrices; and in these languages such operations must be spelled out in loops. Successful attempts have been made to relieve the intolerable tedium with special computing systems for vectors and matrices.

I have had access to an experimental implementation of Iverson's programming language known as APL [3,4], at IBM's Thomas J. Watson Research Center, Yorktown Heights, N.Y. APL is running as a coding language for computation in conversational mode through typewriter terminals. Though the language was not originally developed for statistical work (but rather for the precise and concise expression of any algorithms), it is in fact well adapted to statistical purposes. Two salient reasons are:

(i) APL was designed at the outset to handle (almost indifferently) scalars, vectors, matrices and rectangular arrays in any number of dimensions. All the basic arithmetic operations can be performed on arrays just as well as on scalars, without any loop written in the program. Programs in APL therefore tend to contain few loops. The programmer is encouraged to think of array operations as entities without a logically irrelevent internal sequence; this is aesthetically pleasing, even illuminating.

(ii) There is a high degree of consistency in APL. Syntax is governed ruthlessly by a very few simple rules. Once the basic vocabulary is learned, the language is easy to remember. There is a remarkable absence of arbitrary features that require frequent reference to the manual. The language therefore has a peculiar dignity and reasonableness. One feels it is worth learning.

I have elsewhere [2] prepared a description of APL, with illustrations of its use in statistical work. The above remarks are abstracted from that article.

This implementation of APL as a computer coding language is not yet available for general use. Something like it must surely become available eventually, hopefully soon. I am confident that it will have a profound influence on the development of statistics. The computations mentioned below were done through an APL terminal.

3. LEAST-SQUARES REGRESSION. The data used for this study of regression methods were kindly supplied by Dr. Frank E. Grubbs, of the U.S. Army Ballistic Research Laboratories, Aberdeen Proving Ground. They relate to some 175 mm. gun firings. In Table 1 we see the following information for 35 rounds: Range (metres), Projectile Weight (lb.), Muzzle Velocity (f.p.s.), and four items of weather information taken at the maximum ordinate of the trajectory, namely Temperature (deg. C), Air Density (kgm/1000), Range Wind and Cross Wind (both in metres per second divided by 10). The first 24 rounds were fired on one day, between 13.07 and 15.13 hrs. The remaining 11 rounds were fired the next day between 10.57 and 11.33 hrs.

Let us perform a regression analysis of Range as dependent variable on the other six variables as predictors (or "independent" variables). As usual, we shall begin by considering a linear combination of the predictor variables, and then later consider the possibility of a nonlinear function.

The traditional first step in such a regression analysis is to calculate the matrix of sums of squares and products of deviations of the seven given variables from their means, and then perhaps note various correlation coefficients. What is considerably more informative than the correlation coefficient between two variables, and just as easily obtained from the computer, is a scatter plot of the two variables against each other. Before ever any regression is calculated, a good deal of insight can be obtained by looking at a few such scatter plots. Here we should expect Muzzle Velocity to have a substantial predictive effect on Range, as is verified by plotting these against each other. So plots of the other predictor variables against Muzzle Velocity are of interest. One such plot is shown in Figure 1*, where Cross Wind is the other variable. We shall see that Muzzle Velocity and Cross Wind turn out to be the only two effective predictors, and in retrospect this diagram is the most revealing.

The diagram shows more of the relation between Cross Wind and Muzzle Velocity than is conveyed by the simple correlation coefficient (which happens to be about 0.27). As that correlation coefficient indicates, the two variables are only slightly related, so far as the calculation of linear regression is concerned. But if we should wish to calculate a nonlinear prediction surface with these two variables, it becomes relevant to notice that whereas the abscissas (M.V.) are distributed rather uniformly over an interval, the ordinate (C.W.) are clustered in two bands with a sizable gap between. We shall be able to estimate a quadratic response to M.V., and also a cross-product response (interaction between both variables), relatively well, but

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*The plotting code in the figures is as follows: one observation is represented by a small circle, two coincident observations by a plus sign, three or more coincident observations by a star. The axes are shown by crosses; zero is marked if it occurs.

No machine works perfectly all the time. When I ran off these figures the terminal showed an occasional wobble in the left margin. The fault seemed too trivial to warrant repetition on another terminal.

TABLE 1. 175MM. GUN FIRINGS

RANGE	WEIGHT	M.V.	TEMP.	AIR D.	R.W.	C.W.
20138	147.40	3001	9.1	0.9985	2.3	-5.0
20097	147.80	2995	9.1	0.9983	2.2	-5.0
20096	147.50	3002	9.2	0.9981	2.1	-5.1
20031	147.40	2997	9.3	0.9976	1.9	-5.1
20079	147.00	3003	9.3	0.9974	1.8	-5.1
20081	147.50	3002	9.4	0.9972	1.8	-5.1
19953	147.80	2993	9.4	0.9971	1.7	-5.2
20079	147.20	2999	9.4	0.9969	1.6	-5.2
20391	147.20	3031	9.5	0.9968	1.6	-5.2
20318	147.25	3021	9.5	0.9965	1.5	-5.2
20272	147.10	3020	9.5	0.9963	1.4	-5.3
20319	147.00	3031	9.6	0.9961	1.3	-5.3
20037	147.40	2998	9.9	0.9952	0.8	-4.8
19989	147.60	2993	10.0	0.9952	0.8	-4.7
20025	147.80	2996	10.0	0.9951	0.7	-4.6
20064	147.50	2997	10.1	0.9951	0.7	-4.5
20170	148.20	3004	10.1	0.9950	0.6	-4.4
20363	147.80	3018	10.2	0.9948	0.6	-4.3
20214	148.20	3015	10.2	0.9947	0.6	-4.0
20160	147.40	3016	10.3	0.9947	0.5	-4.0
20261	147.00	3015	10.3	0.9946	0.4	-3.8
20209	146.70	3013	10.4	0.9946	0.4	-3.7
20231	147.40	3016	10.4	0.9946	0.3	-3.6
20254	147.00	3014	10.5	0.9945	0.3	-3.4
20072	147.30	3005	10.3	0.9988	0.1	0.6
20142	147.20	3008	10.3	0.9987	0.1	0.6
20089	147.40	3002	10.2	0.9986	0.2	0.7
20022	147.40	3006	10.1	0.9983	0.3	0.8
20030	146.50	3022	9.9	0.9979	0.5	1.0
20105	147.00	3019	9.9	0.9978	0.5	1.0
20045	147.00	3017	9.8	0.9976	0.5	1.1
20103	147.00	3019	9.8	0.9976	0.5	1.1
20132	147.80	3011	9.8	0.9975	0.6	1.1
20334	147.80	3023	9.7	0.9975	0.6	1.2
19962	148.20	3013	9.7	0.9974	0.6	1.2

EXTREME ABSCISSAS ARE:2993 3031

ABSCISSA INTERVAL WIDTH IS 1.225806452 .

EXTREME ORDINATES ARE:-5.241935484 1.258064516

ORDINATE INTERVAL WIDTH IS 0.2096774194.

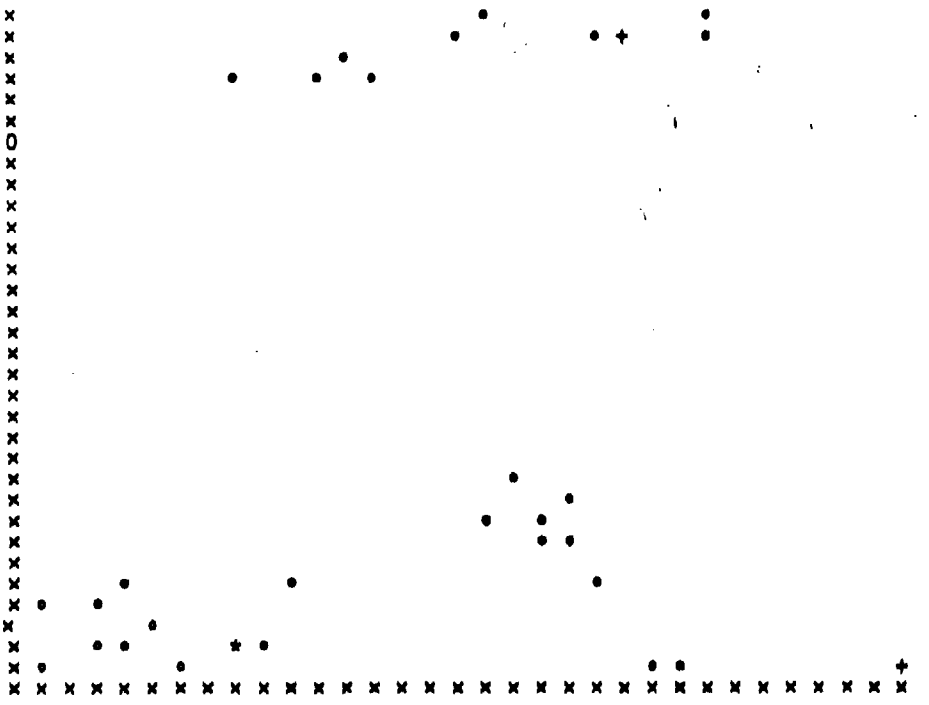

FIGURE 1. CROSS WIND AGAINST MUZZLE VELOCITY

a quadratic response to C.W. less well than if the points in the diagram had been more uniformly distributed between the same extremes.

The six predictor variables are apparently uncontrolled. There is no indication of any deliberate variation in the Projectile Weight or Muzzle Velocity. These could have been intentionally varied, but something approaching an orthogonal pattern of joint variation would presumably have been adopted. The weather characteristics were apparently not deliberately varied either, since the rounds were fired in two short series one afternoon and the following morning. We shall not therefore be surprised to find that some of the variables have no detectable relation to the "dependent" variable Range, even though we may believe that with wider variability and more numerous observations each variable would be seen to have an effect.

In such a situation a step-by-step procedure of introducing one variable at a time into the regression relation suggests itself. A simple computational routine, easily programmed, goes like this. Each time a new predictor variable is introduced, not only the dependent (Range) vector but all the other so-far-unused predictor variables are replaced by their projections at right-angles to the designated predictor vector. All these vectors become vectors of residuals. By the end of this process, if all the predictor variables are used, the matrix of their values will have been completely orthogonalized -- but we shall not necessarily go this far. Each variable has been read to only limited precision (Projectile Weight generally to 0.1 lb. apparently, Muzzle Velocity to 1 f.p.s., Temperature to 0.1 deg. C, etc.) If at any stage the corresponding vector of residuals shows little more variability than this round-off error, that variable should be dropped from further consideration. Usually we shall wish not to introduce any variable into the regression relation unless its presence causes a perceptible lowering of the residual mean square. (The objectives of stepwise regression and possible methods of procedure have been discussed in the literature -- for references see [1].)

The single variable that shows most relation to Range is Muzzle Velocity, and after regression on that has been performed the next most related variable is Cross Wind. The effects may be summarized in the following analysis of variance of Range:

TABLE 2. Analysis of Variance of Range

	Sum of squares	D.f.	Mean square
Muzzle Velocity	239416	1	239416
Cross Wind (after M.V.)	86169	1	86169
Residual	143946	32	4498
Total about mean	469532	34	13830

The corresponding formula for predicting Range is

$$9.227\ (M.V.) - 19.5\ (C.W.) - 7687. \qquad (1)$$

That is about all that seems to be worth doing in the way of simple linear regression on the available predictor variables. The further reduction in the residual sum of squares due to introducing any of the other variables is slight. (Each of the variables has a substantially greater residual mean square, after regression on M.V. and C.W., than would be caused merely by the apparent round-off error in the readings, and so would be usable.)

At this stage it is advisable to make scatterplots of the Range residuals* against (a) the fitted values for Range given by the expression (1) above, and (b) each of the original six predictor variables in turn. The plot against Cross Wind suggests a nonlinear dependence of Range on Cross Wind. This plot and also the plot against Muzzle Velocity suggest that the residual variance of Range is perhaps changing progressively with these variables.

Now if Range depends on Muzzle Velocity and Cross Wind, it need not do so merely linearly. In fact, theory suggests that C.W. should have a quadratic effect. Three more "independent" variables were brought into consideration, the squares of M.V. and of C.W. and their product. Of these new variables, only one, the square of C.W., has a mildly "significant" effect, after the linear regression on M.V. and C.W. already performed. As we saw from Figure 1, the peculiar distribution of the C.W. values does not permit us to determine the shape of the response of Range to C.W. very well. Since theory predicts a quadratic effect we are encouraged to allow for it and replace the "Residual" line in Table 2 above by the following two lines:

TABLE 3. Detail in Analysis of Variance of Range

	Sum of squares	D.f.	Mean square
C.W. squared (after M.V. and C.W.)	16360	1	16360
Residual	127587	31	4116

The corresponding formula for Range is

$$9.224\ (M.V.) - 8.4\ (C.W.)^2 - 52.3\ (C.W.) - 7645 \qquad (2)$$

The effect of Cross Wind is apparently to reduce Range by an amount proportional to $(C.W. + 3.1)^2$. The reduction is <u>not</u> proportional to the simple square of C.W.

Figures 2, 3, 4 show scatterplots of the new Range residuals (after regression on M.V., C.W. and C.W. squared) against fitted values, M.V., and C.W., respectively. They are reasonably satisfactory. We are left with a suggestion that the residual variance of Range is not constant, or possibly that the distribution of the Range errors is nonnormal (slightly leptokurtic).

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*Rather than plot simple residuals one may plot what are known as standardized residuals, in which allowance is made for the different weights arising from the least-squares fitting. In the present case, changing from simple to standardized residuals makes no perceptible difference in the scatterplots.

Sometimes in regression studies it is profitable to consider making simple transformations of the variables. Here, Range and Muzzle Velocity have such small percentage variabilities that no modest power transformation of them, such as squaring or taking logarithms or reciprocals, can noticeably affect the behavior of their residuals. In the absence of some suggestion from theory of a more drastic transformation, we do not pursue the idea.

What have computer facilities done for this regression study that was not available to the desk calculator operator? Any desk calculator man who was willing to contemplate six independent variables in regression, using traditional procedures, would no doubt have reached much the same conclusions. What we have gained, in addition to ease and speed, is some assurance, based on liberal inspection of scatterplots (only a few of which are reproduced here), that our final regression relation fits the data fairly well. That assurance was not provided by desk-calculator practices. When we examine the goodness of fit of a regression relation in this way, we sometimes find clear evidence that a different sort of regression relation ought to be tried instead. Here, on the contrary, the evidence supports the sort of regression relation we began with. What we first think of is not always bad!

4. UNORTHODOX REGRESSION. The method of least squares would be a theoretically perfect means of eliciting information from the observations if we could know that the form of the regression relation being fitted was correct and that the "error" part of the dependent variable, the part not explained by the regression relation, was a random variable independently normally distributed with zero mean and constant variance. When these ideal conditions are not satisfied, the least squares results will be to some extent misleading. Much has been said about least squares estimates' having minimum variance among unbiased linear estimates, independently of a normality assumption, but there is no longer today any good reason for restricting attention to linear estimates. If some method of analysis were known to be better, we should be prepared to use it.

It is widely believed that if the ideal conditions are not grossly violated the least squares method is adequate. One way to check whether this is so is to perform an optimal analysis under weaker conditions, to see whether perceptibly different results are obtained. Various kinds of weaker analysis have been suggested. In [1] I have proposed a particular way of weakening the normality assumption. Instead of assuming that the error part of the dependent variable is normally distributed with constant variance, we assume that the errors are independently distributed in a common distribution belonging to a family having one shape parameter, say α. When $\alpha = 0$ the distribution is normal. When $\alpha > 0$ the distribution is what Karl Pearson called Type VII, with longer tails than the normal, having the same shape as a Student distribution. If σ is a scale parameter, we assume that the errors ε have a density function of the form

$$A\sigma^{-1}\,(1 + C(\varepsilon/\sigma)^2)^{-1/\alpha},$$

where A and C depend on α. If $\alpha < 0$ the distribution is what Pearson called Type II, having shorter tails than the normal distribution and a finite range. (For further details see [1], where α^{-1} is denoted by m.)

EXTREME ABSCISSAS ARE:20005.78601 20359.82654
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EXTREME ORDINATES ARE:-139.5084347 169.4030992
ORDINATE INTERVAL WIDTH IS 9.964888191.

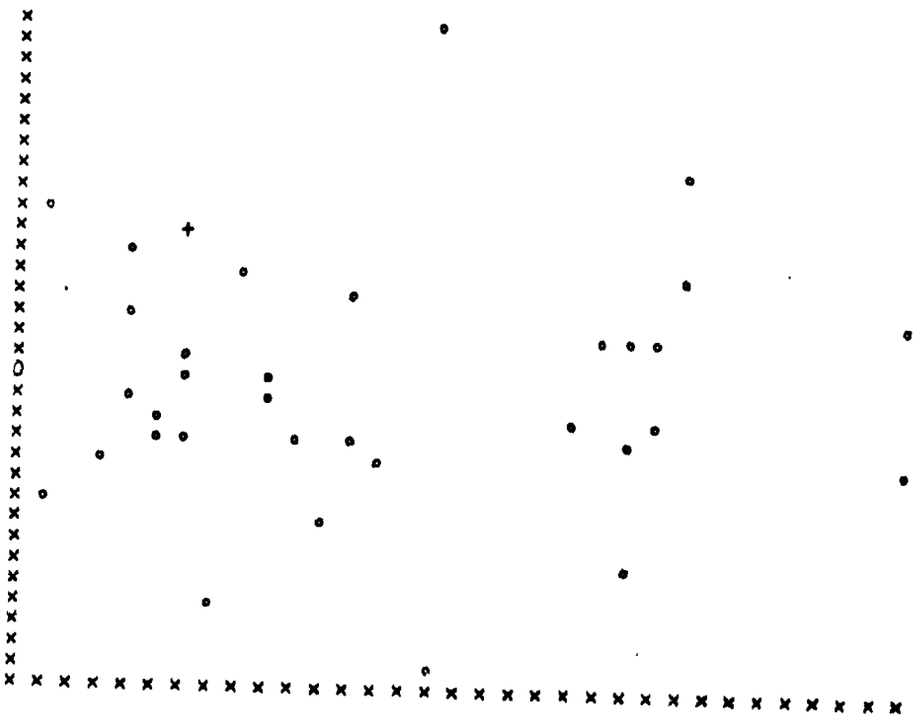

FIGURE 2. RANGE RESIDUALS AGAINST FITTED VALUES

EXTREME ABSCISSAS ARE:2993 3031

ABSCISSA INTERVAL WIDTH IS 1.225806452 .

EXTREME ORDINATES ARE:-139.5084347 169.4030992

ORDINATE INTERVAL WIDTH IS 9.964888191.

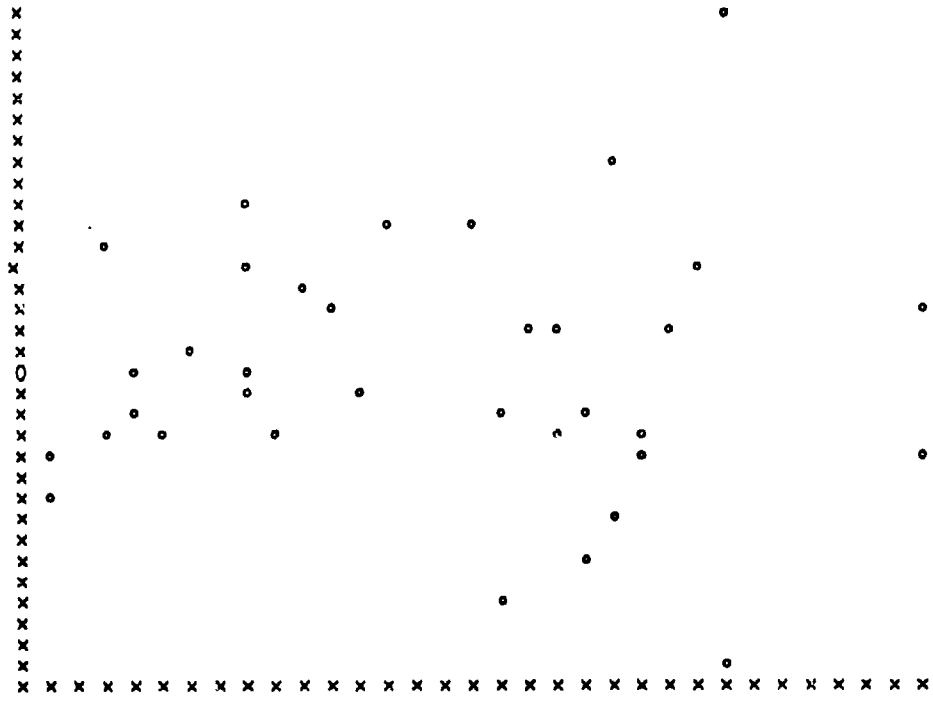

FIGURE 3. RANGE RESIDUALS AGAINST MUZZLE VELOCITY

EXTREME ABSCISSAS ARE: -5.241935484 1.258064516

ABSCISSA INTERVAL WIDTH IS 0.2096774194 .

EXTREME ORDINATES ARE: -139.5084347 169.4030992

ORDINATE INTERVAL WIDTH IS 9.964888191.

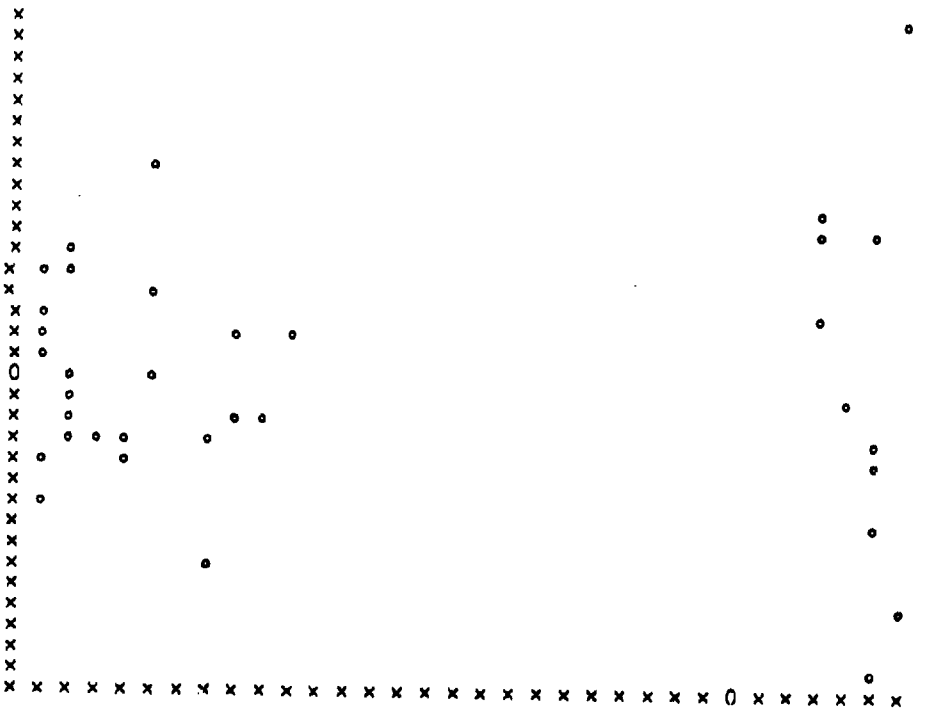

FIGURE 4. RANGE RESIDUALS AGAINST CROSS WIND

The suggested method of fitting a regression relation under this weaker assumption about the distribution of the errors is to investigate the likelihood function, which involves the regression coefficients and also the two nuisance parameters, α and σ. It is suggested that the likelihood function should be integrated with respect to suitable prior distributions for α and σ, yielding a marginal likelihood function of just the regression coefficients; and the latter should if possible be approximated by a multi-variate Student density.

This procedure has been carried out for the above gun firings, with the following particulars. A simple linear regression on Muzzle Velocity and Cross Wind was considered, without a term in C.W. squared. The nuisance parameters σ and α were taken to have independent prior distributions, uniform over the whole real line for $\ln\sigma$, uniform over the interval (-0.25, 0.75) for α. That interval for α was chosen as including the more plausible values for α -- the maximum likelihood estimate of α turns out to be about 0.12 -- and should be broad enough to bring out the qualitative features of this type of analysis. (We should be back at the method of least squares if α were restricted to the single value 0.) Orthogonal independent variables were used as follows:

$$X_1 = (\text{M.V.}) - 3009.6,$$

$$X_2 = (\text{C.W.}) - 0.068666\ (\text{M.V.}) + 209.55.$$

Our task is to fit the linear relation

$$E(\text{Range}) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 .$$

Our previous least squares analysis gave the estimates (equivalent to relation (1) above

$$\hat{\beta}_0 = 20139.1, \quad \hat{\beta}_1 = 7.89, \quad \hat{\beta}_2 = -19.48. \qquad (3)$$

The estimated variance matrix of these three quantities was diagonal, with diagonal elements

129 1.17 19.81

based on the estimated residual variance having 32 degrees of freedom.

In our new analysis we find that the marginal likelihood function of $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$ has its maximum at

$$\hat{\beta}_0 = 20138.9, \quad \hat{\beta}_1 = 7.94, \quad \hat{\beta}_2 = -19.05. \qquad (4)$$

The whole function is fairly well approximated by a multivariate Student density with 33 degrees of freedom and the following estimated variance matrix

$$\begin{pmatrix} 128 & 0.6 & 6.0 \\ 0.6 & 1.14 & 0.28 \\ 6.0 & 0.28 & 21.77 \end{pmatrix}$$

Comparing the new estimates (4) with the previous estimates (3), we see that $\hat{\beta}_0$ has changed by less than 2% of its estimated standard error, $\hat{\beta}_1$ by less than 5%, $\hat{\beta}_2$ by less than 10%. The changes in the estimated variance matrix and number of degrees of freedom are trivial. For most practical purposes, our new analysis has given results indistinguishable from the least squares analysis.

Now if the assumption of normally distributed errors with constant variance, underlying the method of least squares, is false, our weaker assumption of a Type VII - Type II system with α in the range (-0.25, 0.75) may also be false. In particular, the distribution of errors could be skew. But the Type VII - Type II family of error distributions is far broader than the normal family. If an assumption about distribution shape has an important influence on conclusions, we might hope to detect this fact through what we have done. The close agreement of the results of our two types of analysis strongly suggests that the least squares analysis of this particular body of data was not much colored by the implied distribution assumption. Whether the same comforting conclusion would usually be reached in studies of other bodies of data I do not know.

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SOME COMMENTS ON MATCHING*

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My topic today is "matching" in situations where the response is of the (0,1) type, firstly in an experimental situation and secondly in an observational situation. In both cases I wish to advance the suggestion that, frequently, to use a cliché, the game is not worth the candle (whatever that means).

In purely experimental work, in which the response is of the (0,1) type, one may be tempted to use matching. I recall an experiment on weather modification (an activity to which I tend to refer, in general, as rain faking) by Braham, Batten and Byers [1], with the cooperation of the U.S. Air Force. A plane sought out single clouds in the Caribbean. A cloud that looked as if it met certain specifications would be inspected, and if it did then a randomized choice would be made as to whether it was to be seeded. Following the result of the randomization, the plane would fly through the cloud and either release the seeding agent or not, and then the cloud would be observed for an appropriate period to see if it developed radar echoes.

After the completion of this period of observation, the plane would then seek another cloud which met the specifications. This cloud would receive the opposite treatment to that handed out to the first cloud. The two clouds then formed a matched pair, with responses as tabulated below.

		Unseeded	
		+	-
Seeded	+	n_{11}	n_{12}
	-	n_{21}	n_{22}

If time permitted on that day, the plane would make a second mission, but if time or gasoline ran out before the second member of the second pair had been found, then the first member of the second pair had to be abandoned. Of course, it could also happen that on the plane's first flight it found one cloud but failed to find another before running out of gasoline.

The idea of using matched pairs, of course, was an intuitive one based on ideas analogous to those relevant to the concept of randomized blocks. Just as the variation between plots close together in the same block is

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considered likely to be less than that between plots in widely separated blocks, so it was supposed that clouds on the same day were more probable to resemble each other than clouds on different days.

Some work by Jane Worcester [2] is relevant to this. For example, supposing that the variation from day to day is represented by equal numbers of days with probabilities 0.4 and 0.6, and the effect of the treatment is to increase those probabilities by 0.1 to 0.5 and 0.7, and, that a level of significance $\alpha = 0.05$ is used, then for a power of 0.90 the sample sizes necessary for paired and unpaired experiments are reported by Worcester to be 811 and 845 respectively. The use of pairing thus decreases the necessary sample size by 4.0 per cent, a rather inconsequential amount, particularly in the context of the experiment I have referred to, where the use of pairs reduced the number of observations quite appreciably.

If the heterogeneity was more extreme, say equal numbers at probabilities 0.3 and 0.7, then the corresponding sample sizes would be 845 and 709, a reduction of 16.1 per cent.

Of course, in reality the distribution of the probability from day to day would not be a discrete distribution concentrated in equal proportions at two points, but instead presumably a unimodal continuous distribution, with which the effect of heterogeneity would probably be quite modest.

The paired experiment had a further weakness, namely its integrity was compromised if the observer who selected the clouds was aware of which treatment was applied to the first cloud. He would then know ahead of time which treatment would be applied to the second cloud, and could select the second cloud in accordance with his predelictions. The scientists running the experiment maintained that the man selecting the clouds, in the front of the plane, was unable to tell whether the seeding agent was released or not, but nevertheless one wonders whether he could not tell, perhaps subconciously, either from the behavior of the plane (for if the seeding agent was released the plane was appreciably lighter), or from the behavior of the other members of the crew.

The famous calculating horses were able, apparently, to respond to imperceptible gestures on the part of their human accomplice, and it is conceivable that the human cloud selector was as sensitive as these horses.

In general, if matching is employed but actually is ineffective, then the power of the experiment is asymptotically unchanged, but for small samples the matching procedure seems less efficient. For example, for sample sizes of 10 in the two independent samples, the table

	+	−	
A	0	10	10
B	5	5	10

gives a two tailed P value of 0.0326 of Fisher's exact test, but if the data

was to be analyzed as 10 pairs, it would have to be

		A +	A −	
B	+	0	5	5
	−	0	5	5
		0	10	10

for which the two tailed P value is 0.0625.

This question of power when matching is ineffective is explored by L.H. Youkeles [3]. His results show that this loss of power ceases to be appreciable after the two sample sizes have reached 30.

I think that it is clear that the motivation to use matching is provided by its analogy with the idea of randomized blocks. The prestige of this procedure is so great that I rather uncritically assumed that matching would be better without thinking through what might happen. The general robustness of the unmatched completely randomized procedure now seems to me to be preferable to the hypothetical greater power of the matched design. It seems to me to be a common failing of the consulting statistician to automatically recommend the most complicated experimental design he can put over on his client without considering whether it is in reality justified.

Turning to an observational situation, I have observed that in medical and sociological investigations one or another form of "matching" is quite frequently used. One form of matching is the formation of so-called "matched pairs." One such study, which received a great deal of popular interest, is part of a paper by E. Cuyler Hammond [4].

Part of this paper contained a matched pair analysis and the procedure is described in the following quotations:

> "...we decided to investigate the matter by studying the death rates of cigarette smokers and nonsmokers who were alike in many characteristics other than their smoking habits. This was accomplished by a matched pair analysis carried out as follows:
>
> Two groups of subjects were identified: 1) men who never smoked regularly and 2) men currently smoking 20 or more cigarettes a day at enrollment in the study regardless of whether they also smoked cigars or pipes. These two groups were then divided into 5-year age groups. Within each age group, we matched men by pairs, each pair consisting of a nonsmoker and a cigarette smoker. The two men in a pair had to be alike in all the following characteristics: race (white, Negro, Mexican, Indian, or Oriental); height; nativity (native-born or foreign-born); residence (rural or urban); urban occupational exposure to dusts, fumes, vapors, chemical, radioactivity, etc. (yes or no); religion (Protestant, Catholic, Jewish, or none); education;

marital status (single, married, widowed, divorced, or separated); drinking of alcoholic beverages; sleep (under 6 hours, 6-9 hours, or 10 or more hours per night); usual amount of exercise (none or some); severe nervous tension (yes or no); history of cancer other than skin cancer (yes or no); and history of heart disease, stroke, or high blood pressure (yes or no)....

The matching procedure was carried out with an IBM 1410 computer. The records for the nonsmokers were put on one magnetic tape and the records for the cigarette smokers on another. The records in both tapes were then sorted in order by the codes for all the variables under consideration as described. Thus on both tapes the records were arranged in blocks, a block being defined as a group of records identically coded in all the variables under consideration. By use of random numbers, the records within each block were arranged in random order. The 2 tapes were then compared block by block. Blocks found on only one tape (i.e., the same number of cigarette smokers as nonsmokers) were accepted as matching pairs. For example, if a block of 2 cigarette smokers matched a block of 2 nonsmokers, then 2 matched pairs were identified, the first cigarette smoker and the first nonsmoker being the first pair and the second cigarette smoker and the second nonsmoker being the second pair. If the matched blocks were of unequal length, then the excess records in the longer block were discarded. For example, if a block of 5 cigarette smokers matched a block of only 2, then the first 2 smokers formed matched pairs with the 2 nonsmokers, and the last 3 smokers were discarded. Thus the excess (discarded) records were selected at random since, within each block, the records were arranged in random order....

With so many characteristics to be considered, many men could not be matched. However, the computer found 36,975 matched pairs of men (36,975 nonsmokers and 36,975 cigarette smokers), such that the 2 men in each pair were alike in all the specifications outlined.

...Of the 36,975 nonsmokers, 662 (1.8%) died and, of the 36,975 cigarette smokers, 1,385 (3.7%) died between the start of the study and September 30, 1962. This difference is statistically significant ($P < 0.000001$)."

The matching employed by Hammond is very complex: apart from smoking, he employed 15 categorizations, some at two levels only and others at several levels. The number of cells in this 17 dimensional lattice was $2 \times 8 \times 5 \times 2 \times 2 \times 2 \times 4 \times 5 \times 5 \times 5 \times 3 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10} \times 3 \times 4 \times 5^4 \times 8 = 61\ 440\ 000$. However, this overestimates the number of cells, as though education was recorded in 5 categories, "The two men in a pair had to be in the same education category as in adjacent categories."

Let us consider the case where matching is performed on only one categorization at 2 levels. The population is thus cross-classified into a 2 x 2 table: for convenience let us continue to use smoking-nonsmoking as one of the categorizations, represented by the symbols S and T, and baldness-nonbaldness

as the other categorization, represented by B and C. Let the proportion of the population falling into the four classes be θ_{SB}, etc.

Suppose that we have a very large sample, so that sampling fluctuations can be ignored, and let the death rates for each class be ϕ_{SB}, etc. Then if the sample size is N, then the number of bald smokers is $N\theta_{SB}$, etc.

Table 1

	Bald		Nonbald	
	Proportion	Proportion of Deaths	Proportion	Proportion of Deaths
Smokers	θ_{SB}	$\theta_{SB}\phi_{SB}$	θ_{SC}	$\theta_{SC}\phi_{SC}$
Nonsmokers	θ_{TB}	$\theta_{TB}\phi_{TB}$	θ_{TC}	$\theta_{TC}\phi_{TC}$

Matching will consist in matching each bald smoker with a bald nonsmoker (and analogously for the nonbald columns). Since in general $\theta_{SB} \neq \theta_{TB}$, there will be an excess of smokers or an excess of nonsmokers in this category, and the excess will be discarded by random selection. Thus if $\theta_{SB} > \theta_{TB}$ and $\theta_{SC} > \theta_{TC}$, the nonsmokers are left undisturbed, but the number of smokers will be reduced to $N\theta_{TB}$ and $N\theta_{TC}$ and the numbers of deaths amongst smokers to $(N\theta_{TB}\,\phi_{SB} + N\theta_{TC}\,\phi_{SC})$.

The death rate for smokers in the matched sample, say D_{SM}, will thus be

$$D_{SM} = \frac{\theta_{TB}\phi_{SB} + \theta_{TC}\phi_{SC}}{\theta_{TB} + \theta_{TC}} \tag{1}$$

The death rate for nonsmokers in the matched sample, say D_{TM}, will be

$$D_{TM} = \frac{\theta_{TB}\phi_{SB} + \theta_{TC}\phi_{TC}}{\theta_{TB} + \theta_{TC}} \tag{2}$$

and this is, of course, the same as the death rate for nonsmokers in the unmatched sample, say D_{TU}.

For the smokers, however, the death rate in the unmatched sample, say D_{SU}, is

$$D_{SU} = \frac{\theta_{SB}\phi_{SB} + \theta_{SC}\phi_{SC}}{\theta_{SB} + \theta_{SC}} \tag{3}$$

Now suppose that the ratio of smoker to nonsmoker death rates in the unmatched sample is the same as in the matched sample, i.e.

$$D_{SU}/D_{TU} = D_{SM}/D_{TM} \quad (4)$$

Note that this is a rather weak condition. We are not requiring that the matched samples give the "right" answer for the death rates in, e.g., the smoking population, but merely that the ratio of death rates of smokers to nonsmokers be the same in the matched sample as in the population.

The condition implied in equation (4) implies that

$$\frac{\theta_{SB}\phi_{SB} + \theta_{SC}\phi_{SC}}{\theta_{SB} + \theta_{SC}} = \frac{\theta_{TB}\phi_{SB} + \theta_{TC}\phi_{SC}}{\theta_{TB} + \theta_{TC}} \quad (5)$$

which in turn requires that

$$(\phi_{SC} - \phi_{SB})\ (\theta_{SC}\theta_{TB} - \theta_{SB}\theta_{TC}) = 0. \quad (6)$$

Thus the relationship between smokers and nonsmokers is the same in the matched sample as in the original unmatched sample if either

$$(a)\quad \phi_{SC} = \phi_{SB} \quad (7)$$

or

$$(b)\quad \frac{\theta_{SC}}{\theta_{TC}} = \frac{\theta_{SB}}{\theta_{TB}} \quad (8)$$

or (c) both (a) and (b) are simultaneously satisfied.

Condition (a) is that the death rate for bald smokers be the same as the death rate for nonbald smokers; in other words, for smokers baldness or nonbaldness does not affect the death rate.

Condition (b) is one form of the familiar independence criterion for two cross categorizations, for example that the probability of baldness is the same for smokers as for nonsmokers.

Condition (a) is asymmetrical in that it refers only to smokers. This asymmetry occurs, of course, because the operation of matching, in the situation assessed here, alters the relative numbers of bald and nonbald men in the smoking group only, since the nonsmoking group is left unchanged by the matching operation.

If the independence between smoking and baldness implied by (8) is not satisfied, in the particular manner implied by the condition

$$\frac{\theta_{SC}}{\theta_{TC}} < 1 < \frac{\theta_{SB}}{\theta_{TB}}, \quad (9)$$

then the smoking class determines the size of the matched group for nonbald

people and the nonsmoking class determines the size of the matched group for bald people. It is straightforward, but somewhat tedious, to show that (4) in general requires that

$$\phi_{TB} = \phi_{TC} \tag{10}$$

and

$$\phi_{SB} = \phi_{SC} \cdot \tag{11}$$

In other words the death rate for bald smokers must be the same as the death rate for nonbald smokers and also the death rate for bald nonsmokers must be the same as for nonbald nonsmokers.

These two results, (7) and (8), and (10) and (11), both correspond to commonsense. All death rates are in effect weighted averages. The unmatched death rates are weighted averages using as weights the properties of each category in the population. Equation (7) follows from the fact that if the death rates in the two categories are equal it makes no difference what weights are used. Equation (8) follows from the fact that if the proportion bald/nonbald is the same for the smokers as for the nonsmokers, then the matched sample will have the same weights as the unmatched. Equations (10) and (11) are similar to (8).

I should like to illustrate this with a small synthetic numerical example. Imagine a population being matched, smokers against nonsmokers, according to some factor such as baldness.

Suppose that the smokers number 110,000 of whom 100,000 are not bald with a death rate of 1 per cent and 10,000 are bald with a death rate of 5 per cent. Then the overall death rate for smokers is the ratio of the total number of deaths,

$$100{,}000 \times 0.01 + 10{,}000 \times 0.05 = 1000 + 500 = 1500$$

to the total number of smokers,

$$100{,}000 + 10{,}000 = 110{,}000.$$

Thus the death rate for smokers is

$$1100/110{,}000 = 1.36\%.$$

Now suppose that the nonsmokers number 35,000, of whom 20,000 are not bald with a death rate of 1 per cent and 15,000 are bald with a death rate of 2 per cent. Then the death rate for nonsmokers is

$$\frac{20{,}000 \times 0.01 + 15{,}000 \times 0.02}{20{,}000 + 15{,}000} = \frac{500}{35{,}000} = 1.43\%.$$

This is one population the smokers have a slightly lower death rate than the nonsmokers; the ratio of death rates is $1.36/1.43 = 0.95$.

Now suppose that the smokers and nonsmokers are "matched." For the nonbald the device that does the matching will be able to select 20,000 smokers out of

the 100,000 available, and these will have an expected number 20,000 x 0.01 = 200 deaths. The device that does the matching will keep in its matched sample the 10,000 bald smokers who will have an expected number 10,000 x 0.05 = 500 deaths. Thus for smokers in the matched sample the death rate is

$$(200 + 500)/(20,000 + 10,000) = 700/30,000 = 2.33\%.$$

For nonbald nonsmokers the 20,000 in the population stay in the matched sample, producing an expected number 20,000 x 0.01 = 200 deaths, but the bald nonsmokers are reduced in number to 10,000, for which the expected number of deaths is 10,000 x 0.02 = 200. Thus for nonsmokers in the matched sample the death rate is

$$(200 + 200)/(20,000 + 10,000) = 400/30,000 = 1.33\%.$$

Thus in our matched sample the smokers have a somewhat higher death rate than the nonsmokers, the ratio of death rates being 2.33/1.33 = 1.75. The direction of this relationship between the death rates for smokers and nonsmokers in a matched sample is the reverse of what occurred in the population.

I think the reason that matching proves misleading in this observational situation is that it is a close relative of covariance analysis. We know that in a purely experimental situation it is essential that the concommitant variable be independent of the experimental treatments, and the same must hold good in an observational situation. If the concommitant variable is not independent of the treatments, then hideous fallacies may arise when we "adjust" the response means. I think the analogous situation may arise in a discrete matching situation: the frequencies of the matching criterion may be forced into quite unrealistic distributions.

From another point of view, the matching procedure is forming a weighted average for which the weights are quite unrealistic.

In other words, I believe that the matching procedure adds seriously to the difficulties of extracting rigorous inference from observational data and we should be quite hesitant about employing it.

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ON METHODS OF OPTIMIZATION OF A MULTIOBJECTIVE SURVEY

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ABSTRACT. The Wound Data Project is a survey of wounded personnel in which information is collected about the projectile or thermal agent causing the wound, and the incurred physiological and phychological effects, together with the hospital information. The experimental design is not under direct control, since only those cases that do occur can be observed. The control that can be exerted in such work consists of proper questionnaire design and an attempt to continue observation until certain minimum numbers of cases have occurred in designated categories. In this project, the number of categories is in the hundreds, making it highly unlikely that the desired number of cases will be observed in all categories. There is no ability to control the level at which factors occur (e.g., projectile striking velocity). The appropriate statistics to be used are largely those developed in fields specializing in survey work such as epidemiology and social relations. In fact, the computer system to be used for file manipulation in this project, DATA-TEXT, is one developed by the Harvard School of Social Relations.

The areas to be investigated in this study are dictated by Army requirements, and information is now being recorded by the field team. The specific questionnaires from which Hollerith cards will be punched are to be filled out by the CONUS Team from this data. Adequate medical personnel are available in the CONUS Team to insure proper medical interpretation of questions. Areas in which advice is sought from the "clinical" panel include statistical pitfalls in questionnaire design, and optimum selection of subjects where choices exist. If subjects are selected with multiple wounds, individual variation is minimized and direct comparison allowed between or among physical characteristics such as penetrating ability. However, the physiological and phychological effects of a particular wound are unmeasurable in the multiple wound case due to the confounding. The proper target sample size would appear to be better defined on attaining a pre-determined number of cases showing some set of characteristics, rather than by merely observing some total number of cases without regard to the information content of these cases. However, how to select the proper set, or sets, of characteristics in a survey where many such combinations exist, each for some different output of the survey, is difficult. Any selection based on frequency of observed characteristics implies feedback from the evaluation team (CONUS) to the collection team (SEA) which are physically separated by some 10,000 miles. The ability of the collection team to "collect" also depends upon the vagaries of war.

COMPONENTS OF VARIANCE OF A LINEAR FUNCTION IN REPEATED TRIALS

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ABSTRACT. The quality (Q_i) of the i-th batch of a material diminishes with time according to a function which is linear in its parameters, a separate parameter set estimated for each batch. The quality of each batch is extrapolated to a common future date, t_f, by means of its time function. A weighted mean quality is computed, using the known amount of each batch as the weight:

$$\bar{Q}_w = \Sigma w_i \, Q_i / \Sigma w_i$$

The problem is to find the variance of the weighted mean, $V(\bar{Q}_w)$, given the estimated parameters of each time function and the elapsed time to t_f. In case that the time functions have the form

$$Q_i = A_i + B_i \, X + C_i \, Y$$

it is known in a special application that the batch-to-batch distribution of the A_i is normal and independent of B and C. The bivariate distribution of B and C has a high covariance, $\rho = .87$, with markedly skewed marginal distributions, each in the positive direction. It has been acceptable to write C in terms of B as

$$C = d + e \, B$$

for another application not discussed here.

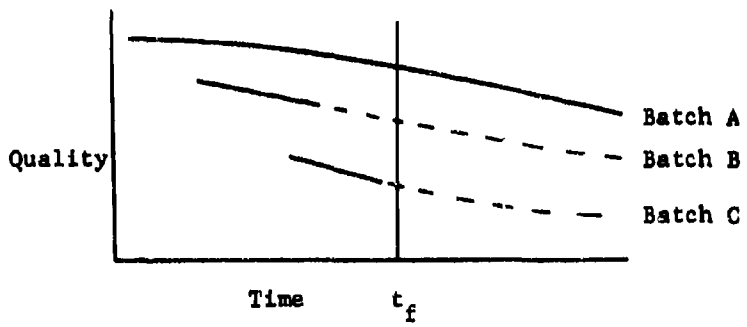
The remainder of this paper was reproduced photographically from the author's copy.

INTRODUCTION

In a manufacturing process, the quality, Q_i, of the i-th batch of material is measured periodically, because quality is known to deteriorate with time. The deterioration function,

$$Q_i = A_i + B_iX + C_iY,$$

is fitted to each batch, resulting in a unique set of statistics, (A, B, C) for each batch. The variable X is time; Y is log(X + 1). The times of observation are not necessarily the same for each batch, nor is each batch manufactured at the same time. The quality of each batch is weighted by the amount of each batch. It is the weighted average of quality, $\bar{Q}_w$, and its variance which are required for a fixed time, t_f. The following schematic is illustrative.



Computationally, it is straight forward to compute Q at time t_f for each batch and to continue over batches to compute

$$\bar{Q}_w = \Sigma w_i Q_i / \Sigma w_i \qquad \text{and}$$

$$V(\bar{Q}_w) = \Sigma w_i (Q_i - \bar{Q}_w)^2 / \Sigma w_i .$$

But this is not the problem. It is desired to find a formulation for $V(\bar{Q}_w)$ involving the distribution of the parameters of the deterioration functions. The distribution of the A_i is known to be normal (with available estimates of mean and variance) and independent of B and C. The bivariate distribution of B and C while not known functionally has a high covariance, r = .87, with markedly skewed marginal distributions.

Three cases are given as successive stages of a possible approach to illustrate the form of a desired solution. General notation applicable to all

three cases includes the following:

Let Q_{ij} = j-th observation of quality on the i-th batch, i = 1--i, j = 1-j

and $Q_{i.}$ = quality of i-th batch averaged over the j observations

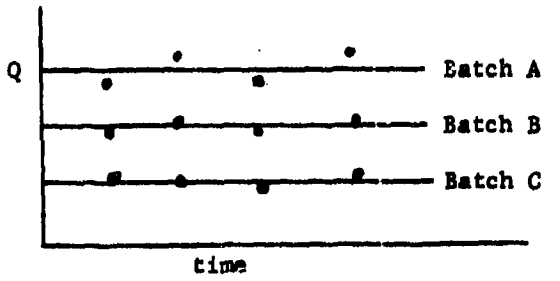
and Q.. = average quality averaged over all batches

w_i = amount of i-th batch, used as a weight factor

t = time of observation

CASE I

No deterioration of batch quality with time. All batches manufactured at the same time with the same dates of surveillance. Variance of assay the same for each batch. Pictorially,



Let the random model,

$$Q_{ij} = m + c_i + e_{ij},$$

$$i = 1---i,$$

$$j = 1---j,$$

with zero covariances represent quality so that

$$E(Q_{i.}) = \mu \qquad \text{which is estimated by}$$

$$Q_{i.} = \Sigma Q_{ij}/j; \quad \text{also}$$

$$Q..{}_w = \Sigma\Sigma w_i Q_{ij}/j \Sigma w_i.$$

Note that the previous computation for the variance of the mean, namely,

$$V(Q..{}_w) = \Sigma w_i (Q_{i.} - Q..{}_w)^2 / \Sigma w_i ,$$

neither partitions the source of variation nor uses the distribution of the parameters of the deterioration functions.

The following results, assuming that the w_i are constant, are obtained from the expected mean squares which are well known to be given by the analysis of variance model,

Source	df	E(MS)
Between Batches	i-1	$\sigma^2 + j\sigma_c^2$
Within Batches	i(j-1)	σ^2

from which we have

$$V(Q_{ij}) = \sigma^2 + \sigma_c^2$$

$$V(Q_{i.}) = \sigma^2/i + \sigma_c^2 \quad \text{and}$$

$$V(Q..) = \sigma^2/ij + \sigma_c^2/j \ .$$

When the amounts, w_i, are known but not equal, the weighted mean is

$$Q..{}_w = \Sigma w_i \, Q_i / \Sigma \, w_i$$

$$= m + \Sigma \, w_i c_i / \Sigma w_i + \Sigma w_i \, \Sigma_j e_{ij} / j \Sigma \, w_i$$

with variance

$$V(Q..{}_w) = (\sigma_c^2 + \sigma^2/j) \, \Sigma w_i^2 / (\Sigma \, w_i)^2$$

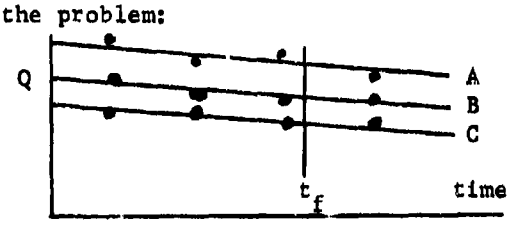
if the covariances are ignored. The partition has a desirable form.

CASE II

The deterioration rate of each batch in time has the same loss coefficient in the model,

$$Q_{ij} = m + c_i - bt + e_{ij},$$

where as before all batches have the same date of manufacture and the same dates of surveillance and the same variance of assay. The following figure indicates the nature of the problem:



Then the quality of the i-th batch at a fixed time, t_f, is given by

$$Q_{i\cdot} = m + c_i - bt_f$$

and the weighted average quality of all batches at time, t_f, is

$$Q_{\cdot\cdot w} = \Sigma w_i(m + c_i - bt_f)/\Sigma w_i$$

$$= m - bt_f + \Sigma w_i c_i/\Sigma w_i$$

with corresponding variances given by

$$V(Q_{i\cdot}) = \sigma^2[\,1/j + (t_f - \bar{t})^2/i\Sigma(t - \bar{t})^2] + \sigma_c^2, \text{ and}$$

$$V(Q_{\cdot\cdot w}) = (\sigma_c^2 + \sigma^2/j)\Sigma w_i^2/(\Sigma w_i)^2 + (t_f - \bar{t})^2\sigma^2/i\Sigma_j(t_j - \bar{t})^2$$

which is partitioned and follows the components-of-variance sense.

CASE III

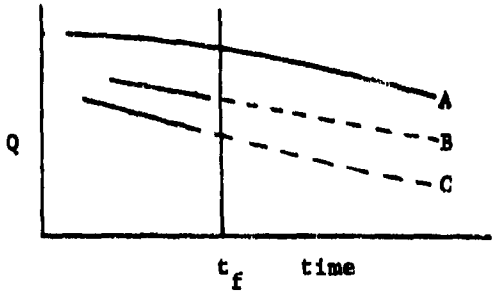
Let the deterioration function be representable by a linear function,

$$Q_{ij} = m + c_i + b_i t + d_i t' + e_{ij},$$

or by a non-linear function such as

$$Q_{ij} = Q_{oi}(r_i + 1)/r_i + s_i^t)$$

whose covariances in both cases are non-zero. Further, the date of manufacture of each batch is neither the same nor is the distribution of manufacture dates constant. Finally, neither the number of surveillance periods or their dates are necessarily the same from one batch to another. However, the variance of assay is constant. A pictorial representation is given below.



A continuation of the approach shown in Cases I and II while desirable may not be tractable. The problem is not so much to estimate

$$Q_{i\cdot} = f_i(t_f) \quad \text{and}$$

$$Q_{\cdot\cdot w} = \Sigma w_i f_i(t_f)/\Sigma w_i$$

which are readily computable as to formulate expressions of their partitioned variances estimable from the distributions of the model parameters in the sense of Cases I and II.

A MODEL
FOR OBTAINING THE OPERATING CHARACTERISTICS
OF A SKIP LOT SAMPLING PROCEDURE

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1.0 INTRODUCTION

Project SKIP is the name given to a ballistic testing procedure developed and administered by the Quality Evaluation Division of the U.S. Army Ammunition Procurement and Supply Agency. The need for such a procedure became evident when a study of ballistic testing revealed substantial savings could be effected by properly lowering ballistic test frequencies. The development of the methodology required to obtain the operating characteristics of the plans covered by the procedure parallelled its implementation at selected loading plants.

Fig. 1 depicts the essential steps of the flow diagram of Project SKIP. The associated verbal transition matrix is contained in Fig. 2. It is seen that we have a Markov model. Throughout the discussion the various steps of the flow diagram will be referred to as the states, i.e. qualification state, restart state, etc. The steady state occupancy probability of a lot being in state i will be P_i. We shall use P_i' to denote the probability of entering state i on the next step. A step is defined as the testing of the lot.

2.0 METHOD OF DERIVING P_i AND P'_i

We shall first restrict ourselves to the case where only tested lots are considered and temporarily ignore the skip lot possibilities in Step One and Step Two. Let

P_o = Prob (being in qualification state)

P_{N1} = Prob (being in normal step one)

P_{N2} = Prob (being in normal step two)

P_{N1*} = Prob (being in retrial step one)

P_{N2*} = Prob (being in retrial step two)

P_R = Prob (being in restart state)

and p = Pr (lot meeting all ballistic tests' requirements except those concerned with critical malfunctions)

γ = Pr (critical malfunction)

hence:

(1) $P_o = P'_o + P'_o (1-\gamma)\, p + \ldots\ldots + P'_o (1-\gamma)^9 p^9$

$$= P'_o \left[\frac{1-(1-\gamma)^{10}\ p^{10}}{1-(1-\gamma)\ p}\right]$$

(2) $P_{N1} = P'_{N1} + P'_{N1} (1-\gamma)\, p + \ldots\ldots + P'_{N1} (1-\gamma)^4 p^4 = P'_{N1} \left[\frac{1-(1-\gamma)^5\ p}{1-(1-\gamma)\ p}\right]$

(3) $P_{N2} = P'_{N2}$

(4) $P_{N1*} = P'_{N1*} + P'_{N1*} (1-\gamma)\, p + \ldots\ldots + P'_{N1*} (1-\gamma)^3 p^3 = P'_{N1*} \left[\frac{1-(1-\gamma)^4\ p^4}{1\ (1-\gamma)\ p}\right]$

(5) $P_{N2*} = P'_{N2*} + P'_{N2*} (1-\gamma)\, p + \ldots\ldots + P'_{N2*} (1-\gamma)^3 p^3 = P'_{N2*} \left[\frac{1-(1-\gamma)^4\ p^4}{1\ (1-\gamma)\ p}\right]$

Define: P_{R1} = Probability (being in restart in process of testing for five consecutive lots free of critical malfunctions while disregarding all other malfunctions).

P_{R2} = Probability (being in restart and testing for ten consecutive lots meeting all requirements).

(6) $P_R = P_{R1} + P_{R2}$

(7) $P_{R1} = P'_{R1} + P'_{R1}(1-\gamma) + \ldots\ldots + P'_{R1}(1-\gamma)^4 = P'_{R1}\left[\frac{1-(1-\gamma)^5}{\gamma}\right]$

(8) $P_{R2} = P'_{R2} + P'_{R2}(1-\gamma)\ p + \ldots\ldots + P'_{R2}(1-\gamma)^9\ p^9 = \left[\frac{1-(1-\gamma)^{10}\ p^{10}}{1-(1-\gamma)\ p}\right] P'_{R2}$

We shall now investigate the derivation of the P'_i. P'_o = Pr (being in 0 two steps ago and rejecting a lot for reasons other than a critical malfunction on the last step) + Pr (being in N1* two steps ago and rejecting a lot for reasons other than a critical malfunction on the last step) + Pr (being in N2* two steps ago and rejecting the lot for reasons other than a critical malfunction on the last step)

$= P_o(1-p)(1-\gamma) + P_{N1*}(1-p)(1-\gamma) + P_{N2*}(1-p)(1-\gamma)$

Utilizing (1), (4) and (5) yields

(9) $P'_o = P'_o\left[\frac{1-(1-\gamma)^{10}\ p^{10}}{1-(1-\gamma)\ p}\right](1-p)(1-\gamma) + P'_{N1*}\left[\frac{1-(1-\gamma)^4\ p^4}{1-(1-\gamma)\ p}\right](1-p)(1-\gamma)$

$+ P'_{N2*}\left[\frac{1-(1-\gamma)^4\ p^4}{1-(1-\gamma)\ p}\right](1-p)(1-\gamma)$

Similar reasoning for P'_{N1}, P'_{N1*}, P'_{N2}, P'_{N2*}, P'_{R1}, P'_{R2} yields:

(10) $P'_{N1} = P'_o(1-\gamma)^{10}\ p^{10} + P'_{N1*}(1-\gamma)^4\ p^4 + P'_{R2}(1-\gamma)^{10}\ p^{10}$

$$(11)\quad P'_{N1*} = \left[\frac{1-(1-\gamma)^5 p^5}{1-(1-\gamma)\ p}\right](1-\gamma)\ (1-p)\ P'_{N1}$$

$$(12)\quad P'_{N2} = P'_{N2*}\, p^4\, (1-\gamma)^4 + P'_{N1}\, (1-\gamma)^5 p^5 + P'_{N2}\, (1-\gamma)\, p$$

$$(13)\quad P'_{N2*} = P'_{N2}\, (1-p)\, (1-\gamma)$$

$$(14)\quad P'_{R1} = \gamma$$

$$(15)\quad P'_{R2} = P'_{R1}\, (1-\gamma)^5 + P'_{R2}\, (1-\gamma)\, (1-p) \left[\frac{1-(1-\gamma)^{10} p^{10}}{1-(1-\gamma)\ p}\right]$$

Equations (9) through (15) define 7 equations in terms of the P'_i. However an additional equation is needed since it can be shown that the coefficient matrix is not of full rank. The needed equation is

$$\sum_i P_i = 1$$

It was found convenient to solve for P'_o and then relate the remaining P'_i in terms of P'_o. The steady state probabilities were then obtained by substitution in (1) through (8).

3.0 INCORPORATION OF SKIP LOT POSSIBILITIES

The preceeding discussion neglected the skipping possibilities in N_1 and N_2. A plausible approach to the skipping anomaly would be to obtain the expected ratio of total to tested lots in the states of concern, multiply the original P_{N1} and P_{N2} by these ratios and then force the modified P_{N1}, P_{N2}, and the remaining P_i to sum to one by normalization.

Let: X_1 = Number of lots tested in N_1

X_2 = Number of lots skipped in N_1

then the ratio of interest is $\left[\frac{X_1 + X_2}{X_1}\right] = \frac{1 + X_2}{X_1}$

taking expectation $E\left[\frac{X_1 + X_2}{X_1}\right] = 1 + E\left(\frac{X_2}{X_1}\right)$

Now $E\left[\frac{X_2}{X_1}\right] = \sum_{X_1=1}^{5} \sum_{X_2=0}^{\infty} \frac{X_2}{X_1} P(X_2,X_1)$

where $P(X_2,X_1)$ is the joint probability density function of X_1,X_2.

However $P(X_2,X_1) = P(X_2|X_1) \; P(X_1)$

hence $E\left[\frac{X_2}{X_1}\right] = \sum_{X_1=1}^{5} P(X_1) \sum_{X_2=0}^{\infty} \frac{X_2}{X_1} P(X_2|X_1)$

$$= \sum_{X_1=1}^{5} \frac{P(X_1)}{X_1} \sum_{X_2=0}^{\infty} X_2 \; P(X_2|X_1)$$

Define i) a skipped lot as a no-test

ii) a tested lot as a test

Then $\sum_{X_2=0}^{\infty} X_2 \; P(X_2,X_1)$ may, for a given X_1, be considered as the expected number of no-tests before the X_1st test. Hence X_2 is distributed as a negative binomial random variable with expectation $\left(\frac{X_1 q}{p}\right)$

where q is the probability of a no-test

p is the probability of a test.

Therefore $E\left[\frac{X_2}{X_1}\right] = \sum_{X_1=1}^{5} \frac{P(X_1)}{X_1}\left(\frac{X_1 q}{p}\right) = q/p.$

For N_1 $q=p=1/2$ Therefore $E\left[\frac{X_2}{X_1}\right] = 1$ and $E\left[\frac{X_1 + X_2}{X_1}\right] = 2.$

A completely analagous procedure yields a ratio of 3 for N_2. Hence P_{N1} is multiplied by 2 and P_{N2} by 3.

The total procedure is then

(1) Obtain $\sum_{i \neq N_1, N_2} P_i$

(2) Multiply P_{N1} by 2 and P_{N2} by 3 and add these products to (1).

(3) Divide P_o, P_R, P_{N1*}, P_{N2*}, by the sum obtained in (2); also divide $2P_{N1}$ and $3\ P_{N2}$ by that sum.

(4) Each quoitient obtained in (3) is defined as P_i^*.

and $\sum_i P_i^* = 1.$

4.0 DERIVATION OF ACCEPTANCE PROBABILITY

Let:

P_A = Expected proportation of accepted lots

(I) $P_A = \sum_i (P_{ti} + P_{si})\ P_i^*$

where: P_{ti} = Pr (lot tested and accepted in state i)

P_{si} = Pr (lot skipped and hence accepted in state i)

Referring to the flow diagram we have:

STATE	P_{ti}	P_{si}
0	$p(1-\gamma)$	0
N_1	$1/2\ p(1-\gamma)$	1/2
N_2	$1/3\ p(1-\gamma)$	2/3
N_{1*}	$p(1-\gamma)$	0
N_{2*}	$p(1-\gamma)$	0
R	$p(1-\gamma)$	0

which may be seen to yield:

$$P_A = p(1-\gamma)\left[P^*_o + P^*_R + P^*_{N1*} + P^*_{N2*} + 1/2\ P^*_{N1} + P^*_{N2}\right] + \frac{P^*_{N1}}{2} + \frac{2\ P^*_{N2}}{3}$$

This formula together with an assumed $\gamma = .0002$ was used to obtain Figure 3.

5.0 EXPECTED REDUCTION IN TESTING

The expected reduction in testing is $\left[\frac{P^*_{N1}}{2} + \frac{2\ P^*_{N2}}{3}\right]$ vs. p, the probability of the lot meeting all ballistic tests' requirements not concerned with critical malfunctions. The asymptote is the maximum reduction possible for $\gamma = .0002$, and was obtained by finding $\lim_{p \to 1} (P^*_{N1}, P^*_{N2})$.

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FIGURE 1

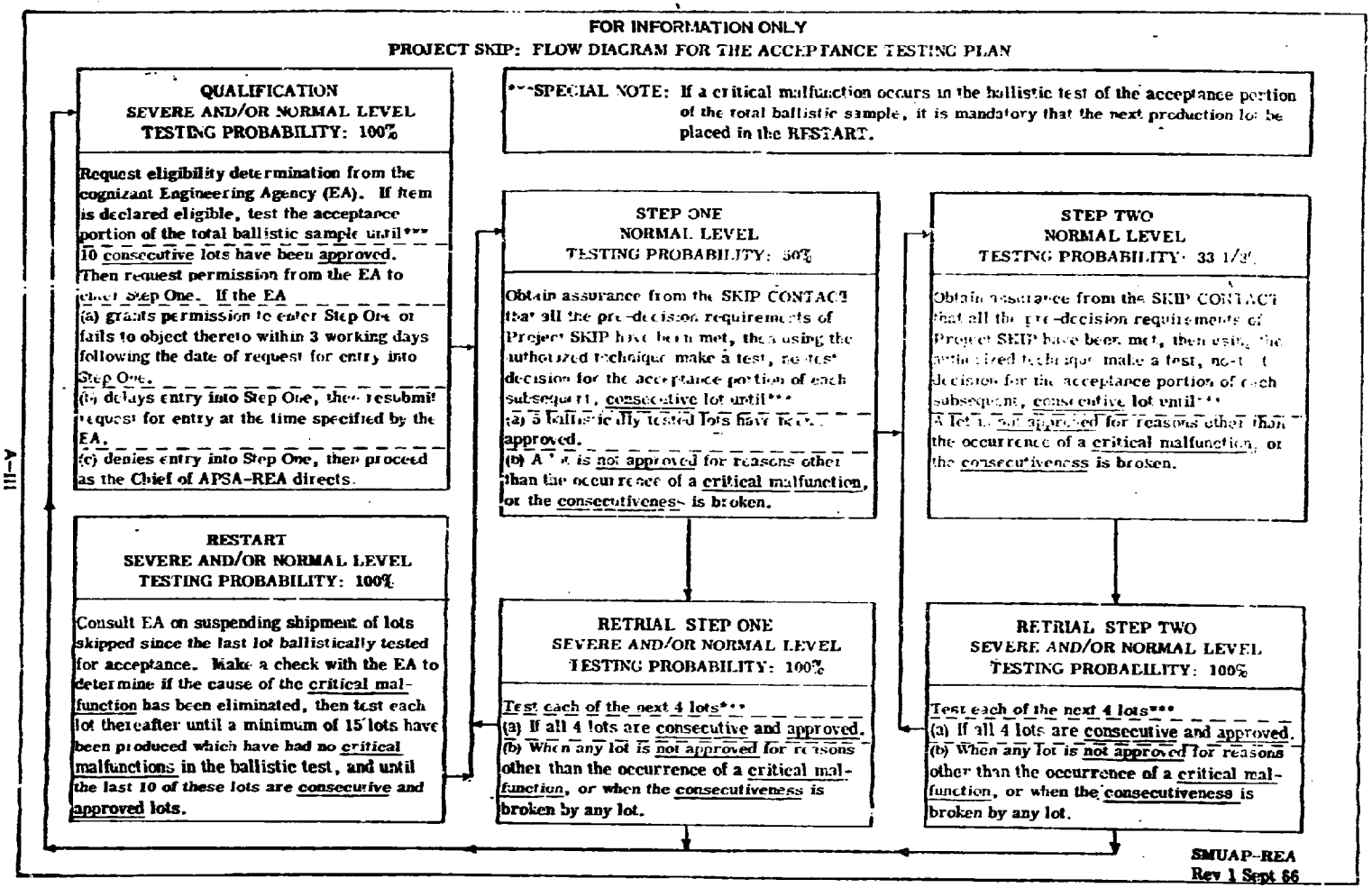
FOR INFORMATION ONLY
PROJECT SKIP: FLOW DIAGRAM FOR THE ACCEPTANCE TESTING PLAN
QUALIFICATION
SEVERE AND/OR NORMAL LEVEL
TESTING PROBABILITY: 100%
Request eligibility determination from the cognizant Engineering Agency (EA). If item is declared eligible, test the acceptance portion of the total ballistic sample until*** 10 consecutive lots have been approved. Then request permission from the EA to enter Step One. If the EA
(a) grants permission to enter Step One or fails to object thereto within 3 working days following the date of request for entry into Step One.
(b) delays entry into Step One, then resubmit request for entry at the time specified by the EA.
(c) denies entry into Step One, then proceed as the Chief of APSA-REA directs.
***SPECIAL NOTE: If a critical malfunction occurs in the ballistic test of the acceptance portion of the total ballistic sample, it is mandatory that the next production lot be placed in the RESTART.
STEP ONE
NORMAL LEVEL
TESTING PROBABILITY: 50%
Obtain assurance from the SKIP CONTACT that all the pre-decision requirements of Project SKIP have been met, then using the authorized technique make a test, request decision for the acceptance portion of each subsequent, consecutive lot until***
(a) 5 ballistically tested lots have been approved.
(b) A lot is not approved for reasons other than the occurrence of a critical malfunction, or the consecutiveness is broken.
STEP TWO
NORMAL LEVEL
TESTING PROBABILITY: 33 1/3%
Obtain assurance from the SKIP CONTACT that all the pre-decision requirements of Project SKIP have been met, then using the authorized technique make a test, request decision for the acceptance portion of each subsequent, consecutive lot until***
A lot is not approved for reasons other than the occurrence of a critical malfunction, or the consecutiveness is broken.
RESTART
SEVERE AND/OR NORMAL LEVEL
TESTING PROBABILITY: 100%
Consult EA on suspending shipment of lots skipped since the last lot ballistically tested for acceptance. Make a check with the EA to determine if the cause of the critical malfunction has been eliminated, then test each lot thereafter until a minimum of 15 lots have been produced which have had no critical malfunctions in the ballistic test, and until the last 10 of these lots are consecutive and approved lots.
RETRIAL STEP ONE
SEVERE AND/OR NORMAL LEVEL
TESTING PROBABILITY: 100%
Test each of the next 4 lots***
(a) If all 4 lots are consecutive and approved.
(b) When any lot is not approved for reasons other than the occurrence of a critical malfunction, or when the consecutiveness is broken by any lot.
RETRIAL STEP TWO
SEVERE AND/OR NORMAL LEVEL
TESTING PROBABILITY: 100%
Test each of the next 4 lots***
(a) If all 4 lots are consecutive and approved.
(b) When any lot is not approved for reasons other than the occurrence of a critical malfunction, or when the consecutiveness is broken by any lot.
SMUAP-REA
Rev 1 Sept 66

FIGURE TWO

Causes of Transition from
State i to State j

From \ To	0	R1	R2	N1	N1*	N2	N2*
0	Reject the lot. No critical malfunction.	Critical malfunction.		10 consecutive lots accepted.			
R1		Critical malfunction.	5 consecutive lots without critical malfunction.				
R2		Critical malfunction.	Reject the lot. No critical malfunction.	10 consecutive lots accepted.			
N1		Critical malfunction.			Reject the lot. No critical malfunction.	5 tested lots are accepted.	
N1*	Reject the lot. No critical malfunction.	Critical malfunction.		4 lots are accepted.			
N2		Critical malfunction.				Lot is accepted.	Reject the lot. No critical malfunction.
N2*	Reject the lot. No critical malfunction.	Critical malfunction.				4 lots are accepted.	

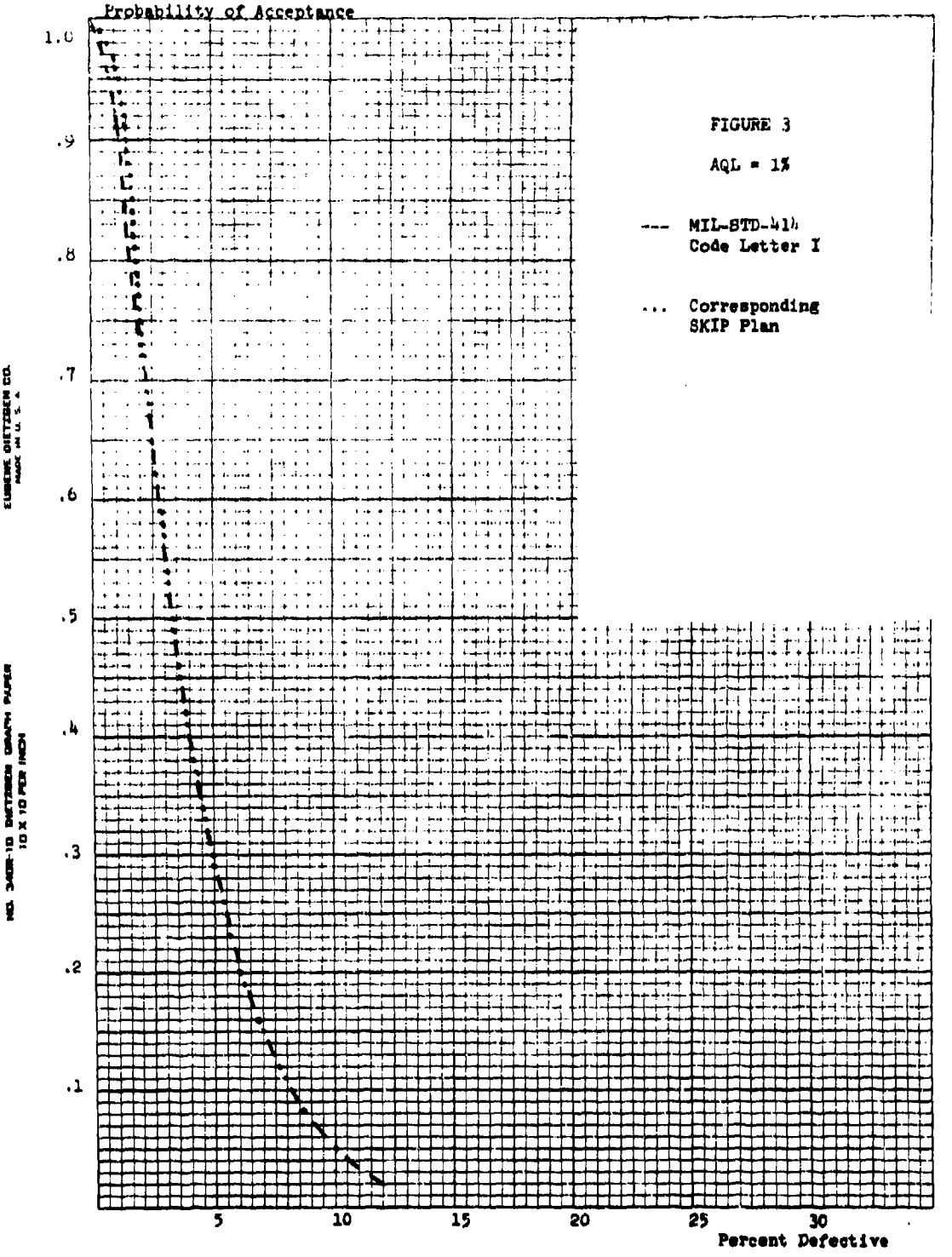

FIGURE 3

AQL = 1%

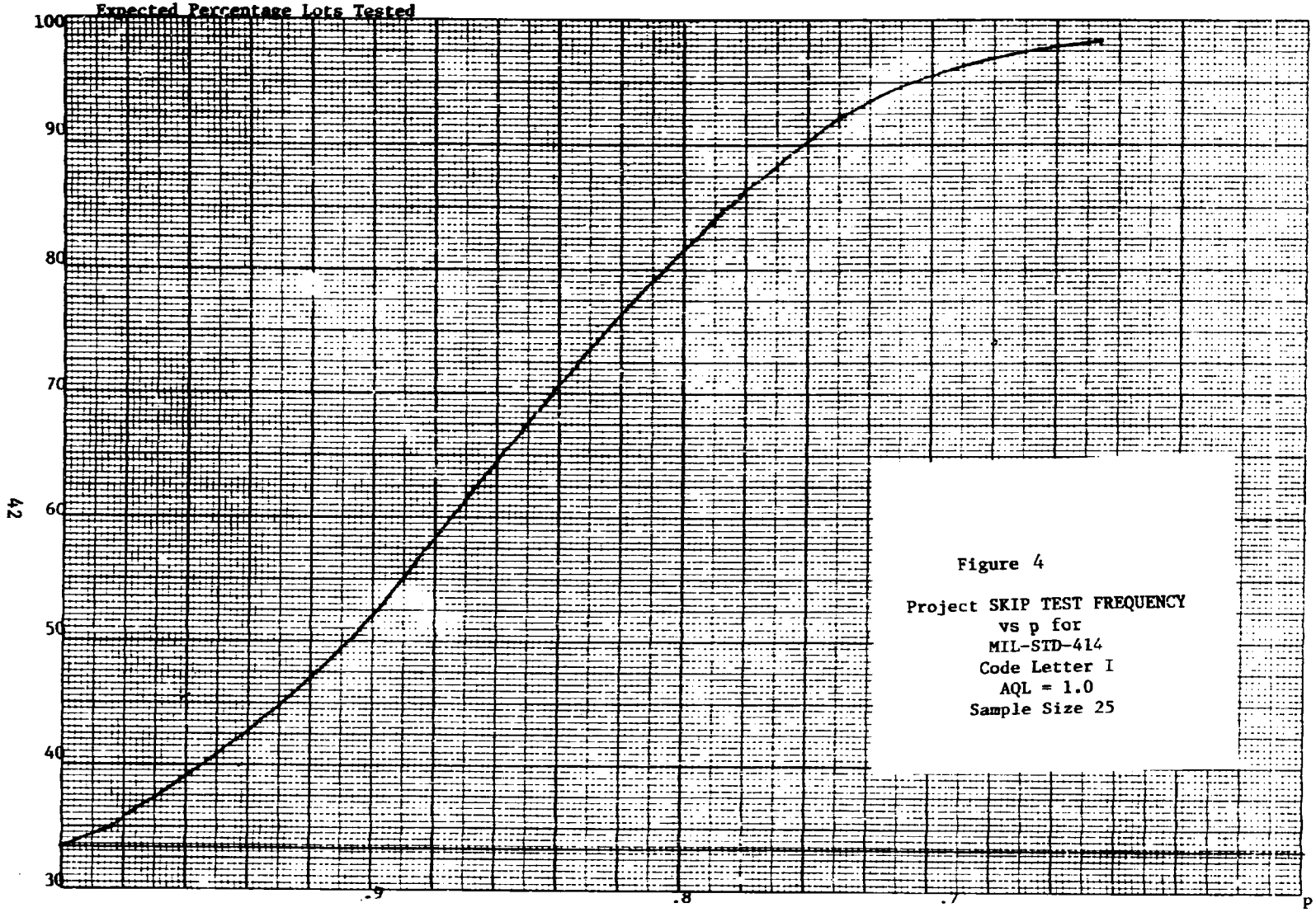

Figure 4

Project SKIP TEST FREQUENCY
vs p for
MIL-STD-414
Code Letter I
AQL = 1.0
Sample Size 25

A MODEL FOR DETERMINING QUALITY INCENTIVE PAYOFFS FOR PROCUREMENT

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INTRODUCTION. The purpose of this paper is to formulate a model for the preparation of Quality Incentive Clauses to be included in Government contracts. The model will concern itself with those items which are procured according to acceptance criteria involving single sampling plans by attributes.

A Quality Incentive Clause is an addition to a supply contract which is designed to benefit both the contractor and the Government. The clause provides for the payment of a bonus to the contractor if product quality is above that designated as acceptable in the product specification.

Changes in product quality will be observed by selecting one or more parameters which reflect item effectiveness; changes in the "relative AQL" of these parameters with respect to the AQL's outlined in the product specification will be used to indicate differences in quality level.

Finally, variations in AQL will be combined with a payoff factor to assign a partial payoff for each parameter. This payoff factor is designed to adjust for the relative importance of each parameter as well as the magnitude of the quality measurement. The sum of the partial payoffs will indicate the total payoff to which the contractor is entitled.

THE GENERAL MODEL. This section represents an outline of the general model proposed for formulating Quality Incentive Clauses. A brief explanation of each of the major segments of the model is presented below. A more elaborate discussion of the development of each of these segments will be presented in a later section.

The remainder of this paper was photographically reproduced from the author's copy.

1. Item Control Parameters.-These parameters are used to measure the "incentive quality" of the item.

2. Ratio Weights.-Weights are assigned to all control parameters to indicate the relative importance of each parameter in determining item effectiveness.

3. Maximum Payoff.-This value represents the maximum amount the purchasing agent is willing to pay for quality in the item.

4. Payoff Factors for Each Control Parameter.-The payoff factor is a multiplier which transforms a given quality measurement into an incentive payoff. It is designed to reflect both the magnitude and importance of measured quality for each parameter.

5. Partial Payoffs for Each Control Parameter.-The partial payoff is a measure of that portion of final payoff which is attributable to each control parameter.

6. Total Payoff.-This value represents the bonus payable to the contractor on the basis of the indicated quality of the item.

DEVELOPMENT OF THE GENERAL MODEL

This section traces the development of the various segments of the general model. Each major segment of the model is expanded and quantified according to the basic assumptions of the model.

Selection of Control Parameters.-Although many parameters may contribute to the performance of a particular item, it is desirable to select only a few parameters to measure quality for incentive purposes. One or two parameters are ideal; any more than three may be unwiedly and impractical.

The parameters selected should be those which most clearly define item effectiveness under operational conditions. Consequently, in addition to minimizing the number of parameters selected, care must also be taken to insure that all parameters which indicate effectiveness are included. Thus, the number of parameters selected should be as restrictive as possible, yet comprehensive enough to include all significant parameters.

Furthermore, it is important that parameter measurements be compatible with acceptance tests as outlined in the product specification. Parameters which require increased sample size or additional testing in order to be measured satisfactorily are not desirable.

Assignment of Weights.-Weights will be assigned to each control parameter in multiples of ten within the range 0 to 100 (10,20,30 .....,100). For example, consider a situation involving two parameters where it is felt that parameter A is 1 1/2 times as important as parameter B. The weights assigned would be $W_a = 30$; $W_b = 20$ or $W_a = 60$; $W_b = 40$. As long as the ratio is maintained it does not matter which combination of weights is selected.

Determination of Maximum Payoff.-The maximum payoff (MPO) is selected as that percentage of unit price the purchasing agent is willing to pay if maximum incentive quality is obtained in all parameters.

Determination of Payoff Factors.-On the basis of the subjective-objective decisions outlined above, a PAY OFF FACTOR (POF_I) is determined for each parameter. The POF_I is a function of:

(1) the individual weight of each parameter relative to the combined weights of all parameters.

(2) maximum pay-off allowed in percent unit price.

Development of POF_T:

Initially define a "quality point" as a measure of incentive quality which will give a pay-off. In order to achieve the maximum pay-off (MPO) incentive, the contractor must achieve the maximum quality points (MQP) which have been assigned for each parameter. Assuming a linear model where Zero "QP" would give Zero Pay-off, the relationship between Pay-off (PO) and "QP" can be shown as figure 1 below:

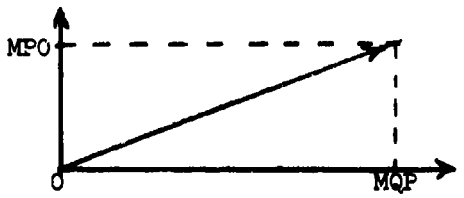

Figure 1

Since each parameter is weighted in its importance to item effectiveness, it shall be defined that the maximum QP for each parameter be equal to the weight assigned to each parameter (ie., if $W_a = 40$, $W_b = 20$; $W_a = 2W_b$ - therefore parameter W_a is twice as important as W_b and will receive twice the number of quality points).

So, on an item basis:

(1) MQP = the sum of the weights for the parameters considered for that item.

$$MQP = W_T = \sum^{N} W_I, \text{ where N is the number of parameters considered.}$$

(2) MPO = Percentage of item unit price.

Therefore Figure 1 becomes:

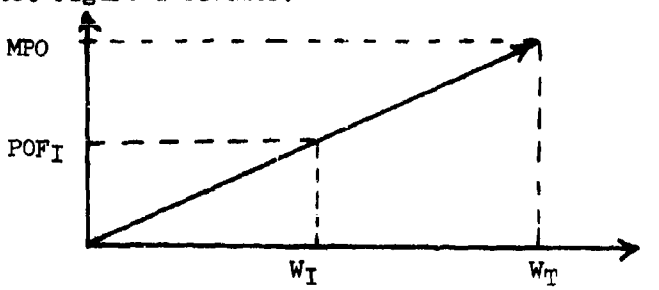

Figure 2

Parameter (I) with weight W_I gives a pay-off factor (POF_I).. POF_I is a proportional part of MPO as expressed below:

$$POF_I = \frac{(MPO) - (0)}{(W_T) - (0)} \quad \frac{(W_I)}{}$$

$$POF_I = \frac{MPO}{1} \quad \frac{(W_I)}{W_T}$$

Where POF_I is a percentage. In a fractional form:

$$POF_I = \frac{(MPO)}{(W_T)} \quad \frac{(W_I)}{100} \qquad (1)$$

A verification of the relationship is given in the appendix.

Determination of Partial Payoff.-Using the payoff factor and the percentage change in quality for each parameter (to be discussed in the next section) a partial payoff may be computed for each parameter. Therefore, for each parameter and its POF_I, Figure 3 shows the corresponding partial payoff.

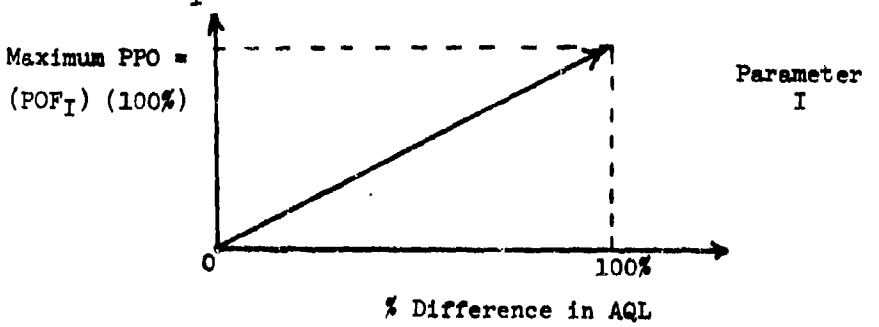

Figure 3

The partial pay-off for Parameter I (PPO_I) is represented as:

$$PPO_I = (POF_I) \quad (\% \text{ difference}) \tag{2}$$

where PPO_I is a percentage of unit price.

Determination of Total Payoff.-It follows that the total payoff (TPO) is the sum of all partial payoffs or

$$\text{Total Pay-off (TPO)} = \sum^{N} PPO_I \tag{3}$$

If we have 100% difference in AQL for each parameter; then

TPO (%) = MPO (%) - see Appendix for verification.

"MEASURING" INCENTIVE QUALITY WHEN THE ACCEPTANCE NUMBER OF ALL PARAMETERS IS FIVE OR GREATER

Incentive payoffs will be made on the basis of changes in quality. For this model, these changes will be measured in terms of AQL. This measure is, in fact, a psuedo-AQL (AQL_p).

As stated in the previous section, incentive quality is indicated by the percentage difference. AQL_s is the AQL for a parameter outlined in the product specification. For brevity, AQL_s is presented in the form of the appropriate sampling plan as follows:

$$(AQL_s \mid \text{code letter; } n, x, x + 1) \text{ where } x \geq 5 \tag{4}$$

and: n = sample size

x = acceptance number

x + 1 = rejection number

All data is in accordance with MIL-STD-105D.

When sampling is conducted according to the specification sampling plan and the number of defects in the observed sample is some $x' < x$, then for convenience AQL_p is defined as follows:

$$(AQL_p \mid \text{code letter; } N;\ x',\ x' + 1) \qquad (5)$$

code letter is the same as in (4)

where: n = sample size

x' = acceptance number

x'+1 = rejection number

All data is in accordance with MIL-STD-105D. AQL_p can be determined using MIL-STD-105D, a Thorndike Chart or Poisson Tables.

It is important to point out that the AQL_p does not mean that the process average is actually equal to the AQL_p. The AQL_p is a "dummy" measure of quality. It merely says that --

if a sampling plan had been used with code letter α, sample size N, decision criteria x', x' + 1 - then the AQL associated with this plan is AQL_p. It is the AQL of the sampling plan that has just been passed.

The pseudo value is used in the incentive model to compute the percentage change in AQL_s or "the change in quality".

The percentage difference (%D) between AQL_s and AQL_p is computed by:

$$\%D_I = \frac{AQL_s - AQL_p}{AQL_s} 100 \qquad (6)$$

EXAMPLE OF DETERMINING AQL_p AND % D_I WHEN THE ACCEPTANCE NUMBER OF ALL PARAMETERS IS FIVE OR GREATER

Simple Case - one parameter considered.

Example: Consider a parameter with AQL_s as follows (1.0 | M; 315; 7, 8).

If in sampling the number of defects observed is 3 then x' = 3. Hence, AQL_p is defined according to equation (5) as (AQL_p | M; 315; 3, 4).

Using MIL-STD-105D, AQL_p = .40

Therefore: $$\%D_I = \frac{AQL_s - AQL_p}{AQL_s} \; 100$$

$$= \frac{(1.0 - .40)}{1.0} \; 100$$

$$= 60\%$$

COMPLETE EXAMPLE FOR COMPUTING INCENTIVE PAYOFF IN WHICH THE ACCEPTANCE NUMBER OF ALL PARAMETERS IS 5 OR GREATER

Two significant parameters, A and B, have been selected for the item in question. Subjective judgment indicates that Parameter A is 1-1/2 times as important as Parameter B.

Step	Information	Value Obtained	Explanation
1	Weight for Parameter A: W_a	30	
2.	Weight for Parameter B: W_b	20	
3	Sum of Weights (W_T): $W_a + W_b$	50	30 + 20

Step	Information	Value Obtained	Explanation
4	Maximum Payoff: MPO	10%	Subjective
5	Payoff Factor A (POF_a):		
	$\frac{(MPO)\ (W_a)}{(W_T)\ 100}$	.06	$\frac{(10)\ (30)}{(50)\ (100)}$
6	Payoff Factor B (POF_b):		
	$\frac{(MPO)\ (W_b)}{(W_T)\ 100}$	.04	$\frac{(10)\ (20)}{(50)\ (100)}$

The incentive clause indicates that two parameters A and B, will be used. Parameter A has an AQL_s of 1.0%, POF_a = .06. Parameter B has an AQL_s of .65%, POF_b = .04. The size of the lot for which a payoff is to be calculated is 15,000. General Inspection level II is to be used.

The number of defectives found in the sample for Parameter A was 3. (x' = 3)

The number of defectives found in the sample for Parameter B was 2. (x' = 2)

Step	Information	Value Obtained	Explanation
7	Sampling Plan	Code Letter M	
	Parameter A: AQL_s; SS (x, x+1)	(1.0%; 315 (7,8))	Product Specification
	Parameter B: AQL_s; SS (x, x+1)	(.65%; 315 (5,6))	Product Specification
8	Pseudo AQL		
	Parameter A: AQL_p	.40%	105-D
	Parameter B: AQL_p	.25%	

Step	Information	Value Obtained	Explanation
9	Percentage Difference (%D): $\frac{(AQL_s - AQL_p)}{AQL_s}$ (100)		
	Parameter A: %D	60%	$\frac{(1.0 - .40)}{1.0}$ 100
	Parameter B: %D	62%	$\frac{(.65 - .25)}{.65}$ 100
10	Partial Payoff (PPO) = (%D) (POF):		
	Parameter A: PPO_a	3.6%	(60) (.06)
	Parameter B: PPO_b	2.5%	(62) (.04)
11	Total Payoff TPO = $PPO_a + PPO_b$:		
		6.1%	3.6 + 2.5

"MEASURING" INCENTIVE QUALITY WHEN THE ACCEPTANCE NUMBER OF AT LEAST ONE PARAMETER IS LESS THAN FIVE

Choose the control parameter which has a sampling plan where X is minimum less than five. The general approach to the problem will be to determine from the requirements:

$(AQL_s \mid$ code letter; N; X, X+1) where $X < 5$. A second sampling plan will be defined as:

$$(AQL_s \mid N'; X_a, X_a + 1) \qquad (7)$$

where $X_a > X$ and N'/N is some whole number greater than one which represents the cumulative number of lots that have to be sampled before an incentive pay-off decision can be made. The conditions of equation (7) can be satisfied by use of a standard Thorndike Chart or Summation of Terms of Poisson's Exponential Binomial Limit.

Example

Consider the specification sampling plan:

$(AQL_s) = 0.25\% \mid N = 315;\ 2,3)$

The AQL can also be expressed as fraction defective:

$(AQL_s = .0025 \mid N = 315;\ 2,3)$

If an equivalent 5 - 6 plan ($X_a = 5$) is required, a Thorndike Chart can be used. Defining the possibility of acceptance at the AQL to be .95 -

N' (AQL expressed as fraction defective) = 2.6

or N' (.0025) = 2.6

$N' = 1040$ items

The AQL could have also been written as a percentage

N' (AQL (%)) = 260 (8)

$N' = 1040$

The equivalent sampling plan expressed as equation (7) is:

$(AQL_s = .25\% \mid N = 1040;\ 5,6)$

In general, equation (8) can be written for both the specification sampling plan (N, X, X + 1) and the second equivalent sampling plan (N', X_a, X_a + 1):

$$\left.\begin{aligned} \text{(i):}\quad (N)\,(AQL_s) &= (\text{Poisson factor } X,\ X+1)\ 100 \\ &= (\pi X,\ x+1)\ 100 \\ \text{(ii):}\quad (N)\,(AQL_s) &= (\text{Poisson factor } X_a,\ X_a+1)\ 100 \\ &= (\pi X_a,\ X_a+1)\ 100 \end{aligned}\right\} \quad (9)$$

$$\frac{N'}{N} = \frac{\dfrac{(\pi X_a,\ X_a+1)100}{AQL_s}}{\dfrac{(\pi X,\ X+1)\ 100}{AQL_s}}$$

or $$\frac{N'}{N} = \frac{(\overline{\Pi} X_a,\ X_a + 1)\ 100}{(\overline{\Pi} X,\ X + 1)\ 100}$$

From equation (9):

$$(\overline{\Pi} X,\ X + 1)\ 100 = (N)\ (AQL_s)$$

So that

$$\frac{N'}{N} = \frac{(\overline{\Pi} X_a,\ X_a + 1)}{(N)\ (AQL_s)} \tag{10}$$

For convenience, modified Poisson factors for all X_a, $X_a + 1$ can be derived for a probability of acceptance of .95. The numerator of equation (10) can be written as:

$(\overline{\Pi} X_a,\ X_a + 1)\ 100 = Z_i$, Equation (10) now becomes:

$$\frac{N'}{N} = \frac{Z_i}{(N)\ (AQL_s)} \tag{11}$$

Values of Z_i for attribute sampling plans X, X + 1 are given below in Table I

TABLE I

X, X + 1	Z_i
5 - 6	260
6 - 7	330
7 - 8	400
8 - 9	470
9 - 10	540
10 - 11	620
11 - 12	700
12 - 13	780
13 - 14	850

X, X + 1	Z_i
14 - 15	930
15 - 16	1000
16 - 17	1070
17 - 18	1170
18 - 19	1250
19 - 20	1320
20 - 21	1400

By letting L equal the number of cumlative lots, such that N'/N is a whole number:

$$L = l_i = \frac{Z_i}{(N)\ (AQL_s)}$$ where the value of l_i which is closest to a whole number is chosen as the value of L.

Example:

Consider the specification sampling plan:

(AQL_s = .25% | N = 315; 2,3)

In this case:

$(N)\ (AQL_s) = (315)\ (.25) = 78.75$

Therefore, applying equation (11) and Table I:

$$l_{5-6} = \frac{260}{78.75} = 3.30$$

$$l_{6-7} = \frac{330}{78.75} = 4.20$$

$$l_{7-8} = \frac{400}{78.75} = 5.08$$

$$l_{8-9} = \frac{470}{78.75} = 5.97$$

which is nearly a whole integer and corresponds to a 8-9 plan. Therefore L = 6 lots will be accumulated before an incentive pay-off decision will be made. If 6 lots of sample size 315 each are accumulated, then the

adjusted sample size is:

$$N' = (L)(N) \qquad (12)$$

or $N' = (6)(315) = 1890$

In summary - the original sampling plan is:

(AQL = .25% | 315; 2,3)

This is replaced with an equivalent plan by accumulating 6 lot samples defined in accordance with equation (7):

(AQL = .25% | 1850; 8,9)

Extention of this example by computing AQL_p and Payoff:

If 6 defects were encountered (X' = 6, X' + 1 = 7) in 6 lots of sampling (total N = 1890), the pseudo AQL (AQL_p) can be determined as follows. The definition of AQL_p is the same as in the previous section if

(AQL_s = .25% | 1890; 8,9)

then (AQL_p = ? | 1890; 6,7)

It is known that

(N) (AQL_s) = (Z_i for X, X + 1 plan). If X, X + 1 and N are known, then AQL_s can be determined. This is also true for AQL_p. In this example, (X', X' + 1) = (6,7) and N' = 1890. Applying equations (9) to this situation:

(N') (AQL_p) = (Z_i for X', X' + 1 plan)

(N') (AQL_p) = (Z_i for 6,7 plan)

(1890) (AQL_p) = 330

AQL_p = .175%

The resultant percentage difference (%D) between AQL_s and AQL_p is:

$$\%D_i = \frac{(.250 - .175)}{.250} 100 = 30\%$$

Note: These adjustments must be made in the sampling plans of all (pertanent) item control parameters. The following example (although repetitious in part) will include the complete computation as well as the TPO.

A COMPLETE EXAMPLE FOR CALCULATING PAYOFF WHEN THE ACCEPTANCE NUMBER OF AT LEAST ONE PARAMETER IS LESS THAN 5

Parameter A: $N = 32$, $(0 - 1)$; $AQL_s = .40$; $W_A = 30$

B: $N = 50$, $(1 - 2)$; $AQL_s = 1.00$; $W_B = 20$

MPO = 10%

$W_T = W_A + W_B = 50$

$$POF_a = \frac{(MPO)\ (W_A)}{(W_T)\ 100} = \frac{(10)\ (30)}{(50)\ (100)} = .06$$

$$POF_b = \frac{(MPO)\ (W_B)}{(W_T)\ (100)} = \frac{(10)\ (20)}{(50)\ (100)} = .04$$

Because both of the sampling plans have acceptance numbers less than 5, the number of cumulative lots (samples) must be determined in order to determine equivalent sampling plans:

$$l_i = \frac{Z_i}{(N)\ \ (AQL_s)}$$

Plan A has X as a minimum (X = 0):

$$(N)\ (AQL_s) = (32)\ (.4) = 12.8$$

Referring to Table I:

$$l_1 = \frac{260}{12.8} = 20.31$$

$$l_2 = \frac{330}{12.8} = 25.78$$

$$l_3 = \frac{400}{12.8} = 31.25$$

$$l_4 = \frac{470}{12.8} = 36.72$$

$$l_5 = \frac{540}{12.8} = 42.19$$

arbitrarily stop

$l_2 = 25.78$ is nearest to a whole number, $\cong(26)$, therefore L = 26 lots will be accumulated. If 26 lots of sample size 32 are accumulated, the adjusted sample size N' is:

$N' = (L)\ (N) = (26)\ (32) = 832$

In summary - the original plan is:

$(AQL_s = .40 \mid 32;\ (0 - 1))$

But Z_2 of 330 (see Table I) corresponds to a 6 - 7 plan (i.e.; $X_a = 6$, $X_a + 1 = 7$) so that the revised plan is:

$(AQL_s = .40 \mid 832;\ (6 - 7))$ for Parameter A

An appropriate adjustment must be made for Parameter B:

The original plan for Parameter B is:

$(AQL_s = 1.00 \mid 50;\ 1 - 2)$

Since L = 26 lots will be accumulated, the adjusted sample size N' is:

$N' = (L)\ (N) = (26)\ (50) = 1300$

Since:

$(N)\ (AQL_s) = Z_1$
$(1300)\ (1.00) = Z_1$
$Z_1 = 1300$

Reference to Table I shows that $Z_1 = 1320$ corresponds with a 19 - 20 plan. The revised plan for Parameter B is therefore:

$(AQL_s = 1.00 \mid 1300;\ 19 - 20)$

In actual sampling the following defects were counted:

Parameter A - 5 defects

Parameter B - 10 defects

The Psuedo AQL_s are computed:

For Parameter A:

$(N)\ (AQL_p) = Z_1$ ($X' = 5$, $X' + 1 = 6$ for 5 - 6 plan)
$(832)\ (AQL_p) = 260$ from Table I
$AQL_p = .31\%$

For Parameter B:

$$(N)\ (AQL_p) = Z_i \quad (X' = 10,\ X' + 1 = 11 \text{ plan})$$
$$(1300)\ (AQL_p) = 620 \text{ from Table I}$$
$$AQL_p = .48\%$$

Therefore:

$$\%D_a = \frac{(.40 - .31)}{.40}\ 100 = 22.5\%$$

$$\%D_b = \frac{(1.00 - .48)}{1.00}\ 100 = 52.0\%$$

The Partial Pay-offs for each Parameter are:

$$PPO_a = (POF_a)\ (\%D_a)$$
$$= (.06)\ (22.5\%)$$
$$= 1.35\%$$

$$PPO_b = (POF_b)\ (\%D_b)$$
$$= (.04)\ (52\%)$$
$$= 2.08\%$$

Therefore the total pay-off awarded to the contractor after 26 lots were produced and sampled was:

$$TPO = PPO_a + PPO_b$$
$$= 1.35\% + 2.08\%$$
$$= 3.43\% \text{ of unit price}$$

CONCLUSIONS

Although some effort has been expended in investigating the development of quality incentive payoffs it is believed that this paper makes a significant contribution in the area. This contribution is evidenced by the investigation and extension of previously formulated concepts and the synthesis of standard statistical techniques. In particular, an effort is made to make provision for the following common situations which normally occur in actual acceptance sampline plans:

1. Items having several parameters which contribute to overall effectiveness in varying degrees.

2. Items/parameters with acceptance sampling plans specifying small

acceptance numbers (i.e., C 5) which inherently lack an expanded range of quality measure (AQL_p and %D).

3. Items/parameters specifying sampling plans with an acceptance number equal to zero.

Provisions for the above situations were established by defining a procedure for selecting and weighting item parameters relating to effectiveness. Furthermore, quality incentive pay-off decisions for sampling plans requiring small acceptance numbers were incorporated into the model by cumulating the results of several product lots.

The procedures presented in this paper are not considered to contribute a sophisticated approach to formulating quality incentive plans. Intuitively, the basic philosophical framework is believed to be workable, however the overall model should certainly lend itself to further refinement and simplication. Some restrictive features of the model which would be adaptable to future work are:

(1) Restricted to single sampling by attributes.

(2) Limited to simple functioning items.

(3) Considerable subjective judgment involved in selection and weighting of parameters.

(4) Loss of incentive impact due to complexity of special procedures for cases in which $C < 5$.

(5) Undesirable time factor due to lot accumulation when acceptance number is very small.

(6) Rounding error in Z_i values.

APPENDIX

I. Property:

$$\sum POF_1 = MPO \text{ (in terms of percentage)}$$

Demonstration:

$$W_t = W_1 + W_2 + \ldots + W_i + \ldots + W_n = \sum W_i$$

Therefore:

$$\frac{W_1}{W_t} + \frac{W_2}{W_t} + \ldots + \frac{W_1}{W_t} + \ldots + \frac{W_N}{W_t} = 1$$

It has been established:

$$POF_1 = MPO \left(\frac{W_1}{W_t}\right) \text{ (in terms of percentage)}$$

$$\sum_{1}^{N} POF_1 = \sum_{1}^{N} MPO \left(\frac{W_1}{W_t}\right)$$

Expanding the summation:

$$\sum_{1}^{N} POF = MPO \left[\left(\frac{W_1}{W_t} + \frac{W_2}{W_t} + \ldots + \frac{W_1}{W_t} + \ldots + \frac{W_n}{W_t}\right)\right]$$

or $$\sum_{1}^{N} POF = MPO$$

II. Property:

If the percent difference between the specification AQL and the pseudo AQL is 100 percent, then:

$$TPO = MPO \text{ (in percent)}$$

Demonstration:

$$\begin{aligned} TPO\ (\%) &= \sum_{1}^{N} PPO_1 \\ &= \Sigma (POF_1)\ (\% \text{ difference}) \\ &= \Sigma (POF_1)\ (100\%) \end{aligned}$$

where POF_1 is a fraction, therefore;

$$TPO\ (\%) = \sum_{1}^{N} (POF_1) \text{ (in percent)}$$

But it has been shown in 1 that:

$$\sum_{1}^{N}(POF_1) = MPO$$

Therefore:

$$TPO\ (\%) = MPO\ (\%)$$

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OPTIMUM SAMPLING PLANS FOR GRADING BINOMIAL POPULATIONS

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INTRODUCTION AND BACKGROUND. In the surveillance evaluation of ammunition an important task is that of grading lots on the basis of attribute characteristics of a sample drawn from larger populations. At the present time, lots are placed into one of three grades based on the performance of a random sample of n items chosen from the lot. It is of obvious importance that the probability of misgrading a lot based on this sample be made a minimum. The basis for the current grading procedure is a BRL report written by Mr. A. Golub entitled "The Determination of Acceptance Numbers for Placing a Lot from which a Single Sample is Drawn into One of Three Grades" published in 1951. In this report, Mr. Golub maximizes the probability of correct grading by differentiating expressions of the following type $\sum_{i=0}^{c} \binom{n}{i} p^i q^{n-i}$, setting the resulting values equal to zero and solving for the c values (acceptance numbers).

Mr. Golub's report serves as a basis for the following paper in which a different method of maximizing the probability of correct grading is developed. A generalized solution is given and tables are developed for lot classification into 2, 3, or 4 grades.

THEORETICAL DISCUSSION. In determining the original acceptability of large quantities of manufactured products or in checking the reliability of items which have been in storage for some time, groups of the product are submitted for inspection (testing) in divisions called lots. These lots can often be characterized by a certain property, or set of properties of the individual members of the lot. For example, a population of artillery projectiles can be divided into those which are defective and those which are not, a group of washer fittings can be divided into those which fit a five-inch setting and those which do not. We let x be a random variable which assumes the value 0 if an individual in the lot has none of the characterizing properties and 1 if the individual possesses one or more of

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these properties. If we now let p=F(x=1), then the lot is defined to be a lot with fraction defective p and an individual which exhibits one or more of the characterizing properties is called a defective item.

In dealing with large lots, it is frequently too expensive or time consuming to examine or test each item in the lot. (In fact, where the procedure calls for the destruction of the item, it is impossible to inspect every item.) Thus, some type of sampling inspection plan must be devised. One of the more common types of sampling plans is the so-called single-sampling plan where the consumer selects a random sample of size n from the lot and if the number of defective items in the sample is less than or equal to a given number c, the lot is accepted and if c + 1 or more defectives is found in the sample, the lot is rejected.

This concept can be readily extended to situations involving classification of a lot into more than two classes, say, three, four, or any number up to k classes or grades. Let us assume that for each of the k grades, an interval has been determined such that, if the lot fraction defective is in this interval, the lot belongs to that grade. These intervals or levels can be determined by a review of the specifications for the item or by considering the requirements established by the user or consumer for the reliability of the item.

Now, let us suppose for convenience, that our stockpile consists of exactly 100 lots which have the corresponding fraction defectives; $y_0 = 0$, $y_1 = .01$, $y_2 = .02$, ... $y_{98} = .98$, $y_{99} = .99$. One of these lots is selected at random and submitted to our sampling plan. We let p be the lot fraction defective for this lot.

We now want to place this lot into one of k grades in accordance with the following: if the lot fraction defective is less than p_1 ($0 \leq p \leq p_1$), the lot is of Grade A quality; if the lot fraction defective is between p_1 and p_2 ($p_1 < p \leq p_2$), the lot is of Grade B quality; if the lot fraction defective is between p_2 and p_3 ($p_2 < p \leq p_3$), the lot is of Grade C quality and so on, out to the final grade, that is, if the lot fraction defective is more than p_k ($p_{k-1} < p \leq 1$), the lot is of Grade K quality.

Our plan now calls for selecting a random sample of n items from the lot, inspecting (testing) each item in the sample and determining the number of items (r) which are defective. The lot will then be placed into one of the k grades using the following rule:

If $0 \leq r \leq c_1$	Place the lot in Grade A
If $c_1 + 1 \leq r \leq c_2$	Place the lot in Grade B
If $c_2 + 1 \leq r \leq c_3$	Place the lot in Grade C
.	.
.	.
.	.
If $c_{k-1} + 1 \leq r \leq n$	Place the lot in Grade K

Under this set of conditions, we can use the formula of total probability* to calculate the probability of placing a given lot into its proper grade, or in other words, we are determining the probability of correctly calling a Grade t lot its actual grade, Grade t. This gives

P = P {of placing the lot in the correct grade} =

$$P\{p = 0\}\, P\{0 \leq r \leq c_1 \mid p = 0\} + P\{p = .01\}\, P\{0 \leq r \leq c_1 \mid p = .01\} + \ldots + P\{p = p_1\}\, P\{0 \leq r \leq c_1 \mid p = p_1\} + P\{p = p_1 + .01\}\, P\{c_1 + 1 \leq r \leq c_2 \mid p = p_1 + .01\} + \ldots + P\{p = p_2\}\, P\{c_1 + 1 \leq r \leq c_2 \mid p = p_2\} + P\{p = p_2 + .01\}\, P\{c_2 + 1 \leq r \leq c_3 \mid p = p_2 + .01\} + \ldots + P\{p = p_3\}\, P\{c_2 + 1 \leq r \leq c_3 \mid p = p_3\} + \ldots + P\{p = p_{k-1} + .01\}\, P\{c_{k-1} + 1 \leq r \leq n \mid p = p_{k-1} + .01\} + \ldots + P\{p = .99\}\, P\{c_{k-1} + 1 \leq r \leq n \mid p = .99\} \quad (1)$$

Because the lot which was submitted to the plan was selected at random from the 100 lots available in the stockpile, we know that

$$P\{p = 0\} = P\{p = .01\} = \ldots = P\{p = .99\} = \frac{1}{100} \quad (2)$$

The probability expressed in the second bracket of each product can be written as the sum of a binomial probability function of the form

$$\sum_{r=0}^{c} \binom{n}{r} (p)^r (1-p)^{n-r} \quad (3)$$

---

* B. V. Gnedenko, "The Theory of Probability", page 64.

Using expressions (2) and (3), we can rewrite (1) in the following form.

$$P = \frac{1}{100} \sum_{r=0}^{c_1} \binom{n}{r} (0)^r (1)^{n-r} + \frac{1}{100} \sum_{r=0}^{c_1} \binom{n}{r} (.01)^r (.99)^{n-r} + \ldots +$$

$$\frac{1}{100} \sum_{r=0}^{c_1} \binom{n}{r} (p_1)^r (1-p_1)^{n-r} + \frac{1}{100} \sum_{r=c_1+1}^{c_2} \binom{n}{r} (p_1+.01)^r (1-p_1-.01)^{n-r} + \ldots +$$

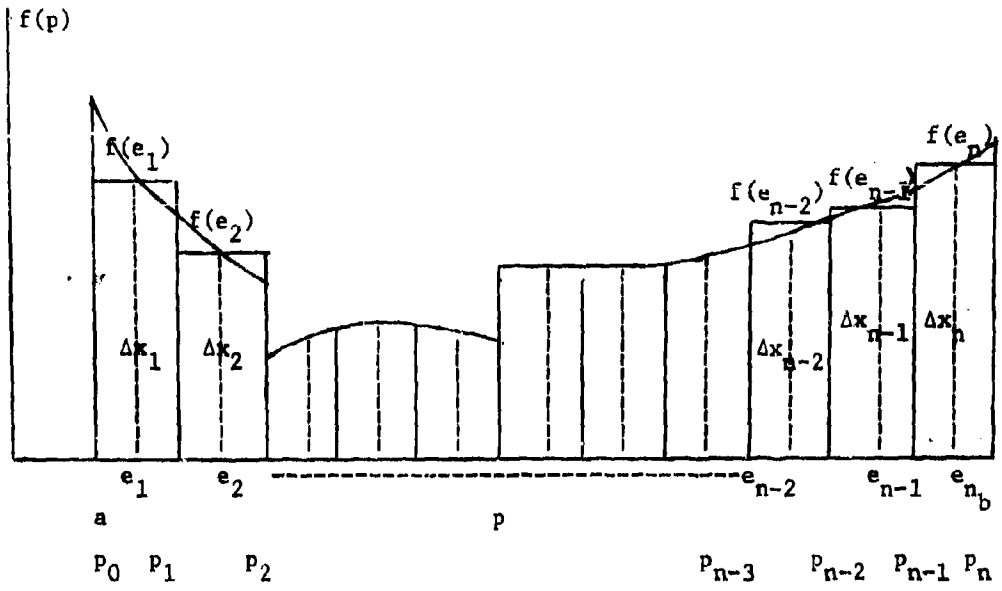
$$\frac{1}{100} \sum_{r=c_1+1}^{c_2} \binom{n}{r} (p_2)^r (1-p_2)^{n-r} + \frac{1}{100} \sum_{r=c_2+1}^{c_3} \binom{n}{r} (p_2+.01)^r (1-p_2-.01)^{n-r} + \ldots +$$

$$\frac{1}{100} \sum_{r=c_2+1}^{c_3} \binom{n}{r} (p_3)^r (1-p_3)^{n-r} + \ldots + \frac{1}{100} \sum_{r=c_{k-1}+1}^{n} \binom{n}{r} (p_{k-1}+.01)^r (1-p_{k-1}-.01) +$$

$$\ldots + \frac{1}{100} \sum_{r=c_{k-1}+1}^{n} \binom{n}{r} (.99)^r (.01)^{n-r} \qquad (4)$$

In order to maximize the probability of putting a lot into the correct category, we must maximize (4) with respect to c_1, c_2, c_3, ... c_{k-1}. However, obviously, we do not want to limit ourselves to the case where N (number of lots in the stockpile) = 100, but rather want to generalize our approach so that N can go to infinity and thus consider the case where p can assume any value on the closed interval [0,1] with equal probability.

First, we must look briefly at the definition of a definite integral. Consider a function f(p) which is continuous on the interval [a,b], (a < b), except at one or more points of the form p = a + t/N, where t = 1,2, ... N(b-a) and is everywhere non-negative on this interval. The graph of this function (using three grades as an example) can be represented by the following sketch.



We now divide [a,b] into N equal intervals, with the length of each interval = Δx_i. In each segment choose points e_1, e_2 ... e_n and consider the sum $f(e_1)\,\Delta x_1 + f(e_2)\,\Delta x_2 + \ldots + f(e_n)\,\Delta x_n$ (5)

which is equal to $\sum_{i=1}^{n} f(e_i)\,\Delta x_i$

Since all the intervals are equal, (5) can be written as

$$\sum_{i=1}^{n} f(e_i)\,\Delta x_i = \frac{b-a}{n} \sum_{i=1}^{n} f(e_i) \quad \text{and by definition} \tag{6}$$

$$\lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(e_i)\,\Delta x_i = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \frac{b-a}{n} \sum_{i=1}^{n} f(e_i) = \int_a^b f(p)\,dp$$

This is exactly the form of (4), the sum of which we are seeking to maximize with respect to c_1, c_2, c_3, ... c_{k-1}, if we let $n\to\infty$. Thus, if we maximize

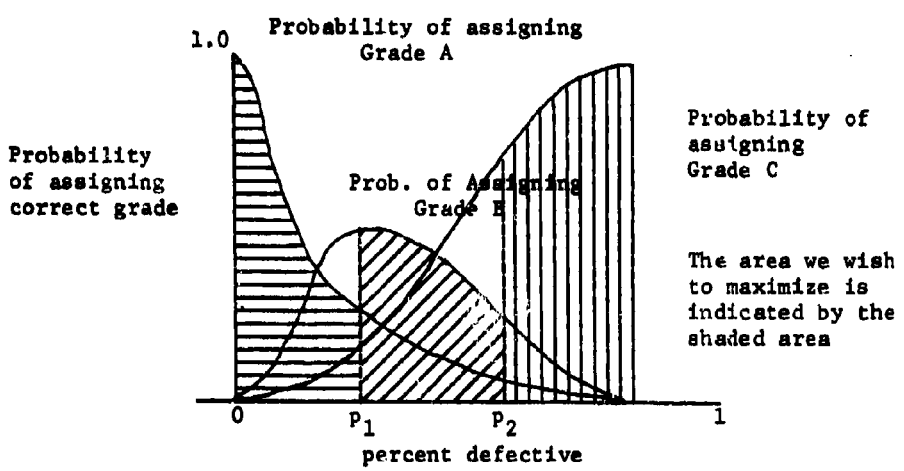
$$s = \int_0^{p_1} \sum_{r=0}^{c_1} \binom{n}{r} p^r (1-p)^{n-r}\,dp + \int_{p_1}^{p_2} \sum_{r=c_1+1}^{c_2} \binom{n}{r} p^r (1-p)^{n-r}\,dp +$$

$$\int_{p_2}^{p_3} \sum_{r=c_2+1}^{c_3} \binom{n}{r} p^r (1-p)^{n-r} dp + \ldots + \int_{p_{k-1}}^{1} \sum_{r=c_{k-1}+1}^{n} \binom{n}{r} p^r (1-p)^{n-r} dp \quad (7)$$

with respect to $c_1, c_2, c_3, \ldots c_{k-1}$, we have maximized the probability of placing a lot in the correct grade given that it was selected at random from a population of lots whose fraction defective has a uniform distribution on the unit interval. Thus, the use which we have made of integration is equivalent to placing a uniform prior distrubtion on p, the true lot fraction defective.

Our problem now becomes one of choosing those values of $c_1, c_2, \ldots c_{k-1}$ which maximize (7) for given values of $p_1, p_2, p_3 \ldots p_{k-1}$ and n.

As an example for the case k=3 (3 grades) we can illustrate graphically by "operating-characteristic" curves the area which we wish to be a maximum.



We now express s in (7) as

$$s = \int_0^{p_1} \sum_{i=0}^{c_1} \binom{n}{i} p^i (1-p)^{n-i} dp + \int_0^{p_2} \sum_{i=c_1+1}^{c_2} \binom{n}{i} p^i (1-p)^{n-i} dp - \int_0^{p_1} \sum_{i=c_1+1}^{c_2}$$

$$\binom{n}{i} p^i (1-p)^{n-i} dp + \int_0^{p_3} \sum_{i=c_2+1}^{c_3} \binom{n}{i} p^i (1-p)^{n-i} dp - \int_0^{p_2} \sum_{i=c_2+1}^{c_3} \binom{n}{i} p^i (1-p)^{n-i} dp + \ldots +$$

$$\int_0^1 \sum_{i=c_{k-1}+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} dp - \int_0^{p_{k-1}} \sum_{i=c_{k-1}+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} dp \qquad (8)$$

Let $Q = \int_0^{p_1} \binom{n}{i} p^i (1-p)^{n-i} dp = \binom{n}{i} \int_0^{p_1} p^i (1-p)^{n-i} dp$

Integrate by parts: $u = p^i$ $\quad dv = (1-p)^{n-i} dp$

$du = i\, p^{i-1} dp$ $\quad v = \frac{-(1-p)^{n-i+1}}{(n-i+1)}$

$$Q = \binom{n}{i} \left\{ \frac{-p^i (1-p)^{n-i+1}}{n-i+1} \Big|_0^{p_1} + \int_0^{p_1} \frac{i\, p^{i-1} (1-p)^{n-i+1}}{n-i+1} dp \right\}^{(n-i+1)}$$

$$= \binom{n}{i} \left\{ \frac{-1}{n-i+1} p_1^i (1-p_1)^{n-i+1} + \int_0^{p_1} \frac{i}{n-i+1} p^{i-1} (1-p)^{n-i+1} dp \right\}$$

$$= \binom{n}{i} \left\{ \frac{-1}{n-i+1} p_1^i (1-p_1)^{n-i+1} + \frac{i}{n-i+1} \int_0^{p_1} p^{i-1} (1-p)^{n-i+1} dp \right\}$$

Integrate by parts: $i = p^{i-1}$ $\quad dv = (1-p)^{n-i+1}$

$du = (i-1)\, p^{i-2} dp$ $\quad v = \frac{(1-p)^{n-i+2}}{(n-i+2)}$

Now $Q = \binom{n}{i} \left\{ \frac{-1}{n-i+1} p_1^i (1-p_1)^{n-i+1} + \frac{i}{n-i+1} \left[\frac{-1}{n-i+2} p^{i-1} (1-p)^{n-i+2} \right]_0^{p_1} + \right.$

$\left. \frac{i}{n-i+1} \int_0^{p_1} \frac{i-1}{n-i+2} p^{i-2} (1-p)^{n-i+2} dp \right\}$

which equals, after simplification

$$Q = -\frac{1}{n+1} \binom{n+1}{i} p_1^i (1-p_1)^{n-i+1} - \frac{1}{n+1} \binom{n+1}{i-1} p_1^{i-1} (1-p_1)^{n-i+2} +$$

$$\frac{i-1}{n+1} \binom{n+1}{i-1} \int_0^{p_1} p^{i-2} (1-p)^{n-i+2} dp$$

which can be written as

$$Q = -\frac{1}{n+1}\sum_{j=i-1}^{i}\binom{n+1}{j}p_1^{\,j}(1-p_1)^{n+1-j} + \frac{i-1}{n+1}\binom{n+1}{i-1}\int_0^{p_1} p^{i-2}(1-p)^{n-i+2}dp$$

Again integrate by parts: $u = p^{i-2}$ $\quad dv = (1-p)^{n-i+2}dp$

$$du = (i-2)\,p^{i-3}dp \qquad v = -\frac{(1-p)^{n-i+3}}{(n-i+3)}$$

$$Q = -\frac{1}{n+1}\sum_{j=i-1}^{i}\binom{n+1}{j}p_1^{\,j}(1-p_1)^{n+1-j} - \frac{i-1}{n+1}\binom{n+1}{i-1}\left[\frac{1}{n+3-i}p^{i-2}(1-p)^{n-i+3}\right]_0^{p_1} +$$

$$\left(\frac{i-1}{n+1}\right)\binom{n+1}{i-1}\int_0^{p_1}\frac{i-2}{n+3-i}\,p^{i-3}(1-p)^{n+3-i}dp$$

which equals, after some simplification

$$Q = -\frac{1}{n+1}\sum_{j=i-1}^{i}\binom{n+1}{j}p_1^{\,j}(1-p_1)^{n+1-j} - \frac{1}{n+1}\binom{n+1}{i-2}p_1^{\,i-2}(1-p_1)^{n+3-i} +$$

$$\frac{i-2}{n+1}\binom{n+1}{i-2}\int_0^{p_1}p^{i-3}(1-p)^{n+3-i}dp$$

and, combining terms

$$Q = -\frac{1}{n+1}\sum_{j=i-2}^{i}\binom{n+1}{j}p_1^{\,j}(1-p_1)^{n+1-j} + \frac{i-2}{n+1}\binom{n+1}{i-2}\int_0^{p_1}p^{i-3}(1-p)^{n+3-i}dp$$

Continuing to evaluate the integral by integrating by parts, we come to this term

$$Q = -\frac{1}{n+1}\sum_{j=1}^{i}\binom{n+1}{j}p_1^{\,j}(1-p_1)^{n+1-j} + \frac{1}{n+1}\binom{n+1}{1}\int_0^{p_1}p^0(1-p)^n dp$$

$$= -\frac{1}{n+1}\sum_{j=1}^{i}\binom{n+1}{j}p_1^{\,j}(1-p_1)^{n+1-j} + \frac{1}{n+1}\binom{n+1}{1} - \frac{(1-p)^{n+1}}{n+1}\Bigg|_0^{p_1}$$

$$= -\frac{1}{n+1}\sum_{j=1}^{i}\binom{n+1}{j}p_1^{\,j}(1-p_1)^{n+1-j} - \frac{1}{n+1}\binom{n+1}{0}(1-p_1)^{n+1}p_1^{\,0} + \frac{1}{n+1}\binom{n+1}{0}$$

$$= -\frac{1}{n+1}\sum_{j=0}^{i}\binom{n+1}{j} p_1^{\,j}\,(1-p_1)^{n+1-j} + \frac{1}{n+1}$$

$$= -\frac{1}{n+1}\left[1 - \sum_{j=0}^{i}\binom{n+1}{j} p_1^{\,j}\,(1-p_1)^{n+1-j}\right]$$

$$= \frac{1}{n+1}\left[\sum_{j=i+1}^{n+1}\binom{n+1}{j} p_1^{\,j}\,(1-p_1)^{n+1-j}\right]$$

Thus, finally we have

$$\int_0^{p_1}\binom{n}{i} p^i\,(1-p)^{n-i}dp = \frac{1}{n+1}\sum_{j=i+1}^{n+1}\binom{n+1}{j} p_1^{\,j}\,(1-p_1)^{n+1-j}$$

and we have conveniently gone from the integral of a binomial to the sum of another binomial expansion.

Now, making use of this expression in the original "s" equation, we have

$$\int_0^{p_1}\sum_{i=0}^{c}\binom{n}{i} p^i\,(1-p)^{n-i}dp = \frac{1}{n+1}\sum_{i=0}^{c}\sum_{j=i+1}^{n+1}\binom{n+1}{j} p_1^{\,j}\,(1-p_1)^{n+1-j}$$

$$\int_0^{1}\binom{n}{i} p^i\,(1-p)^{n-i}dp = \frac{1}{n+1}\sum_{j=i+1}^{n+1}\binom{n+1}{j}(1)^j\,(0)^{n+1-j}$$

All terms in the summation vanish (= 0) except for the last term when $j = n+1$, where we have $\frac{1}{n+1}(1)^{n+1}(0)^0 = \frac{1}{n+1}$

and $$\int_0^{1}\sum_{i=0}^{c}\binom{n}{i} p^i\,(1-p)^{n-1}dp = \frac{c+1}{n+1}$$

$$\int_0^{p_2}\sum_{i=a}^{b}\binom{n}{i} p^i\,(1-p)^{n-i}dp = \frac{1}{n+1}\sum_{i=a}^{b}\sum_{j=i+1}^{n+1}\binom{n+1}{j} p_2^{\,j}\,(1-p_2)^{n+1-j}$$

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And using these expressions in our original expression (8), we have

$$s = \frac{1}{n+1} \sum_{i=0}^{c_1} \sum_{j=i+1}^{n+1} \binom{n+1}{j} p_1^j (1-p_1)^{n+1-j} + \frac{1}{n+1} \sum_{i=c_1+1}^{c_2} \sum_{j=i+1}^{n+1} \binom{n+1}{j} p_2^j$$

$$(1-p_2)^{n+1-j} - \frac{1}{n+1} \sum_{i=c_1+1}^{c_2} \sum_{j=i+1}^{n+1} \binom{n+1}{j} p_1^j (1-p_1)^{n+1-j} + \frac{1}{n+1} \sum_{i=c_2+1}^{c_3}$$

$$\sum_{j=i+1}^{n+1} \binom{n+1}{j} p_3^j (1-p_3)^{n+1-j} - \frac{1}{n+1} \sum_{i=c_2+1}^{c_3} \sum_{j=i+1}^{n+1} \binom{n+1}{j} p_2^j (1-p_2)^{n+1-j} + \ldots +$$

$$\frac{n-c_{k-1}}{n+1} - \frac{1}{n+1} \sum_{i=c_{k-1}+1}^{n} \sum_{j=i+1}^{n+1} \binom{n+1}{j} p_{k-1}^j (1-p_{k-1})^{n+1-j}$$

$$s = \frac{1}{n+1} \left[\sum_{i=0}^{c_1} \sum_{j=i+1}^{n+1} \binom{n+1}{j} p_1^j (1-p_1)^{n+1-j} + \sum_{i=c_1+1}^{c_2} \sum_{j=i+1}^{n+1} \binom{n+1}{j} p_2^j \right.$$

$$(1-p_2)^{n+1-j} - \sum_{i=c_1+1}^{c_2} \sum_{j=i+1}^{n+1} \binom{n+1}{j} p_1^j (1-p_1)^{n+1-j} + \sum_{i=c_2+1}^{c_3} \sum_{j=i+1}^{n+1}$$

$$\binom{n+1}{j} p_3^j (1-p_3)^{n+1-j} - \sum_{i=c_2+1}^{c_3} \sum_{j=i+1}^{n+1} \binom{n+1}{j} p_2^j (1-p_2)^{n+1-j} + \ldots +$$

$$\left. n-c_{k-1} - \sum_{i=c_{k-1}+1}^{n} \sum_{j=i+1}^{n+1} \binom{n+1}{j} p_{k-1}^j (1-p_{k-1})^{n+1-j} \right]$$

$$s = \frac{1}{n+1}\Big[\sum_{i=1}^{c_1+1}\sum_{j=1}^{n+1}\binom{n+1}{j}p_1^{\,j}(1-p_1)^{n+1-j} + \sum_{i=c_1+2}^{c_2+1}\sum_{j=i}^{n+1}\binom{n+1}{j}p_2^{\,j}$$

$$(1-p_2)^{n+1-j} - \sum_{i=c_1+2}^{c_2+1}\sum_{j=i}^{n+1}\binom{n+1}{j}p_1^{\,j}(1-p_1)^{n+1-j} + \sum_{i=c_2+2}^{c_3+1}\sum_{j=i}^{n+1}\binom{n+1}{j}p_3^{\,j}$$

$$(1-p_3)^{n+1-j} - \sum_{i=c_2+2}^{c_3+1}\sum_{j=i}^{n+1}\binom{n+1}{j}p_2^{\,j}(1-p_2)^{n+1-j} + \ldots + n-c_{k-1}$$

$$- \sum_{i=c_{k-1}+2}^{n+1}\sum_{j=i}^{n+1}\binom{n+1}{j}p_{k-1}^{\,j}(1-p_{k-1})^{n+1-j}\Big]$$

which can be written as

$$s = 1/n+1\Big[\sum_{i=1}^{c_1+1}p(x \geq i;\ n+1,\ p_1) + \sum_{i=c_1+2}^{c_2+1}p(x \geq i;\ n+1,\ p_2) - \sum_{i=c_1+2}^{c_2+1}$$

$$p(x \geq i;\ n+1,\ p_1) + \sum_{i=c_2+2}^{c_3+1}p(x \geq i;\ n+1,\ p_3) - \sum_{i=c_2+2}^{c_3+1}p(x \geq i;\ n+1,\ p_2) +$$

$$\ldots + n-c_{k-1} - \sum_{i=c_{k-1}+2}^{n+1}p(x \geq i;\ n+1,\ p_{k-1})\Big]$$

where $p(x \geq i;\ n+1,\ p_t)$ is defined to be the probability that the random variable x is greater than or equal i if it has the binomial distribution with parameters $n+1$ and p_t.

We now make use of any convenient table of cumulative binomial probabilities for several different values of n. Thus, for a fixed sample size and given quality levels, $p_1, p_2, p_3, \ldots p_{k-1}$ we can, by use of a high-speed electronic computer, compute values of s for every $c_1, c_2, \ldots c_{k-1}$ combination and choose that combination which gives the maximum value of s.

GENERAL APPLICATIONS AND EXAMPLES.

For the Case K = 2

The classification of a lot into two grades is, for most situations, equivalent to either accepting or rejecting the lot. For example, a quality control analyst might be willing to accept as satisfactory a 10% defect rate for flash bulbs. Thus, if he took a sample of 20 bulbs from one hour's production, the appropriate table indicates he would allow one defective sample before rejecting that hour's output.

For the Case K = 3

Here, the purpose might be to place a given lot of artillery fuzes which have been in storage for some time into one of three grades, Grade A, indicating those lots acceptable for unrestricted use; Grade B, those lots generally acceptable with certain restrictions; and Grade C, those lots unacceptable for future use.

Given a sample size of 45 and prescribed quality levels of 15% and 30%, the appropriate table indicates we would allow 6 defects for a Grade A lot and up to 13 defects for a Grade B lot.

For the Case K = 4

An example here might be the case where an electronics dealer would be willing to pay x dollars for a lot of batteries which are of Grade A quality, y dollars ($y < x$) for a lot of Grade B quality, z dollars ($z < y < x$) for a lot of Grade C quality and reject as unacceptable, lots of Grade D quality.

If for a sample of size 200, the respective quality levels are 1% (Grade A), 10% (Grade B), and 25% (Grade C), the appropriate table calls for acceptance numbers 2, 20, and 50.

FURTHER RECOMMENDATIONS. The use of the uniform prior distribution is a fairly conservative approach but would seem to have realistic applications for newly manufactured items or items for which little is known of the functioning characteristics.

It would be interesting to consider some other prior distributions. A simple one, which seems both reasonable and easy to handle mathematically would be to assume p is uniformly distributed on the interval [0, 0.50], i.e., assume that no lot is more than 50% defective and guard against misgrading any lot with fraction defective between 0 and 50% with equal protection.

Another interesting distribution to consider would be

$$f(p) = 2(1-p) \qquad 0 \leq p \leq 1$$

$$= 0 \qquad \text{otherwise}$$

This distribution assumes lots with p almost zero are most likely in the stockpile, lots with p almost equal one are quite rare and the probability that $a \leq p \leq b$ increases linearly as a and b increase.

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APPENDIX

TABLES OF ACCEPTANCE NUMBERS

	SAMPLE SIZE								
P	10	15	20	25	30	35	40	45	50
.01	0	0	0	0	0	0	0	0	0
.02	0	0	0	0	0	0	0	0	0
.03	0	0	0	0	0	0	0	0	0
.04	0	0	0	0	0	0	0	1	1
.05	0	0	0	0	0	1	1	1	1
.06	0	0	0	0	1	1	1	2	2
.07	0	0	0	1	1	1	2	2	2
.08	0	0	1	1	1	2	2	3	3
.09	0	0	1	1	2	2	3	3	3
.10	0	0	1	1	2	2	3	3	4
.11	0	1	1	2	2	3	3	4	4
.12	0	1	1	2	3	3	4	4	5
.13	0	1	2	2	3	4	4	5	6
.14	0	1	2	3	3	4	5	5	6
.15	1	1	2	3	4	4	5	6	7
.16	1	1	2	3	4	5	5	6	7
.17	1	2	2	3	4	5	6	7	8
.18	1	2	3	4	4	5	6	7	8
.19	1	2	3	4	5	6	7	8	9
.20	1	2	3	4	5	6	7	8	9
.21	1	2	3	4	5	6	8	9	10
.22	1	2	4	5	6	7	8	9	10
.23	1	3	4	5	6	7	8	9	11
.24	2	3	4	5	6	8	9	10	11
.25	2	3	4	5	7	8	9	10	12
.26	2	3	4	6	7	8	10	11	12
.27	2	3	5	6	7	9	10	11	13
.28	2	3	5	6	8	9	10	12	13
.29	2	4	5	6	8	9	11	12	14
.30	2	4	5	7	8	10	11	13	14
.31	2	4	5	7	9	10	12	13	15
.32	2	4	6	7	9	10	12	14	15
.33	3	4	6	8	9	11	12	14	16
.34	3	4	6	8	9	11	13	15	16
.35	3	5	6	8	10	12	13	15	17
.36	3	5	7	8	10	12	14	16	17
.37	3	5	7	9	10	12	14	16	18
.38	3	5	7	9	11	13	14	16	18
.39	3	5	7	9	11	13	15	17	19
.40	3	5	7	9	11	13	15	17	19
.41	3	6	8	10	12	14	16	18	20
.42	4	6	8	10	12	14	16	18	20
.43	4	6	8	10	12	14	17	19	21
.44	4	6	8	10	13	15	17	19	21
.45	4	6	8	11	13	15	17	20	22
.46	4	6	9	11	13	16	18	20	22
.47	4	7	9	11	14	16	18	21	23
.48	4	7	9	11	14	16	19	21	23
.49	4	7	9	12	14	17	19	22	24
.50	4	7	9	12	14	17	19	22	24

THREE GRADES N=10

P2 \ P1	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15
.02	0,1														
.03	0,1	0,1													
.04	0,1	0,1	0,1												
.05	0,1	0,1	0,1	0,1											
.06	0,1	0,1	0,1	0,1	0,1										
.07	0,1	0,1	0,1	0,1	0,1	0,1									
.08	0,1	0,1	0,1	0,1	0,1	0,1	0,1								
.09	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1							
.10	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1						
.11	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1					
.12	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1				
.13	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1			
.14	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1		
.15	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	
.16	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1
.17	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1
.18	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1
.19	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1
.20	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1
.21	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1
.22	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1
.23	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1
.24	0,2	0,2	0,2	0,2	0,2	0,2	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	1,2
.25	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,1	0,1	0,1	1,2	1,2
.26	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2
.27	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2
.28	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2
.29	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2
.30	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2
.31	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2
.32	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2
.33	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,2	1,2	1,2
.34	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3
.35	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3
.36	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3
.37	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3
.38	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3
.39	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3
.40	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3
.41	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3
.42	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,3	0,4	1,4
.43	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,3	0,4	1,4
.44	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,3	0,4	1,4
.45	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,3	0,4	1,4
.46	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,3	0,4	1,4
.47	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,3	0,4	1,4
.48	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,3	0,4	1,4
.49	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,3	0,4	1,4
.50	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,3	0,4	1,4

THREE GRADES N=10

P2 \ P1	.16	.17	.18	.19	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30
.02															
.03															
.04															
.05															
.06															
.07															
.08															
.09															
.10															
.11															
.12															
.13															
.14															
.15															
.16															
.17	0,1														
.18	0,1	0,1													
.19	0,1	0,1	0,1												
.20	0,1	0,1	1,2	1,2											
.21	0,1	1,2	1,2	1,2	1,2										
.22	1,2	1,2	1,2	1,2	1,2	1,2									
.23	1,2	1,2	1,2	1,2	1,2	1,2	1,2								
.24	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2							
.25	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2						
.26	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2					
.27	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2				
.28	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2			
.29	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	2,3	2,3		
.30	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	2,3	2,3	2,3	2,3	
.31	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	2,3	2,3	2,3	2,3	2,3	2,3
.32	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.33	1,2	1,2	1,2	1,2	1,2	1,2	1,2	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.34	1,3	1,2	1,2	1,2	1,2	1,2	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.35	1,3	1,3	1,3	1,3	1,2	1,2	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.36	1,3	1,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.37	1,3	1,3	1,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3	2,3	23,	2,3	2,3
.38	1,3	1,3	1,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	,3	2,3
.39	1,3	1,3	1,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.40	1,3	1,3	1,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.41	1,3	1,3	1,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.42	1,4	1,4	1,4	1,4	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.43	1,4	1,4	1,4	1,4	1,4	1,4	1,4	2,4	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.44	1,4	1,4	1,4	1,4	1,4	1,4	1,4	2,4	2,4	2,4	2,4	2,4	2,4	2,3	3,4
.45	1,4	1,4	1,4	1,4	1,4	1,4	1,4	2,4	2,4	2,4	2,4	2,4	2,4	2,4	3,4
.46	1,4	1,4	1,4	1,4	1,4	1,4	1,4	2,4	2,4	2,4	2,4	2,4	2,4	2,4	3,4
.47	1,4	1,4	1,4	1,4	1,4	1,4	1,4	2,4	2,4	2,4	2,4	2,4	2,4	2,4	3,4
.48	1,4	1,4	1,4	1,4	1,4	1,4	1,4	2,4	2,4	2,4	2,4	2,4	2,4	2,4	3,4
.49	1,4	1,4	1,4	1,4	1,4	1,4	1,4	2,4	2,4	2,4	2,4	2,4	2,4	2,4	3,4
.50	1,4	1,4	1,4	1,4	1,4	1,4	1,4	2,4	2,4	2,4	2,4	2,4	2,4	2,4	3,4

THREE GRADES N=15

P2 \ P1	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15
.02	0,1														
.03	0,1	0,1													
.04	0,1	0,1	0,1												
.05	0,1	0,1	0,1	0,1											
.06	0,1	0,1	0,1	0,1	0,1										
.07	0,1	0,1	0,1	0,1	0,1	0,1									
.08	0,1	0,1	0,1	0,1	0,1	0,1	0,1								
.09	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1							
.10	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1						
.11	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1					
.12	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1				
.13	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1			
.14	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	1,2		
.15	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	1,2	1,2	1,2	
.16	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	1,2	1,2	1,2	1,2	1,2
.17	0,2	0,2	0,2	0,2	0,2	0,1	0,1	0,1	0,1	1,2	1,2	1,2	1,2	1,2	1,2
.18	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2
.19	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2
.20	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	1,2
.21	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	1,2
.22	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	1,2
.23	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,2	1,2	1,2	1,2	1,2	1,2	1,2
.24	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3	1,3	1,2	1,2	2,3
.25	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3	1,3	1,3	1,3	2,3
.26	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3	1,3	1,3	1,3	2,3
.27	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3	1,3	1,3	1,3	2,3
.28	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3	1,3	1,3	1,3	2,3
.29	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,3	1,3	1,3	1,3
.30	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	1,4	1,4
.31	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	1,4	1,4
.32	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	1,4	1,4
.33	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	1,4	1,4
.34	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	1,4	1,4
.35	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,4
.36	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,5
.37	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,5
.38	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,5
.39	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,5
.40	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,5
.41	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	1,6
.42	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	1,6
.43	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	1,6
.44	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	1,6
.45	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	1,6
.46	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	1,6
.47	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	1,7
.48	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	1,7
.49	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	1,7
.50	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	1,7

THREE GRADES N=15

P2 \ P1	.16	.17	.18	.19	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30
.02															
.03															
.04															
.05															
.06															
.07															
.08															
.09															
.10															
.11															
.12															
.13															
.14															
.15															
.16															
.17	1,2														
.18	1,2	1,2													
.19	1,2	1,2	1,2												
.20	1,2	1,2	1,2	2,3											
.21	1,2	1,2	2,3	2,3	2,3										
.22	1,2	2,3	2,3	2,3	2,3	2,3									
.23	2,3	2,3	2,3	2,3	2,3	2,3	2,3								
.24	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3							
.25	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3						
.26	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	3,4					
.27	2,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3	3,4	3,4	3,4				
.28	2,3	2,3	2,3	2,3	2,3	2,3	2,3	3,4	3,4	3,4	3,4	3,4			
.29	2,3	2,3	2,3	2,3	2,3	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4		
.30	2,4	2,3	2,3	2,3	2,3	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	
.31	2,4	2,4	2,4	2,3	2,3	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4
.32	2,4	2,4	2,4	2,4	2,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4
.33	2,4	2,4	2,4	2,4	2,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4
.34	2,4	2,4	2,4	2,4	2,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	4,5
.35	2,4	2,4	2,4	2,4	2,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	4,5	4,5
.36	2,5	2,5	2,5	2,5	2,5	3,4	3,4	3,4	3,4	3,4	3,4	3,4	4,5	4,5	4,5
.37	1,5	2,5	2,5	2,5	2,5	3,5	3,5	3,5	3,4	3,4	3,4	4,5	4,5	4,5	4,5
.38	1,5	2,5	2,5	2,5	2,5	3,5	3,5	3,5	3,5	3,5	4,5	4,5	4,5	4,5	4,5
.39	1,5	2,5	2,5	2,5	2,5	3,5	3,5	3,5	3,5	3,5	3,5	4,5	4,5	4,5	4,5
.40	1,5	2,5	2,5	2,5	2,5	3,5	3,5	3,5	3,5	3,5	3,5	4,5	4,5	4,5	4,5
.41	1,6	2,6	2,5	2,5	2,5	3,5	3,5	3,5	3,5	3,5	3,5	4,5	4,5	4,5	4,5
.42	1,6	2,6	2,6	2,6	2,6	3,6	3,6	3,6	3,6	3,5	3,5	4,5	4,5	4,5	4,5
.43	1,6	2,6	2,6	2,6	2,6	3,6	3,6	3,6	3,6	3,6	3,6	3,6	4,6	4,5	4,5
.44	1,6	2,6	2,6	2,6	2,6	2,6	2,6	3,6	3,6	3,6	3,6	3,6	4,6	4,6	4,6
.45	1,6	2,6	2,6	2,6	2,6	2,6	2,6	3,6	3,6	3,6	3,6	3,6	4,6	4,6	4,6
.46	1,6	2,6	2,6	2,6	2,6	2,6	2,6	3,6	3,6	3,6	3,6	3,6	4,6	4,6	4,6
.47	1,7	2,7	2,7	2,6	2,6	2,6	2,6	3,6	3,6	3,6	3,6	3,6	4,6	4,6	4,6
.48	1,7	2,7	2,7	2,7	2,7	2,7	2,7	3,7	3,7	3,7	3,7	3,7	4,7	4,7	4,6
.49	1,7	2,7	2,7	2,7	2,7	2,7	2,7	3,7	3,7	3,7	3,7	3,7	4,7	4,7	4,7
.50	1,7	2,7	2,7	2,7	2,7	2,7	2,7	3,7	3,7	3,7	3,7	3,7	3,7	4,7	4,7

<<

THREE GRADES N=20

P2 \ P1	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15
.02	0,1														
.03	0,1	0,1													
.04	0,1	0,1	0,1												
.05	0,1	0,1	0,1	0,1											
.06	0,1	0,1	0,1	0,1	0,1										
.07	0,1	0,1	0,1	0,1	0,1	0,1									
.08	0,1	0,1	0,1	0,1	0,1	0,1	0,1								
.09	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1							
.10	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1						
.11	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	1,2					
.12	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	1,2	1,2	1,2				
.13	0,2	0,2	0,2	0,2	0,1	0,1	0,1	1,2	1,2	1,2	1,2	1,2			
.14	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2		
.15	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	
.16	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	2,3	2,3
.17	0,2	0,2	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	2,3	2,3	2,3
.18	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,2	1,2	1,2	1,2	2,3	2,3	2,3	2,3
.19	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3
.20	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3
.21	0,3	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3
.22	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3
.23	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	1,4	2,3	2,3	2,3	2,3
.24	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	1,4	2,4	2,4	2,4	2,3
.25	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	1,4	2,4	2,4	2,4	2,4
.26	0,4	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	1,4	2,4	2,4	2,4	2,4
.27	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,5	2,4	2,4	2,4
.28	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,5	2,5	2,5	2,5
.29	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,5	2,5	2,5	2,5
.30	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,5	2,5	2,5	2,5
.31	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	1,5	1,5	2,5	2,5	2,5
.32	0,6	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	1,6	2,6	2,6	2,6
.33	0,6	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	1,6	2,6	2,6	2,6
.34	0,6	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	1,6	2,6	2,6	2,6
.35	0,6	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	1,6	2,6	2,6	2,6
.36	0,7	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	1,7	2,7	2,6	2,6
.37	0,7	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	1,7	2,7	2,7	2,7
.38	0,7	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	1,7	2,7	2,7	2,7
.39	0,7	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	1,7	2,7	2,7	2,7
.40	0,7	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	1,7	2,7	2,7	2,7
.41	0,8	0,8	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	1,8	1,8	2,8	2,8	2,8
.42	0,8	0,8	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	1,8	1,8	2,8	2,8	2,8
.43	0,8	0,8	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	1,8	1,8	2,8	2,8	2,8
.44	0,8	0,8	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	1,8	1,8	2,8	2,8	2,8
.45	0,8	0,8	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	1,8	1,8	2,8	2,8	2,8
.46	0,9	0,9	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	1,9	1,9	2,9	2,9	2,9
.47	0,9	0,9	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	1,9	1,9	2,9	2,9	2,9
.48	0,9	0,9	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	1,9	1,9	2,9	2,9	2,9
.49	0,9	0,9	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	1,9	1,9	2,9	2,9	2,9
.50	0,9	0,9	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	1,9	1,9	2,9	2,9	2,9

THREE GRADES N=20

P2 \ P1	.16	.17	.18	.19	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30
.02															
.03															
.04															
.05															
.06															
.07															
.08															
.09															
.10															
.11															
.12															
.13															
.14															
.15															
.16															
.17	2,3														
.18	2,3	2,3													
.19	2,3	2,3	2,3												
.20	2,3	2,3	2,3	3,4											
.21	2,3	2,3	3,4	3,4	3,4										
.22	2,3	2,3	3,4	3,4	3,4	3,4									
.23	2,3	3,4	3,4	3,4	3,4	3,4	3,4								
.24	2,3	3,4	3,4	3,4	3,4	3,4	3,4	3,4							
.25	2,4	3,4	3,4	3,4	3,4	3,4	3,4	3,4	4,5						
.26	2,4	3,4	3,4	3,4	3,4	3,4	3,4	4,5	4,5	4,5					
.27	2,4	3,4	3,4	3,4	3,4	3,4	4,5	4,5	4,5	4,5	4,5				
.28	2,5	3,4	3,4	3,4	3,4	4,5	4,5	4,5	4,5	4,5	4,5	4,5			
.29	2,5	3,5	3,5	3,5	4,5	4,5	4,5	4,5	4,5	4,5	4,5	4,5	4,5		
.30	2,5	3,5	3,5	3,5	4,5	4,5	4,5	4,5	4,5	4,5	4,5	4,5	5,6	5,6	
.31	2,5	3,5	3,5	3,5	4,5	4,5	4,5	4,5	4,5	4,5	4,5	5,6	5,6	5,6	5,6
.32	2,6	3,6	3,5	3,5	3,5	4,5	4,5	4,5	4,5	4,5	5,6	5,6	5,6	5,6	5,6
.33	2,6	3,6	3,6	3,6	3,6	4,6	4,5	4,5	4,5	5,6	5,6	5,6	5,6	5,6	5,6
.34	2,6	3,6	3,6	3,6	3,6	4,6	4,6	4,6	5,6	5,6	5,6	5,6	5,6	5,6	5,6
.35	2,6	3,6	3,6	3,6	3,6	4,6	4,6	4,6	4,6	5,6	5,6	5,6	5,6	5,6	5,6
.36	2,6	3,6	3,6	3,6	3,6	3,6	4,6	4,6	4,6	5,6	5,6	5,6	5,6	5,6	5,6
.37	2,7	3,7	3,7	3,7	3,7	3,7	4,7	4,6	4,6	5,6	5,6	5,6	5,6	5,6	5,6
.38	2,7	2,7	3,7	3,7	3,7	3,7	4,7	4,7	4,7	4,7	5,7	5,6	5,6	5,6	6,7
.39	2,7	2,7	3,7	3,7	3,7	3,7	4,7	4,7	4,7	4,7	5,7	5,7	5,7	6,7	6,7
.40	2,7	2,7	3,7	3,7	3,7	3,7	4,7	4,7	4,7	4,7	5,7	5,7	5,7	6,7	6,7
.41	2,8	2,8	3,8	3,8	3,8	3,8	4,8	4,8	4,7	4,7	5,7	5,7	5,7	5,7	6,7
.42	2,8	2,8	3,8	3,8	3,8	3,8	4,8	4,8	4,8	4,8	5,8	5,8	5,7	5,7	6,7
.43	2,8	2,8	3,8	3,8	3,8	3,8	4,8	4,8	4,8	4,8	5,8	5,8	5,8	5,8	6,8
.44	2,8	2,8	3,8	3,8	3,8	3,8	4,8	4,8	4,8	4,8	4,8	5,8	5,8	5,8	6,8
.45	2,8	2,8	3,8	3,8	3,8	3,8	4,8	4,8	4,8	4,8	4,8	5,8	5,8	5,8	5,8
.46	2,9	2,9	3,9	3,9	3,9	3,9	4,9	4,9	4,9	4,9	4,9	5,9	5,9	5,8	5,8
.47	2,9	2,9	3,9	3,9	3,9	3,9	4,9	4,9	4,9	4,9	4,9	5,9	5,9	5,9	5,9
.48	2,9	2,9	3,9	3,9	3,9	3,9	4,9	4,9	4,9	4,9	4,9	5,9	5,9	5,9	5,9
.49	2,9	2,9	3,9	3,9	3,9	3,9	4,9	4,9	4,9	4,9	4,9	5,9	5,9	5,9	5,9
.50	2,9	2,9	3,9	3,9	3,9	3,9	4,9	4,9	4,9	4,9	4,9	5,9	5,9	5,9	5,9

<<

THREE GRADES N=25

P2 \ P1	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15
.02	0,1														
.03	0,1	0,1													
.04	0,1	0,1	0,1												
.05	0,1	0,1	0,1	0,1											
.06	0,1	0,1	0,1	0,1	0,1										
.07	0,1	0,1	0,1	0,1	0,1	0,1									
.08	0,1	0,1	0,1	0,1	0,1	0,1	0,1								
.09	0,1	0,1	0,1	0,1	0,1	0,1	0,1	1,2							
.10	0,1	0,1	0,1	0,1	0,1	0,1	1,2	1,2	1,2						
.11	0,2	0,2	0,2	0,2	0,1	0,1	1,2	1,2	1,2	1,2					
.12	0,2	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	1,2				
.13	0,2	0,2	0,2	0,2	0,2	1,2	1,2	12,	1,2	1,2	2,3	2,3			
.14	0,3	0,3	0,2	0,2	0,2	1,2	1,2	1,2	1,2	2,3	2,3	2,3	2,3		
.15	0,3	0,3	0,3	0,3	0,3	1,3	1,3	1,2	1,2	2,3	2,3	2,3	2,3	2,3	
.16	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3	2,3	2,3
.17	0,3	0,3	0,3	0,3	0,3	0,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3	3,4
.18	0,4	0,4	0,4	0,4	0,4	0,4	1,3	1,3	1,3	2,3	2,3	2,3	2,3	3,4	3,4
.19	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	2,4	2,3	2,3	3,4	3,4	3,4
.20	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	2,4	2,4	2,4	3,4	3,4	3,4
.21	0,4	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	2,4	2,4	2,4	3,4	3,4	3,4
.22	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	2,5	2,4	2,4	3,4	3,4	3,4
.23	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	2,5	2,5	2,5	3,5	3,4	3,4
.24	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	2,5	2,5	2,5	2,5	3,5	3,5
.25	0,5	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	2,5	2,5	2,5	2,5	3,5	3,5
.26	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	2,6	2,6	2,6	3,6	3,5
.27	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	2,6	2,6	2,6	3,6	3,6
.28	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	2,6	2,6	2,6	3,6	3,6
.29	0,6	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	1,6	2,6	2,6	2,6	3,6	3,6
.30	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	2,7	2,7	2,7	3,7	3,7
.31	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	2,7	2,7	2,7	3,7	3,7
.32	0,7	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	1,7	2,7	2,7	2,7	3,7	3,7
.33	0,8	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	1,8	2,8	2,8	2,8	3,8	3,7
.34	0,8	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	1,8	2,8	2,8	2,8	3,8	3,8
.35	0,8	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	1,8	2,8	2,8	2,8	3,8	3,8
.36	0,8	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	1,8	2,8	2,8	2,8	3,8	3,8
.37	0,9	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9
.38	0,9	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9
.39	0,9	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9
.40	0,9	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9
.41	0,10	0,10	0,10	0,10	0,10	0,10	1,10	1,10	1,10	1,10	2,10	2,10	2,10	3,10	3,10
.42	0,10	0,10	0,10	0,10	0,10	0,10	1,10	1,10	1,10	1,10	2,10	2,10	2,10	3,10	3,10
.43	0,10	0,10	0,10	0,10	0,10	0,10	1,10	1,10	1,10	1,10	2,10	2,10	2,10	3,10	3,10
.44	0,10	0,10	0,10	0,10	0,10	0,10	1,10	1,10	1,10	1,10	2,10	2,10	2,10	3,10	3,10
.45	0,11	0,11	0,11	0,11	0,11	0,11	1,11	1,11	1,11	1,11	2,11	2,11	2,11	3,11	3,11
.46	0,11	0,11	0,11	0,11	0,11	0,11	1,11	1,11	1,11	1,11	2,11	2,11	2,11	3,11	3,11
.47	0,11	0,11	0,11	0,11	0,11	0,11	1,11	1,11	1,11	1,11	2,11	2,11	2,11	3,11	3,11
.48	0,11	0,11	0,11	0,11	0,11	0,11	1,11	1,11	1,11	1,11	2,11	2,11	2,11	3,11	3,11
.49	0,12	0,12	0,12	0,12	0,12	0,12	1,12	1,12	1,12	1,12	2,12	2,12	2,12	3.12	3,12
.50	0,12	0,12	0,12	0,12	0,12	0;12	1,12	1,12	1,12	1,12	2,12	2,12	2,12	3,12	3,12

THREE GRADES N=25

P2 \ P1	.16	.17	.18	.19	.20	.21	.22	.23	2.4	.25	2.6	.27	.28	.29	.30
.02															
.03															
.04															
.05															
.06															
.07															
.08															
.09															
.10															
.11															
.12															
.13															
.14															
.15															
.16															
.17	3,4														
.18	3,4	3,4													
.19	3,4	3,4	3,4												
.20	3,4	3,4	3,4	3,4											
.21	3,4	3,4	3,4	4,5	4,5										
.22	3,4	3,4	4,5	4,5	4,5	4,5									
.23	3,4	4,5	4,5	4,5	4,5	4,5	4,5								
.24	4,5	4,5	4,5	4,5	4,5	4,5	4,5	5,6							
.25	3,5	4,5	4,5	4,5	4,5	4,5	5,6	5,6	5,6						
.26	3,5	4,5	4,5	4,5	4,5	5,6	5,6	5,6	5,6	5,6					
.27	3,6	4,6	4,5	4,5	5,6	5,6	5,6	5,6	5,6	5,6	5,6				
.28	3,6	4,6	4,6	4,6	5,6	5,6	5,6	5,6	5,6	5,6	5,6	6,7			
.29	3,6	4,6	4,6	4,6	5,6	5,6	5,6	5,6	5,6	5,6	6,7	6,7	6,7		
.30	3,7	3,7	4,6	4,6	5,6	5,6	5,6	5,6	5,6	6,7	6,7	6,7	6,7	6,7	
.31	3,7	3,7	4,7	4,7	4,7	5,7	5,6	5,6	6,7	6,7	6,7	6,7	6,7	6,7	6,7
.32	3,7	3,7	4,7	4,7	4,7	5,7	5,7	6,7	6,7	6,7	6,7	6,7	6,7	6,7	7,8
.33	3,7	3,7	4,7	4,7	4,7	5,7	5,7	5,7	6,7	6,7	6,7	6,7	6,7	6,7	7,8
.34	3,8	3,8	4,8	4,8	4,8	5,8	5,7	5,7	6,7	6,7	6,7	6,7	6,7	7,8	7,8
.35	3,8	3,8	4,8	4,8	4,8	5,8	5,8	5,8	6,8	6,7	6,7	6,7	7,8	7,8	7,8
.36	3,8	3,8	4,8	4,8	4,8	4,8	5,8	5,8	5,8	6,8	6,8	7,8	7,8	7,8	7,8
.37	3,9	3,9	4,9	4,9	4,9	4,8	5,8	5,8	5,8	6,8	6,8	7,8	7,8	7,8	7,8
.38	3,9	3,9	4,9	4,9	4,9	4,9	5,9	5,9	5,9	6,9	6,8	6,8	7,8	7,8	7,8
.39	3,9	3,9	4,9	4,9	4,9	4,9	5,9	5,9	5,9	6,9	6,9	6,9	7,9	7,8	7,8
.40	3,9	3,9	4,9	4,9	4,9	4,9	5,9	5,9	5,9	6,9	6,9	6,9	7,9	7,9	7,9
.41	3,10	3,10	4,10	4,10	4,10	4,10	5,10	5,10	5,10	6,10	6,9	6,9	7,9	7,9	7,9
.42	3,10	3,10	4,10	4,10	4,10	4,10	5,10	5,10	5,10	6,10	6,10	6,10	6,10	7,10	7,9
.43	3,10	3,10	4,10	4,10	4,10	4,10	5,10	5,10	5,10	5,10	6,10	6,10	6,10	7,10	7,10
.44	3,10	3,10	4,10	4,10	4,10	4,10	5,10	5,10	5,10	5,10	6,10	6,10	6,10	7,10	7,10
.45	3,11	3,11	4,11	4,11	4,11	4,11	5,11	5,11	5,11	5,11	6,11	6,11	6,11	7,11	7,10
.46	3,11	3,11	4,11	4,11	4,11	4,11	5,11	5,11	5,11	5,11	6,11	6,11	6,11	7,11	7,11
.47	3,11	3,11	4,11	4,11	4,11	4,11	5,11	5,11	5,11	5,11	6,11	6,11	6,11	7,11	7,11
.48	3,11	3,11	4,11	4,11	4,11	4,11	5,11	5,11	5,11	5,11	6,11	6,11	6,11	7,11	7,11
.49	3,12	3,12	4,12	4,12	4,12	4,12	5,12	5,12	5,12	5,12	6,12	6,12	6,12	7,12	7,12
.50	3,12	3,12	4,12	4,12	4,12	4,12	5,12	5,12	5,12	5,12	6,12	6,12	6,12	7,12	7,12

<<

THREE GRADES N=30

P2 \ P1	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15
.02	0,1														
.03	0,1	0,1													
.04	0,1	0,1	0,1												
.05	0,1	0,1	0,1	0,1											
.06	0,1	0,1	0,1	0,1	0,1										
.07	0,1	0,1	0,1	0,1	0,1	0,1									
.08	0,1	0,1	0,1	0,1	0,1	1,2	1,2								
.09	0,2	0,2	0,2	0,1	1,2	1,2	1,2	1,2							
.10	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2						
.11	0,2	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2	2,3					
.12	0,3	0,3	0,3	0,3	1,2	1,2	1,2	1,2	2,3	2,3	2,3				
.13	0,3	0,3	0,3	0,3	1,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3			
.14	0,3	0,3	0,3	0,3	0,3	1,3	1,3	2,3	2,3	2,3	2,3	2,3	3,4		
.15	0,4	0,4	0,4	0,4	0,3	1,3	1,3	2,3	2,3	2,3	2,3	3,4	3,4	3,4	
.16	0,4	0,4	0,4	0,4	0,4	1,4	1,4	2,4	2,3	2,3	3,4	3,4	3,4	3,4	3,4
.17	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	2,4	2,4	3,4	3,4	3,4	3,4	3,4
.18	0,4	0,4	0,4	0,4	0,4	1,4	1,4	1,4	2,4	2,4	3,4	3,4	3,4	3,4	4,5
.19	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	2,5	2,5	3,4	3,4	3,4	3,4	4,5
.20	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	2,5	2,5	3,5	3,5	3,5	4,5	4,5
.21	0,5	0,5	0,5	0,5	0,5	1,5	1,5	1,5	2,5	2,5	2,5	3,5	3,5	4,5	4,5
.22	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	2,6	2,6	2,6	3,6	3,5	4,5	4,5
.23	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	2,6	2,6	2,6	3,6	3,6	4,6	4,6
.24	0,6	0,6	0,6	0,6	0,6	1,6	1,6	1,6	2,6	2,6	2,6	3,6	3,6	3,6	4,6
.25	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	2,7	2,7	2,7	3,7	3,7	3,6	4,6
.26	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	2,7	2,7	2,7	3,7	3,7	3,7	4,7
.27	0,7	0,7	0,7	0,7	0,7	1,7	1,7	1,7	2,7	2,7	2,7	3,7	3,7	3,7	4,7
.28	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	2,8	2,8	2,8	3,8	3,8	3,8	4,7
.29	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	2,8	2,8	2,8	3,8	3,8	3,8	4,8
.30	0,8	0,8	0,8	0,8	0,8	1,8	1,8	1,8	2,8	2,8	2,8	3,8	3,8	3,8	4,8
.31	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9	3,9	4,9
.32	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9	3,9	4,9
.33	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9	3,9	4,9
.34	0,9	0,9	0,9	0,9	0,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9	3,9	4,9
.35	0,10	0,10	0,10	0,10	0,10	1,10	1,10	1,10	2,10	2,10	2,10	3,10	3,10	3,10	4,10
.36	0,10	0,10	0,10	0,10	0,10	1,10	1,10	1,10	2,10	2,10	2,10	3,10	3,10	3,10	4,10
.37	0,10	0,10	0,10	0,10	0,10	1,10	1,10	1,10	2,10	2,10	2,10	3,10	3,10	3,10	4,10
.38	0,11	0,11	0,11	0,11	0,11	1,11	1,11	1,11	2,11	2,11	2,11	3,11	3,11	3,11	4,11
.39	0,11	0,11	0,11	0,11	0,11	1,11	1,11	1,11	2,11	2,11	2,11	3,11	3,11	3,11	4,11
.40	0,11	0,11	0,11	0,11	0,11	1,11	1,11	1,11	2,11	2,11	2,11	3,11	3,11	3,11	4,11
.41	0,12	0,12	0,12	0,12	0,12	1,12	1,12	1,12	2,12	2,12	2,12	3,12	3,12	3,12	4,12
.42	0,12	0,12	0,12	0,12	0,12	1,12	1,12	1,12	2,12	2,12	2,12	3,12	3,12	3,12	4,12
.43	0,12	0,12	0,12	0,12	0,12	1,12	1,12	1,12	2,12	2,12	2,12	3,12	3,12	3,12	4,12
.44	0,13	0,13	0,13	0,13	0,13	1,13	1,13	1,13	2,13	2,13	2,13	3,13	3,13	3,13	4,13
.45	0,13	0,13	0,13	0,13	0,13	1,13	1,13	1,13	2,13	2,13	2,13	3,13	3,13	3,13	4,13
.46	0,13	0,13	0,13	0,13	0,13	1,13	1,13	1,13	2,13	2,13	2,13	3,13	3,13	3,13	4,13
.47	0,14	0,14	0,14	0,14	0,14	1,14	1,14	1,14	2,14	2,14	2,14	3,14	3,14	3,14	4,14
.48	0,14	0,14	0,14	0,14	0,14	1,14	1,14	1,14	2,14	2,14	2,14	3,14	3,14	3,14	4,14
.49	0,14	0,14	0,14	0,14	0,14	1,14	1,14	1,14	2,14	2,14	2,14	3,14	3,14	3,14	4,14
.50	0,14	0,14	0,14	0,14	0,14	1,14	1,14	1,14	2,14	2,14	2,14	3,14	3,14	3,14	4,14

THREE GRADES N=30

P2 \ P1	.16	.17	.18	.19	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30
.02															
.03															
.04															
.05															
.06															
.07															
.08															
.09															
.10															
.11															
.12															
.13															
.14															
.15															
.16															
.17	4,5														
.18	4,5	4,5													
.19	4,5	4,5	4,5												
.20	4,5	4,5	4,5	4,5											
.21	4,5	4,5	4,5	5,6	5,6										
.22	4,5	4,5	5,6	5,6	5,6	5,6									
.23	4,5	5,6	5,6	5,6	5,6	5,6	5,6								
.24	4,6	5,6	5,6	5,6	5,6	5,6	6,7	6,7							
.25	4,6	5,6	5,6	5,6	5,6	6,7	6,7	6,7	6,7						
.26	4,7	5,7	5,6	5,6	6,7	6,7	6,7	6,7	6,7	6,7					
.27	4,7	4,7	5,7	5,7	6,7	6,7	6,7	6,7	6,7	6,7	7,8				
.28	4,7	4,7	5,7	5,7	6,7	6,7	6,7	6,7	6,7	7,8	7,8	7,8			
.29	4,8	4,8	5,8	5,8	6,7	6,7	6,7	6,7	7,8	7,8	7,8	7,8	7,8		
.30	4,8	4,8	5,8	5,8	5,8	6,8	6,7	7,8	7,8	7,8	7,8	7,8	7,8	8,9	
.31	4,8	4,8	5,8	5,8	5,8	6,8	6,8	7,8	7,8	7,8	7,8	7,8	8,9	8,9	8,9
.32	4,9	4,9	5,9	5,9	5,9	6,9	6,8	7,8	7,8	7,8	7,8	8,9	8,9	8,9	8,9
.33	4,9	4,9	5,9	5,9	5,9	6,9	6,9	6,9	7,9	7,8	8,9	8,9	8,9	8,9	8,9
.34	4,9	4,9	5,9	5,9	5,9	6,9	6,9	6,9	7,9	7,9	8,9	8,9	8,9	8,9	8,9
.35	4,10	4,10	5,10	5,10	5,10	6,10	6,10	6,10	7,9	7,9	8,9	8,9	8,9	8,9	9,10
.36	4,10	4,10	5,10	5,10	5,10	6,10	6,10	6,10	7,10	7,10	7,10	8,9	8,9	9,10	9,10
.37	4,10	4,10	4,10	5,10	5,10	5,10	6,10	6,10	7,10	7,10	7,10	8,10	8,10	9,10	9,10
.38	4,11	4,11	4,11	5,11	5,11	5,11	6,11	6,11	7,11	7,11	7,10	8,10	8,10	9,10	9,10
.39	4,11	4,11	4,11	5,11	5,11	5,11	6,11	6,11	7,11	7,11	7,11	8,11	8,11	8,11	9,10
.40	4,11	4,11	4,11	5,11	5,11	5,11	6,11	6,11	6,11	7,11	7,11	8,11	8,11	8,11	9,11
.41	4,12	4,12	4,12	5,12	5,12	5,12	6,12	6,12	6,12	7,12	7,12	7,12	8,11	8,11	9,11
.42	4,12	4,12	4,12	5,12	5,12	5,12	6,12	6,12	6,12	7,12	7,12	7,12	8,12	8,12	9,12
.43	4,12	4,12	4,12	5,12	5,12	5,12	6,12	6,12	6,12	7,12	7,12	7,12	8,12	8,12	8,12
.44	4,13	4,13	4,13	5,13	5,13	5,13	6,13	6,13	6,13	7,13	7,13	7,13	8,13	8,12	8,12
.45	4,13	4,13	4,13	5,13	5,13	5,13	6,13	6,13	6,13	7,13	7,13	7,13	8,13	8,13	8,13
.46	4,13	4,13	4,13	5,13	5,13	5,13	6,13	6,13	6,13	7,13	7,13	7,13	8,13	8,13	8,13
.47	4,14	4,14	4,14	5,14	5,14	5,14	6,14	6,14	6,14	7,14	7,14	7,14	8,14	8,13	8,13
.48	4,14	4,14	4,14	5,14	5,14	5,14	6,14	6,14	6,14	7,14	7,14	7,14	8,14	8,14	8,14
.49	4,14	4,14	4,14	5,14	5,14	5,14	6,14	6,14	6,14	7,14	7,14	7,14	8,14	8,14	8,14
.50	4,14	4,14	4,14	5,14	5,14	5,14	6,14	6,14	6,14	7,14	7,14	7,14	8,14	8,14	8,14

<<

THREE GRADES N=35

P2 \ P1	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15
.02	0,1														
.03	0,1	0,1													
.04	0,1	0,1	0,1												
.05	0,1	0,1	0,1	0,1											
.06	0,1	0,1	0,1	0,1	0,1										
.07	0,1	0,1	0,1	0,1	1,2	1,2									
.08	0,2	0,2	0,2	0,1	1,2	1,2	1,2								
.09	0,2	0,2	0,2	1,2	1,2	1,2	1,2	1,2							
.10	0,2	0,2	0,2	0,2	1,2	1,2	1,2	2,3	2,3						
.11	0,3	0,3	0,3	0,3	1,3	1,2	2,3	2,3	2,3	2,3					
.12	0,3	0,3	0,3	0,3	1,3	1,3	2,3	2,3	2,3	2,3	3,4				
.13	0,4	0,4	0,4	0,4	1,3	1,3	2,3	2,3	2,3	3,4	3,4	3,4			
.14	0,4	0,4	0,4	0,4	1,4	1,4	2,4	2,4	2,3	3,4	3,4	3,4	3,4		
.15	0,4	0,4	0,4	0,4	1,4	1,4	2,4	2,4	3,4	3,4	3,4	3,4	3,4	4,5	
.16	0,5	0,5	0,5	0,5	1,5	1,5	1,5	2,4	2,4	3,4	3,4	3,4	4,5	4,5	4,5
.17	0,5	0,5	0,5	0,5	1,5	1,5	1,5	2,5	2,5	3,5	3,4	4,5	4,5	4,5	4,5
.18	0,5	0,5	0,5	0,5	1,5	1,5	1,5	2,5	2,5	3,5	3,5	4,5	4,5	4,5	4,5
.19	0,6	0,6	0,6	0,6	1,6	1,6	1,6	2,6	2,6	3,6	3,5	4,5	4,5	4,5	5,6
.20	0,6	0,6	0,6	0,6	1,6	1,6	1,6	2,6	2,6	3,6	3,6	4,6	4,6	4,5	5,6
.21	0,6	0,6	0,6	0,6	1,6	1,6	1,6	2,6	2,6	3,6	3,6	3,6	4,6	4,6	5,6
.22	0,7	0,7	0,7	0,7	1,7	1,7	1,7	2,7	2,7	3,7	3,7	3,7	4,7	4,6	5,6
.23	0,7	0,7	0,7	0,7	1,7	1,7	1,7	2,7	2,7	3,7	3,7	3,7	4,7	4,7	5,7
.24	0,8	0,8	0,8	0,8	1,8	1,8	1,8	2,8	2,8	2,8	3,8	3,7	4,7	4,7	5,7
.25	0,8	0,8	0,8	0,8	1,8	1,8	1,8	2,8	2,8	2,8	3,8	3,8	4,8	4,8	5,8
.26	0,8	0,8	0,8	0,8	1,8	1,8	1,8	2,8	2,8	2,8	3,8	3,8	4,8	4,8	4,8
.27	0,9	0,9	0,9	0,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9	4,9	4,9	4,9
.28	0,9	0,9	0,9	0,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9	4,9	4,9	4,9
.29	0,9	0,9	0,9	0,9	1,9	1,9	1,9	2,9	2,9	2,9	3,9	3,9	4,9	4,9	4,9
.30	0,10	0,10	0,10	0,10	1,10	1,10	1,10	2,10	2,10	2,10	3,10	3,10	4,10	4,10	4,10
.31	0,10	0,10	0,10	0,10	1,10	1,10	1,10	2,10	2,10	2,10	3,10	3,10	4,10	4,10	4,10
.32	0,10	0,10	0,10	0,10	1,10	1,10	1,10	2,10	2,10	2,10	3,10	3,10	4,10	4,10	4,10
.33	0,11	0,11	0,11	0,11	1,11	1,11	1,11	2,11	2,11	2,11	3,11	3,11	4,11	4,11	4,11
.34	0,11	0,11	0,11	0,11	1,11	1,11	1,11	2,11	2,11	2,11	3,11	3,11	4,11	4,11	4,11
.35	0,12	0,12	0,12	0,12	1,12	1,12	1,12	2,12	2,12	2,12	3,12	3,12	4,12	4,12	4,12
.36	0,12	0,12	0,12	0,12	1,12	1,12	1,12	2,12	2,12	2,12	3,12	3,12	4,12	4,12	4,12
.37	0,12	0,12	0,12	0,12	1,12	1,12	1,12	2,12	2,12	2,12	3,12	3,12	4,12	4,12	4,12
.38	0,13	0,13	0,13	0,13	1,13	1,13	1,13	2,13	2,13	2,13	3,13	3,13	4,13	4,13	4,13
.39	0,13	0,13	0,13	0,13	1,13	1,13	1,13	2,13	2,13	2,13	3,13	3,13	4,13	4,13	4,13
.40	0,13	0,13	0,13	0,13	1,13	1,13	1,13	2,13	2,13	2,13	3,13	3,13	4,13	4,13	4,13
.41	0,14	0,14	0,14	0,14	1,14	1,14	1,14	2,14	2,14	2,14	3,14	3,14	4,14	4,14	4,14
.42	0,14	0,14	0,14	0,14	1,14	1,14	1,14	2,14	2,14	2,14	3,14	3,14	4,14	4,14	4,14
.43	0,14	0,14	0,14	0,14	1,14	1,14	1,14	2,14	2,14	2,41	3,14	3,14	4,14	4,14	4,14
.44	0,15	0,15	0,15	0,15	1,15	1,15	1,15	2,15	2,15	2,15	3,15	3,15	4,15	4,15	4,15
.45	0,15	0,15	0,15	0,15	1,15	1,15	1,15	2,15	2,15	2,15	3,15	3,15	4,15	4,15	4,15
.46	0,16	0,16	0,16	0,16	1,16	1,16	1,16	2,16	2,16	2,16	3,16	3,16	4,16	4,16	4,16
.47	0,16	0,16	0,16	0,16	1,16	1,16	1,16	2,16	2,16	2,16	3,16	3,16	4,16	4,16	4,16
.48	0,16	0,16	0,16	0,16	1,16	1,16	1,16	2,16	2,16	2,16	3,16	3,16	4,16	4,16	4,16
.49	0,17	0,17	0,17	0,17	1,17	1,17	1,17	2,17	2,17	2,17	3,17	3,17	4,17	4,17	4,17
.50	0,17	0,17	0,17	0,17	1,17	1,17	1,17	2,17	2,17	2,17	3,17	3,17	4,17	4,17	4,17

THREE GRADES N=35

P2 \ P1	.16	.17	.18	.19	.20	.21	.22	.23
.02								
.03								
.04								
.05								
.06								
.07								
.08								
.09								
.10								
.11								
.12								
.13								
.14								
.15								
.16								
.17	4,5							
.18	5,6	5,6						
.19	5,6	5,6	5,6					
.20	5,6	5,6	5,6	5,6				
.21	5,6	5,6	5,6	6,7	6,7			
.22	5,6	5,6	6,7	6,7	6,7	6,7		
.23	5,7	6,7	6,7	6,7	6,7	6,7	7,8	
.24	5,7	6,7	6,7	6,7	6,7	7,8	7,8	7,8
.25	5,8	5,7	6,7	6,7	7,8	7,8	7,8	7,8
.26	5,8	5,8	6,8	6,8	7,8	7,8	7,8	7,8
.27	5,8	5,8	6,8	6,8	7,8	7,8	7,8	7,8
.28	5,9	5,9	6,9	6,9	7,8	7,8	7,8	8,9
.29	5,9	5,9	6,9	6,9	7,9	7,9	8,9	8,9
.30	5,10	5,10	6,10	6,10	6,9	7,9	7,9	8,9
.31	5,10	5,10	5,10	6,10	6,10	7,10	7,10	8,9
.32	5,10	5,10	5,10	6,10	6,10	7,10	7,10	8,10
.33	5,11	5,11	5,11	6,11	6,11	7,11	7,11	8,10
.34	5,11	5,11	5,11	6,11	6,11	7,11	7,11	7,11
.35	5,12	5,12	5,12	6,12	6,11	7,11	7,11	7,11
.36	5,12	5,12	5,12	6,12	6,12	7,12	7,12	7,12
.37	5,12	5,12	5,12	6,12	6,12	7,12	7,12	7,12
.38	5,13	5,13	5,13	6,13	6,13	7,13	7,13	7,13
.39	5,13	5,13	5,13	6,13	6,13	6,13	7,13	7,13
.40	5,13	5,13	5,13	6,13	6,13	6,13	7,13	7,13
.41	5,14	5,14	5,14	6,14	6,14	6,14	7,14	7,14
.42	5,14	5,14	5,14	6,14	6,14	6,14	7,14	7,14
.43	5,14	5,14	5,14	6,14	6,14	6,14	7,14	7,14
.44	5,15	5,15	5,15	6,15	6,15	6,15	7,15	7,15
.45	5,15	5,15	5,15	6,15	6,15	6,15	7,15	7,15
.46	5,16	5,16	5,16	6,16	6,16	6,16	7,16	7,16
.47	5,16	5,16	5,16	6,16	6,16	6,16	7,16	7,16
.48	5,16	5,16	5,16	6,16	6,16	6,16	7,16	7,16
.49	5,17	5,17	5,17	6,17	6,17	6,17	7,17	7,17
.50	5,17	5,17	5,17	6,17	6,17	6,17	7,17	7,17

THREE GRADES N=35

P2 \ P1	.24	.25	.26	.27	.28	.29	.30
.02							
.03							
.04							
.05							
.06							
.07							
.08							
.09							
.10							
.11							
.12							
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.15							
.16							
.17							
.18							
.19							
.20							
.21							
.22							
.23							
.24							
.25	7,8						
.26	7,8	8,9					
.27	8,9	8,9	8,9				
.28	8,9	8,9	8,9	8,9			
.29	8,9	8,9	8,9	9,10	9,10		
.30	8,9	8,9	9,10	9,10	9,10	9,10	
.31	8,9	8,9	9,10	9,10	9,10	9,10	9,10
.32	8,10	9,10	9,10	9,10	9,10	9,10	10,11
.33	8,10	9,10	9,10	9,10	9,10	10,11	10,11
.34	8,11	8,11	9,10	9,10	10,11	10,11	10,11
.35	8,11	8,11	9,11	9,11	10,11	10,11	10,11
.36	8,12	8,12	9,11	9,11	10,11	10,11	10,11
.37	8,12	8,12	9,12	9,12	10,12	10,11	10,11
.38	8,13	8,12	9,12	9,12	9,12	10,12	11,12
.39	8,13	8,13	8,13	9,13	9,13	10,13	10,12
.40	8,13	8,13	8,13	9,13	9,13	10,13	10,13
.41	8,14	8,14	8,14	9,14	9,14	10,13	10,13
.42	8,14	8,14	8,14	9,14	9,14	10,14	10,14
.43	8,14	8,14	8,14	9,14	9,14	10,14	10,14
.44	8,15	8,15	8,15	9,15	9,15	9,15	10,15
.45	8,15	8,15	8,15	9,15	9,15	9,15	10,15
.46	8,16	8,16	8,16	9,16	9,16	9,15	10,15
.47	8,16	8,16	8,16	9,16	9,16	9,16	10,16
.48	8,16	8,16	8,16	9,16	9,16	9,16	10,16
.49	8,17	8,17	8,17	9,17	9,17	9,17	10,17
.50	8,17	8,17	8,17	9,17	9,17	9,17	10,17

THREE GRADES N=40

P2 \ P1	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15
.02	0,1														
.03	0,1	0,1													
.04	0,1	0,1	0,1												
.05	0,1	0,1	0,1	0,1											
.06	0,1	0,1	0,1	0,1	1,2										
.07	0,2	0,2	0,1	1,2	1,2	1,2									
.08	0,2	0,2	0,2	1,2	1,2	1,2	1,2								
.09	0,3	0,3	0,2	1,2	1,2	1,2	2,3	2,3							
.10	0,3	0,3	0,3	1,3	1,3	2,3	2,3	2,3	2,3						
.11	0,3	0,3	0,3	1,3	1,3	2,3	2,3	2,3	2,3	3,4					
.12	0,4	0,4	0,4	1,4	1,4	2,4	2,3	2,3	3,4	3,4	3,4				
.13	0,4	0,4	0,4	1,4	1,4	1,4	2,4	3,4	3,4	3,4	3,4	4,5			
.14	0,5	0,5	0,5	0,5	1,5	1,5	2,4	2,4	3,4	3,4	4,5	4,5	4,5		
.15	0,5	0,5	0,5	0,5	1,5	1,5	2,5	2,5	3,5	3,4	4,5	4,5	4,5	4,5	
.16	0,5	0,5	0,5	0,5	1,5	1,5	2,5	2,5	3,5	3,5	4,5	4,5	4,5	5,6	5,6
.17	0,6	0,6	0,6	0,6	1,6	1,6	2,6	2,6	3,6	3,6	4,5	4,5	5,6	5,6	5,6
.18	0,6	0,6	0,6	0,6	1,6	1,6	2,6	2,6	3,6	3,6	4,6	4,6	5,6	5,6	5,6
.19	0,7	0,7	0,7	0,7	1,7	1,7	2,7	2,7	3,7	3,7	4,6	4,6	5,6	5,6	5,6
.20	0,7	0,7	0,7	0,7	1,7	1,7	2,7	2,7	3,7	3,7	4,7	4,7	5,7	5,6	6,7
.21	0,8	0,8	0,8	0,8	1,8	1,8	2,8	2,7	3,7	3,7	4,7	4,7	5,7	5,7	6,7
.22	0,8	0,8	0,8	0,8	1,8	1,8	2,8	2,8	3,8	3,8	3,8	4,8	4,8	5,8	6,7
.23	0,8	0,8	0,8	0,8	1,3	1,8	2,8	2,8	3,8	3,8	3,8	4,8	4,8	5,8	5,8
.24	0,9	0,9	0,9	0,9	1,9	1,9	2,9	2,9	3,9	3,9	3,9	4,9	4,9	5,9	5,8
.25	0,9	0,9	0,9	0,9	1,9	1,9	2,9	2,9	3,9	3,9	3,9	4,9	4,9	5,9	5,9
.26	0,10	0,10	0,10	0,10	1,10	1,10	2,10	2,10	3,10	3,10	3,10	4,10	4,10	5,9	5,9
.27	0,10	0,10	0,10	0,10	1,10	1,10	2,10	2,10	3,10	3,10	3,10	4,10	4,10	5,10	5,10
.28	0,10	0,10	0,10	0,10	1,10	1,10	2,10	2,10	3,10	3,10	3,10	4,10	4,10	5,10	5,10
.29	0,11	0,11	0,11	0,11	1,11	1,11	2,11	2,11	3,11	3,11	3,11	4,11	4,11	5,11	5,11
.30	0,11	0,11	0,11	0,11	1,11	1,11	2,11	2,11	3,11	3,11	3,11	4,11	4,11	5,11	5,11
.31	0,12	0,12	0,12	0,12	1,12	1,12	2,12	2,12	3,12	3,12	3,12	4,12	4,12	5,12	5,12
.32	0,12	0,12	0,12	0,12	1,12	1,12	2,12	2,12	3,12	3,12	3,12	4,12	4,12	5,12	5,12
.33	0,12	0,12	0,12	0,12	1,12	1,12	2,12	2,12	3,12	3,12	3,12	4,12	4,12	5,12	5,12
.34	0,13	0,13	0,13	0,13	1,13	1,13	2,13	2,13	3,13	3,13	3,13	4,13	4,13	5,13	5,13
.35	0,13	0,13	0,13	0,13	1,13	1,13	2,13	2,13	3,13	3,13	3,13	4,13	4,13	5,13	5,13
.36	0,14	0,14	0,14	0,14	1,14	1,14	2,14	2,14	3,14	3,14	3,14	4,14	4,14	5,14	5,14
.37	0,14	0,14	0,14	0,14	1,14	1,14	2,14	2,14	3,14	3,14	3,14	4,14	4,14	5,14	5,14
.38	0,15	0,15	0,15	0,15	1,15	1,15	2,15	2,15	3,15	3,15	3,15	4,15	4,15	5,15	5,15
.39	0,15	0,15	0,15	0,15	1,15	1,15	2,15	2,15	3,15	3,15	3,15	4,15	4,15	5,15	5,15
.40	0,15	0,15	0,15	0,15	1,15	1,15	2,15	2,15	3,15	3,15	3,15	4,15	4,15	5,15	5,15
.41	0,16	0,16	0,16	0,16	1,16	1,16	2,16	2,16	3,16	3,16	3,16	4,16	4,16	5,16	5,16
.42	0,16	0,16	0,16	0,16	1,16	1,16	2,16	2,16	3,16	3,16	3,16	4,16	4,16	5,16	5,16
.43	0,17	0,17	0,17	0,17	1,17	1,17	2,17	2,17	3,17	3,17	3,17	4,17	4,17	5,17	5,17
.44	0,17	0,17	0,17	0,17	1,17	1,17	2,17	2,17	3,17	3,17	3,17	4,17	4,17	5,17	5,17
.45	0,17	0,17	0,17	0,17	1,17	1,17	2,17	2,17	3,17	3,17	3,17	4,17	4,17	5,17	5,17
.46	0,18	0,18	0,18	0,18	1,18	1,18	2,18	2,18	3,18	3,18	3,18	4,18	4,18	5,18	5,18
.47	0,18	0,18	0,18	0,18	1,18	1,18	2,18	2,18	3,18	3,18	3,18	4,18	4,18	5,18	5,18
.48	0,19	0,19	0,19	0,19	1,19	1,19	2,19	2,19	3,19	3,19	3,19	4,19	4,19	5,19	5,19
.49	0,19	0,19	0,19	0,19	1,19	1,19	2,19	2,19	3,19	3,19	3,19	4,19	4,19	5,19	5,19
.50	0,19	0,19	0,19	0,19	1,19	1,19	2,19	2,19	3,19	3,19	3,19	4,19	4,19	5,19	5,19

THREE GRADES N=40

P2 \ P1	.16	.17	.18	.19	.20	.21	.22	.23
.02								
.03								
.04								
.05								
.06								
.07								
.08								
.09								
.10								
.11								
.12								
.13								
.14								
.15								
.16								
.17	5,6							
.18	5,6	6,7						
.19	6,7	6,7	6,7					
.20	6,7	6,7	6,7	6,7				
.21	6,7	6,7	6,7	7,8	7,8			
.22	6,7	6,7	7,8	7,8	7,8	7,8		
.23	6,8	7,8	7,8	7,8	7,8	7,8	8,9	
.24	6,8	6,8	7,8	7,8	7,8	8,9	8,9	8,9
.25	6,9	6,9	7,8	7,8	8,9	8,9	8,9	8,9
.26	6,9	6,9	7,9	7,9	8,9	8,9	8,9	8,9
.27	6,10	6,10	7,10	7,9	8,9	8,9	8,9	9,10
.28	6,10	6,10	7,10	7,10	8,10	8,10	9,10	9,10
.29	6,11	6,11	6,11	7,11	7,10	8,10	9,10	9,10
.30	6,11	6,11	6,11	7,11	7,11	8,11	9,11	9,10
.31	5,12	6,12	6,12	7,12	7,11	8,11	8,11	9,11
.32	5,12	6,12	6,12	7,12	7,12	8,12	8,12	9,12
.33	5,12	6,12	6,12	7,12	7,12	8,12	8,12	9,12
.34	5,13	6,13	6,13	7,13	7,13	8,13	8,13	9,13
.35	5,13	6,13	6,13	7,13	7,13	8,13	8,13	9,13
.36	5,14	6,14	6,14	7,14	7,14	8,14	8,14	8,14
.37	5,14	6,14	6,14	7,14	7,14	8,14	8,14	8,14
.38	5,15	6,15	6,15	7,15	7,15	8,15	8,14	8,14
.39	5,15	6,15	6,15	7,15	7,15	8,15	8,15	8,15
.40	5,15	6,15	6,15	7,15	7,15	8,15	8,15	8,15
.41	5,16	6,16	6,16	7,16	7,16	8,16	8,16	8,16
.42	5,16	6,16	6,16	7,16	7,16	8,16	8,16	8,16
.43	5,17	6,17	6,17	7,17	7,17	8,17	8,17	8,17
.44	5,17	6,17	6,17	7,17	7,17	8,17	8,17	8,17
.45	5,17	6,17	6,17	7,17	7,17	8,17	8,17	8,17
.46	5,18	6,18	6,18	7,18	7,18	8,18	8,18	8,18
.47	5,18	6,18	6,18	7,18	7,18	8,18	8,18	8,18
.48	5,19	6,19	6,19	7,19	7,19	8,19	8,19	8,19
.49	5,19	6,19	6,19	7,19	7,19	8,19	8,19	8,19
.50	5,19	6,19	6,19	7,19	7,19	8,19	3,19	8,19

THREE GRADES N=40

P2 \ P1	.24	.25	.26	.27	.28	.29	.30
.02							
.03							
.04							
.05							
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.17							
.18							
.19							
.20							
.21							
.22							
.23							
.24							
.25	8,9						
.26	9,10	9,10					
.27	9,10	9,10	9,10				
.28	9,10	9,10	9.10	10,11			
.29	9,10	9,10	10,11	10,11	10,11		
.30	9,10	10,11	10,11	10,11	10,11	11,12	
.31	10,11	10,11	10,11	10,11	11,12	11,12	11,12
.32	9,11	10,11	10,11	11,12	11,12	11,12	11,12
.33	9,12	10,12	10,12	11,12	11,12	11,12	11,12
.34	9,13	10,12	10,12	11,12	11,12	11,12	12,13
.35	9,13	10,13	10,13	11,13	11,12	12,13	12,13
.36	9,14	9,13	10,13	11,13	11,13	12,13	12,13
.37	9,14	9,14	10,14	10,14	11,14	12,13	12,13
.38	9,14	9,14	10,14	10,14	11,14	11,14	12,14
.39	9,15	9,15	10,15	10,15	11,15	11,15	12,14
.40	9,15	9,15	10,15	10,15	11,15	11,15	12,15
.41	9,16	9,16	10,16	10,16	11,16	11,16	12,15
.42	9,16	9,16	10,16	10,16	11,16	11,16	11,16
.43	9,17	9,17	10,17	10,17	10,17	11,16	11,16
.44	9,17	9,17	10,17	10,17	10,17	11,17	11,17
.45	9,17	9,17	10,17	10,17	10,17	11,17	11,17
.46	9,18	9,18	10,18	10,18	10,18	11,18	11,18
.47	9,18	9,18	10,18	10,18	10,18	11,18	11,18
.48	9,19	9,19	10,19	10,19	10,19	11,19	11,19
.49	9,19	9,19	10,19	10,19	10,19	11,19	11,19
.50	9,19	9,19	10,19	10,19	10,19	11,19	11,19

THREE GRADES N=45

P2 \ P1	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15
.02	0,1														
.03	0,1	0,1													
.04	0,1	0,1	0,1												
.05	0,1	0,1	0,1	0,1											
.06	0,2	0,1	0,1	1,2	1,2										
.07	0,2	0,2	0,2	1,2	1,2	1,2									
.08	0,3	0,3	0,2	1,2	1,2	2,3	2,3								
.09	0,3	0,3	0,3	1,3	2,3	2,3	2,3	2,3							
.10	0,3	0,3	0,3	1,3	1,3	2,3	2,3	3,4	3,4						
.11	0,4	0,4	0,4	1,4	1,4	2,4	2,3	3,4	3,4	3,4					
.12	0,4	0,4	0,4	1,4	1,4	2,4	2,4	3,4	3,4	3,4	4,5				
.13	0,5	0,5	0,5	1,5	1,5	2,5	2,5	3,4	3,4	4,5	4,5	4,5			
.14	0,5	0,5	0,5	1,5	1,5	2,5	2,5	3,5	4,5	4,5	4,5	4,5	5,6		
.15	0,6	0,6	0,6	1,6	1,6	2,6	2,6	3,6	3,5	4,5	4,5	5,6	5,6	5,6	
.16	0,6	0,6	0,6	1,6	1,6	2,6	2,6	3,6	3,6	4,6	5,6	5,6	5,6	5,6	6,7
.17	0,7	0,7	0,7	1,7	1,7	2,7	2,7	3,7	3,7	4,6	4,6	5,6	5,6	6,7	6,7
.18	0,7	0,7	0,7	1,7	1,7	2,7	2,7	3,7	3,7	4,7	4,7	5,7	5,6	6,7	6,7
.19	0,8	0,8	0,8	1,8	1,8	2,8	2,8	3,8	3,8	4,8	4,7	5,7	5,7	6,7	6,7
.20	0,8	0,8	0,8	1,8	1,8	2,8	2,8	3,8	3,8	4,8	4,8	5,8	5,8	6,7	6,7
.21	0,9	0,9	0,9	1,9	1,9	2,9	2,9	3,9	3,9	4,9	4,8	5,8	5,8	6,8	6,8
.22	0,9	0,9	0,9	1,9	1,9	2,9	2,9	3,9	3,9	4,9	4,9	5,9	5,9	6,9	6,9
.23	0,9	0,9	0,9	1,9	1,9	2,9	2,9	3,9	3,9	3,9	4,9	4,9	5,9	6,9	6,9
.24	0,10	0,10	0,10	1,10	1,10	2,10	2,10	3,10	3,10	3,10	4,10	4,10	5,10	5,10	6,10
.25	0,10	0,10	0,10	1,10	1,10	2,10	2,10	3,10	3,10	3,10	4,10	4,10	5,10	5,10	6,10
.26	0,11	0,11	0,11	1,11	1,11	2,11	2,11	3,11	3,11	3,11	4,11	4,11	5,11	5,11	6,11
.27	0,11	0,11	0,11	1,11	1,11	2,11	2,11	3,11	3,11	3,11	4,11	4,11	5,11	5,11	6,11
.28	0,12	0,12	0,12	1,12	1,12	2,12	2,12	3,12	3,12	3,12	4,12	4,12	5,12	5,12	6,12
.29	0,12	0,12	0,12	1,12	1,12	2,12	2,12	3,12	3,12	3,12	4,12	4,12	5,12	5,12	6,12
.30	0,13	0,13	0,13	1,13	1,13	2,13	2,13	3,13	3,13	3,13	4,13	4,13	5,13	5,13	6,13
.31	0,13	0,13	0,13	1,13	1,13	2,13	2,13	3,13	3,13	3,13	4,13	4,13	5,13	5,13	6,13
.32	0,14	0,14	0,14	1,14	1,14	2,14	2,14	3,14	3,14	3,14	4,14	4,14	5,14	5,14	6,14
.33	0,14	0,14	0,14	1,14	1,14	2,14	2,14	3,14	3,14	3,14	4,14	4,14	5,14	5,14	6,14
.34	0,15	0,15	0,15	1,15	1,15	2,15	2,15	3,15	3,15	3,15	4,15	4,15	5,15	5,15	6,15
.35	0,15	0,15	0,15	1,15	1,15	2,15	2,15	3,15	3,15	3,15	4,15	4,15	5,15	5,15	6,15
.36	0,16	0,16	0,16	1,16	1,16	2,16	2,16	3,16	3,16	3,16	4,16	4,16	5,16	5,16	6,16
.37	0,16	0,16	0,16	1,16	1,16	2,16	2,16	3,16	3,16	3,16	4,16	4,16	5,16	5,16	6,16
.38	0,16	0,16	0,16	1,16	1,16	2,16	2,16	3,16	3,16	3,16	4,16	4,16	5,16	5,16	6,16
.39	0,17	0,17	0,17	1,17	1,17	2,17	2,17	3,17	3,17	3,17	4,17	4,17	5,17	5,17	6,17
.40	0,17	0,17	0,17	1,17	1,17	2,17	2,17	3,17	3,17	3,17	4,17	4,17	5,17	5,17	6,17
.41	0,18	0,18	0,18	1,18	1,18	2,18	2,18	3,18	3,18	3,18	4,18	4,18	5,18	5,18	6,18
.42	0,18	0,18	0,18	1,18	1,18	2,18	2,18	3,18	3,18	3,18	4,18	4,18	5,18	5,18	6,18
.43	0,19	0,19	0,19	1,19	1,19	2,19	2,19	3,19	3,19	3,19	4,19	4,19	5,19	5,19	6,19
.44	0,19	0,19	0,19	1,19	1,19	2,19	2,19	3,19	3,19	3,19	4,19	4,19	5,19	5,19	6,19
.45	0,20	0,20	0,20	1,20	1,20	2,20	2,20	3,20	3,20	3,20	4,20	4,20	5,20	5,20	6,20
.46	0,20	0,20	0,20	1,20	1,20	2,20	2,20	3,20	3,20	3,20	4,20	4,20	5,20	5,20	6,20
.47	0,21	0,21	0,21	1,21	1,21	2,21	2,21	3,21	3,21	3,21	4,21	4,21	5,21	5,21	6,21
.48	0,21	0,21	0,21	1,21	1,21	2,21	2,21	3,21	3,21	3,21	4,21	4,21	5,21	5,21	6,21
.49	0,22	0,22	0,22	1,22	1,22	2,22	2,22	3,22	3,22	3,22	4,22	4,22	5,22	5,22	6,22
.50	0,22	0,22	0,22	1,22	1,22	2,22	2,22	3,22	3,22	3,22	4,22	4,22	5,22	5,22	6,22

THREE GRADES N=45

P2 \ P1	.16	.17	.18	.19	.20	.21	.22	.23
.02								
.03								
.04								
.05								
.06								
.07								
.08								
.09								
.10								
.11								
.12								
.13								
.14								
.15								
.16								
.17	6,7							
.18	6,7	6,7						
.19	6,7	7,8	7,8					
.20	7,8	7,8	7,8	7,8				
.21	7,8	7,8	7,8	8,9	8,9			
.22	7,8	7,8	8,9	8,9	8,9	8,9		
.23	7,9	7,9	8,9	8,9	8,9	9,10	9,10	
.24	7,10	7,9	8,9	8,9	9,10	9,10	9,10	9,10
.25	7,10	7,10	8,10	9,10	9,10	9,10	9,10	9,10
.26	6,11	7,11	8,10	8,10	9,10	9,10	9,10	10,11
.27	6,11	7,11	7,11	8,11	9,11	9,10	10,11	10,11
.28	6,12	7,12	7,12	8,11	9,11	9,11	10,11	10,11
.29	6,12	7,12	7,12	8,12	8,12	9,12	10,12	10,11
.30	6,13	7,13	7,13	8,13	8,12	9,12	10,12	10,12
.31	6,13	7,13	7,13	8,13	8,13	9,13	9,13	10,13
.32	6,14	7,14	7,14	8,14	8,14	9,13	9,13	10,13
.33	6,14	7,14	7,14	8,14	8,14	9,14	9,14	10,14
.34	6,15	7,15	7,15	8,15	8,15	9,14	9,14	10,14
.35	6,15	7,15	7,15	8,15	8,15	9,15	9,15	10,15
.36	6,16	7,16	7,16	8,15	8,15	9,15	9,15	10,15
.37	6,16	7,16	7,16	8,16	8,16	9,16	9,16	10,16
.38	6,16	7,16	7,16	8,16	8,16	9,16	9,16	10,16
.39	6,17	7,17	7,17	8,17	8,17	9,17	9,17	10,17
.40	6,17	7,17	7,17	8,17	8,17	9,17	9,17	10,17
.41	6,18	7,18	7,18	8,18	8,18	9,18	9,18	10,18
.42	6,18	7,18	7,18	8,18	8,18	9,18	9,18	10,18
.43	6,19	7,19	7,19	8,19	8,19	9,19	9,19	9,19
.44	6,19	7,19	7,19	8,19	8,19	9,19	9,19	9,19
.45	6,20	7,20	7,20	8,20	8,20	9,20	9,20	9,20
.46	6,20	7,20	7,20	8,20	8,20	9,20	9,20	9,20
.47	6,21	7,21	7,21	8,21	8,21	9,21	9,21	9,21
.48	6,21	7,21	7,21	8,21	8,21	9,21	9,21	9,21
.49	6,22	7,22	7,22	8,22	8,22	9,22	9,22	9,22
.50	6,22	7,22	7,22	8,22	8,22	9,22	9,22	9,22

THREE GRADES N=45

P2 \ P1	.24	.25	.26	.27	.28	.29	.30
.02							
.03							
.04							
.05							
.06							
.07							
.08							
.09							
.10							
.11							
.12							
.13							
.14							
.15							
.16							
.17							
.18							
.19							
.20							
.21							
.22							
.23							
.24							
.25	10,11						
.26	10,11	10,11					
.27	10,11	10,11	11,12				
.28	10,11	11,12	11,12	11,12			
.29	11,12	11,12	11,12	11,12	12,13		
.30	11,12	11,12	11,12	12,13	12,13	12,13	
.31	11,12	11,12	12,13	12,13	12,13	12,13	12,13
.32	11,13	11,13	12,13	12,13	12,13	12,13	13,14
.33	10,14	11,14	12,13	12,13	12,13	13,14	13,14
.34	10,14	11,14	12,14	12,14	13,14	13,14	13,14
.35	10,15	11,15	11,15	12,14	13,14	13,14	13,14
.36	10,15	11,15	11,15	12,15	12,15	13,15	14,15
.37	10,16	11,16	11,16	12,16	12,15	13,15	14,15
.38	10,16	11,16	11,16	12,16	12,16	13,16	13,16
.39	10,17	11,17	11,17	12,17	12,17	13,17	13,16
.40	10,17	10,17	11,17	11,17	12,17	13,17	13,17
.41	10,18	10,18	11,18	11,18	12,18	12,18	13,18
.42	10,18	10,18	11,18	11,18	12,18	12,18	13,18
.43	10,19	10,19	11,19	11,19	12,19	12,19	13,19
.44	10,19	10,19	11,19	11,19	12,19	12,19	13,19
.45	10,20	10,20	11,20	11,20	12,20	12,20	13,20
.46	10,20	10,20	11,20	11,20	12,20	12,20	13,20
.47	10,21	10,21	11,21	11,21	12,21	12,21	13,21
.48	10,21	10,21	11,21	11,21	12,21	12,21	13,21
.49	10,22	10,22	11,22	11,22	12,22	12,22	13,22
.50	10,22	10,22	11,22	11,22	12,22	12,22	13,22

THREE GRADES N=50

P2 \ P1	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	.11	.12	.13	.14	.15
.02	0,1														
.03	0,1	0,1													
.04	0,1	0,1	0,1												
.05	0,1	0,1	0,1	1,2											
.06	0,2	0,2	1,2	1,2	1,2										
.07	0,2	0,2	1,2	1,2	1,2	2,3									
.08	0,3	0,3	0,3	1,3	2,3	2,3	2,3								
.09	0,3	0,3	0,3	1,3	2,3	2,3	2,3	3,4							
.10	0,4	0,4	0,4	1,4	2,4	2,3	3,4	3,4	3,4						
.11	0,4	0,4	0,4	1,4	2,4	2,4	3,4	3,4	3,4	4,5					
.12	0,5	0,5	0,5	1,5	1,5	2,5	3,5	3,4	4,5	4,5	4,5				
.13	0,6	0,6	0,6	1,5	1,5	2,5	3,5	3,5	4,5	4,5	4,5	5,6			
.14	0,6	0,6	0,6	1,6	1,6	2,6	3,6	3,6	4,5	4,5	5,6	5,6	5,6		
.15	0,7	0,7	0,7	1,7	1,7	2,6	3,6	3,6	4,6	5,6	5,6	5,6	6,7	6,7	
.16	0,7	0,7	0,7	1,7	1,7	2,7	2,7	3,7	4,7	4,7	5,6	6,7	6,7	6,7	6,7
.17	0,8	0,8	0,8	1,8	1,8	2,8	2,8	3,7	4,7	4,7	5,7	6,7	6,7	6,7	7,8
.18	0,8	0,8	0,8	1,8	1,8	2,8	2,8	3,8	4,8	4,8	5,8	5,8	6,7	7,8	7,8
.19	0,9	0,9	0,9	1,9	1,9	2,9	2,9	3,9	4,9	4,8	5,8	5,8	6,8	7,8	7,8
.20	0,9	0,9	0,9	1,9	1,9	2,9	2,9	3,9	3,9	4,9	5,9	5,9	6,9	7,8	7,8
.21	0,10	0,10	0,10	1,10	1,10	2,10	2,10	3,10	3,10	4,10	5,10	5,9	6,9	6,9	7,9
.22	0,10	0,10	0,10	1,10	1,10	2,10	2,10	3,10	3,10	4,10	5,10	5,10	6,10	6,10	7,10
.23	0,11	0,11	0,11	1,11	1,11	2,11	2,11	3,11	3,11	4,11	5,11	5,11	6,11	6,10	7,10
.24	0,11	0,11	0,11	1,11	1,11	2,11	2,11	3,11	3,11	4,11	5,11	5,11	6,11	6,11	7,11
.25	0,12	0,12	0,12	1,12	1,12	2,12	2,12	3,12	3,12	4,12	4,12	5,12	6,12	6,12	7,12
.26	0,12	0,12	0,12	1,12	1,12	2,12	2,12	3,12	3,12	4,12	4,12	5,12	6,12	6,12	7,12
.27	0,13	0,13	0,13	1,13	1,13	2,13	2,13	3,13	3,13	4,13	4,13	5,13	6,13	6,13	7,13
.28	0,13	0,13	0,13	1,13	1,13	2,13	2,13	3,13	3,13	4,13	4,13	5,13	6,13	6,13	7,13
.29	0,14	0,14	0,14	1,14	1,14	2,14	2,14	3,14	3,14	4,14	4,14	5,14	6,14	6,14	7,14
.30	0,14	0,14	0,14	1,14	1,14	2,14	2,14	3,14	3,14	4,14	4,14	5,14	6,14	6,14	7,14
.31	0,15	0,15	0,15	1,15	1,15	2,15	2,15	3,15	3,15	4,15	4,15	5,15	6,15	6,15	7,15
.32	0,15	0,15	0,15	1,15	1,15	2,15	2,15	3,15	3,15	4,15	4,51	5,15	6,15	6,15	7,15
.33	0,16	0,16	0,16	1,16	1,16	2,16	2,16	3,16	3,16	4,16	4,16	5,16	6,16	6,16	7,16
.34	0,16	0,16	0,16	1,16	1,16	2,16	2,16	3,16	3,16	4,16	4,16	5,16	6,16	6,16	7,16
.35	0,17	0,17	0,17	1,17	1,17	2,17	2,17	3,17	3,17	4,17	4,17	5,17	6,17	6,17	7,17
.36	0,17	0,17	0,17	1,17	1,17	2,17	2,17	3,17	3,17	4,17	4,17	5,17	6,17	6,17	7,17
.37	0,18	0,18	0,18	1,18	1,18	2,18	2,18	3,18	3,18	4,18	4,18	5,18	6,18	6,18	7,18
.38	0,18	0,18	0,18	1,18	1,18	2,18	2,18	3,18	3,18	4,18	4,18	5,18	6,18	6,18	7,18
.39	0,19	0,19	0,19	1,19	1,19	2,19	2,19	3,19	3,19	4,19	4,19	5,19	6,19	6,19	7,19
.40	0,19	0,19	0,19	1,19	1,19	2,19	2,19	3,19	3,19	4,19	4,19	5,19	6,19	6,19	7,19
.41	0,20	0,20	0,20	1,20	1,20	2,20	2,20	3,20	3,20	4,20	4,20	5,20	6,20	6,20	7,20
.42	0,20	0,20	0,20	1,20	1,20	2,20	2,20	3,20	3,20	4,20	4,20	5,20	6,20	6,20	7,20
.43	0,21	0,21	0,21	1,21	1,21	2,21	2,21	3,21	3,21	4,21	4,21	5,21	6,21	6,21	7,21
.44	0,21	0,21	0,21	1,21	1,21	2,21	2,21	3,21	3,21	4,21	4,21	5,21	6,21	6,21	7,21
.45	0,22	0,22	0,22	1,22	1,22	2,22	2,22	3,22	3,22	4,22	4,22	5,22	6,22	6,22	7,22
.46	0,22	0,22	0,22	1,22	1,22	2,22	2,22	3,22	3,22	4,22	4,22	5,22	6,22	6,22	7,22
.47	0,23	0,23	0,23	1,23	1,23	2,23	2,23	3,23	3,23	4,23	4,23	5,23	6,23	6,23	7,23
.48	0,23	0,23	0,23	1,23	1,23	2,23	2,23	3,23	3,23	4,23	4,23	5,23	6,23	6,23	7,23
.49	0,24	0,24	0,24	1,24	1,24	2,24	2,24	3,24	3,24	4,24	4,24	5,24	6,24	6,24	7,24
.50	0,24	0,24	0,24	1,24	1,24	2,24	2,24	3,24	3,24	4,24	4,24	5,24	6,24	6,24	7,24

THREE GRADES N=50

P2 \ P1	.16	.17	.18	.19	.20	.21	.22	.23
.02								
.03								
.04								
.05								
.06								
.07								
.08								
.09								
.10								
.11								
.12								
.13								
.14								
.15								
.16								
.17	7,8							
.18	7,8	7,8						
.19	7,8	8,9	8,9					
.20	8,9	8,9	8,9	8,9				
.21	8,9	8,9	8,9	9,10	9,10			
.22	8,9	8,9	9,10	9,10	9,10	9,10		
.23	8,10	8,10	9,10	9,10	9,10	10,11	10,11	
.24	7,11	8,11	9,10	9,10	10,11	10,11	10,11	10,11
.25	7,11	8,11	9,11	9,11	10,11	10,11	10,11	11,12
.26	7,12	8,12	8,12	9,12	10,11	10,11	11,12	11,12
.27	7,13	8,13	8,12	9,12	10,12	11,12	11,12	11,12
.28	7,13	8,13	8,13	9,13	10,13	10,13	11,12	11,12
.29	7,14	8,14	8,14	9,14	9,13	10,13	11,13	12,13
.30	7,14	8,14	8,14	9,14	9,14	10,14	11,14	11,14
.31	7,15	8,15	8,15	9,15	9,15	10,15	10,14	11,14
.32	7,15	8,15	8,15	9,15	9,15	10,15	10,15	11,15
.33	7,16	8,16	8,16	9,16	9,16	10,16	10,16	11,16
.34	7,16	8,16	8,16	9,16	9,16	10,16	10,16	11,16
.35	7,17	8,17	8,17	9,17	9,17	10,17	10,17	11,17
.36	7,17	8,17	8,17	9,17	9,17	10,17	10,17	11,17
.37	7,18	8,18	8,18	9,18	9,18	10,18	10,18	11,18
.38	7,18	8,18	8,18	9,18	9,18	10,18	10,18	11,18
.39	7,19	8,19	8,19	9,19	9,19	10,19	10,19	11,19
.40	7,19	8,19	8,19	9,19	9,19	10,19	10,19	11,19
.41	7,20	8,20	8,20	9,20	9,20	10,20	10,20	11,20
.42	7,20	8,20	2,20	9,20	9,20	10,20	10,20	11,20
.43	7,21	8,21	8,21	9,21	9,21	10,21	10,21	11,21
.44	7,21	8,21	8,21	9,21	9,21	10,21	10,21	11,21
.45	7,22	8,22	8,22	9,22	9,22	10,22	10,22	11,22
.46	7,22	8,22	8,22	9,22	9,22	10,22	10,22	11,22
.47	7,23	8,23	8,23	9,23	9,23	10,23	10,23	11,23
.48	7,23	8,23	8,23	9,23	9,23	10,23	10,23	11,23
.49	7,24	8,24	8,24	9,24	9,24	10,24	10,24	11,24
.50	7,24	8,24	8,24	9,24	9,24	10,24	10,24	11,24

THREE GRADES N=50

P2 \ P1	.24	.25	.26	.27	.28	.29	.30
.02							
.03							
.04							
.05							
.06							
.07							
.08							
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.17							
.18							
.19							
.20							
.21							
.22							
.23							
.24							
.25	11,12						
.26	11,12	11,12					
.27	11,12	12,13	12,13				
.28	12,13	12,13	12,13	12,13			
.29	12,13	12,13	12,13	13,14	13,14		
.30	12,13	12,13	13,14	13,14	13,14	13,14	
.31	12,14	13,14	13,14	13,14	13,14	14,15	14,15
.32	12,15	12,14	13,14	13,14	14,15	14,15	14,15
.33	12,15	12,15	13,15	14,15	14,15	14,15	15,16
.34	11,16	12,16	13,16	14,15	14,15	14,15	15,16
.35	11,17	12,17	13,16	13,16	14,16	15,16	15,16
.36	11,17	12,17	12,17	13,17	14,17	15,16	15,16
.37	11,18	12,18	12,18	13,18	14,17	14,17	15,17
.38	11,18	12,18	12,18	13,18	14,18	14,18	15,18
.39	11,19	12,19	12,19	13,19	13,19	14,19	15,18
.40	11,19	12,19	12,19	13,19	13,19	14,19	15,19
.41	11,20	12,20	12,20	13,20	13,20	14,20	14,20
.42	11,20	12,20	12,20	13,20	13,20	14,20	14,20
.43	11,21	12,21	12,21	13,21	13,21	14,21	14,21
.44	11,21	12,21	12,21	13,21	13,21	14,21	14,21
.45	11,22	12,22	12,22	13,22	13,22	14,22	14,22
.46	11,22	12,22	12,22	13,22	13,22	14,22	14,22
.47	11,23	12,23	12,23	13,23	13,23	14,23	14,23
.48	11,23	12,23	12,23	13,23	13,23	14,23	14,23
.49	11,24	12,24	12,24	13,24	13,24	14,24	14,24
.50	11,24	12,24	12,24	13,24	13,24	14,24	14,24

FOUR GRADES N=50

P1	P2	P3	ACCEPT. NOS.
.01	.05	.10	0,2,4
		.15	0,1,7
		.20	0,1,9
		.25	0,1,12
		.30	0,1,14
		.35	0,1,17
		.40	0,1,19
		.45	0,1,22
		.50	0,1,24
	.10	.15	0,5,6
		.20	0,4,9
		.25	0,4,12
		.30	0,4,14
		.35	0,4,17
		.40	0,4,19
		.45	0,4,22
		.50	0,4,24
	.15	.20	0,7,8
		.25	0,7,12
		.30	0,7,14
		.35	0,7,17
		.40	0,7,19
		.45	0,7,22
		.50	0,7,24
	.20	.25	0,10,11
		.30	0,9,14
		.35	0,9,17
		.40	0,9,19
		.45	0,9,22
		.50	0,9,24
	.25	.30	0,12,13
		.35	0,12,17
		.40	0,12,19
		.45	0,12,22
		.50	0,12,24
	.30	.35	0,15,16
		.40	0,15,19
		.45	0,14,22
		.50	0,14,24

P1	P2	P3	ACCEPT. NOS.
.05	.10	.15	2,4,6
		.20	2,4,9
		.25	2,4,12
		.30	2,4,14
		.35	2,4,17
		.40	2,4,19
		.45	2,4,22
		.50	2,4,24
	.15	.20	1,7,8
		.25	1,7,12
		.30	1,7,14
		.35	1,7,17
		.40	1,7,19
		.45	1,7,22
		.50	1,7,24
	.20	.25	1,10,11
		.30	1,9,14
		.35	1,9,17
		.40	1,9,19
		.45	1,9,22
		.50	1,9,24
	.25	.30	1,12,13
		.35	1,12,17
		.40	1,12,19
		.45	1,12,22
		.50	1,12,24
	.30	.35	1,15,16
		.40	1,15,19
		.45	1,14,22
		.50	1,14,24
.10	.15	.20	5,7,8
		.25	5,6,12
		.30	5,6,14
		.35	5,6,17
		.40	5,6,19
		.45	5,6,22
		.50	5,6,24

P1	P2	P3	ACCEPT. NOS.
.10	.20	.25	4,10,11
		.30	4,9,14
		.35	4,9,17
		.40	4,9,19
		.45	4,9,22
		.50	4,9,24
	.25	.30	4,12,13
		.35	4,12,17
		.40	4,12,19
		.45	4,12,22
		.50	4,12,24
	.30	.35	4,15,16
		.40	4,15,19
		.45	4,14,22
		.50	4,14,24
.15	.20	.25	7,9,10
		.30	7,8,14
		.35	7,8,17
		.40	7,8,19
		.45	7,8,22
		.50	7,8,24
	.25	.30	7,12,13
		.35	7,12,17
		.40	7,12,19
		.45	7,12,22
		.50	7,12,24
	.30	.35	7,15,16
		.40	7,15,19
		.45	7,14,22
		.50	7,14,24
.20	.25	.30	10,12,13
		.35	10,11,17
		.40	10,11,19
		.45	10,11,22
		.50	10,11,24
	.30	.35	9,15,16
		.40	9,14,19
		.45	9,14,22
		.50	9,14,24

FOUR GRADES N=60

P1	P2	P3	ACCEPT. NOS.
.01	.05	.10	0,2,5
		.15	0,2,8
		.20	0,2,11
		.25	0,2,14
		.30	0,2,17
		.35	0,2,20
		.40	0,2,23
		.45	0,2,26
		.50	0,2,29
	.10	.15	0,5,8
		.20	0,5,11
		.25	0,5,14
		.30	0,5,17
		.35	0,5,20
		.40	0,5,23
		.45	0,5,26
		.50	0,5,29
	.15	.20	0,9,10
		.25	0,8,14
		.30	0,8,17
		.35	0,8,20
		.40	0,8,23
		.45	0,8,26
		.50	0,8,29
	.20	.25	0,12,13
		.30	0,11,17
		.35	0,11,20
		.40	0,11,23
		.45	0,11,26
		.50	0,11,29
	.25	.30	0,15,16
		.35	0,14,20
		.40	0,14,23
		.45	0,14,26
		.50	0,14,29
	.30	.35	0,18,19
		.40	0,18,23
		.45	0,17,26
		.50	0,17,29

P1	P2	P3	ACCEPT. NOS.
.05	.10	.15	2,5,8
		.20	2,5,11
		.25	2,5,14
		.30	2,5,17
		.35	2,5,20
		.40	2,5,23
		.45	2,5,26
		.50	2,5,29
	.15	.20	2,9,10
		.25	2,8,14
		.30	2,8,17
		.35	2,8,20
		.40	2,8,23
		.45	2,8,26
		.50	2,8,29
	.20	.25	2,12,13
		.30	2,11,17
		.35	2,11,20
		.40	2,11,23
		.45	2,11,26
		.50	2,11,29
	.25	.30	2,15,16
		.35	2,14,20
		.40	2,14,23
		.45	2,14,26
		.50	2,14,29
	.30	.35	2,18,19
		.40	2,18,23
		.45	2,17,26
		.50	2,17,29
.10	.15	.20	5,8,10
		.25	5,8,14
		.30	5,8,17
		.35	5,8,20
		.40	5,8,23
		.45	5,8,26
		.50	5,8,29

P1	P2	P3	ACCEPT. NOS.
.10	.20	.25	5,12,13
		.30	5,11,17
		.35	5,11,20
		.40	5,11,23
		.45	5,11,26
		.50	5,11,29
	.25	.30	5,15,16
		.35	5,14,20
		.40	5,14,23
		.45	5,14,26
		.50	5,14,29
	.30	.35	5,18,19
		.40	5,18,23
		.45	5,17,26
		.50	5,17,29
.15	.20	.25	9,11,13
		.30	9,10,17
		.35	9,10,20
		.40	9,10,23
		.45	9,10,26
		.50	9,10,29
	.25	.30	8,15,16
		.35	8,14,20
		.40	8,14,23
		.45	8,14,26
		.50	8,14,29
	.30	.35	8,18,19
		.40	8,17,23
		.45	8,17,26
		.50	8,17,29
.20	.25	.30	12,14,16
		.35	12,13,20
		.40	12,13,23
		.45	12,13,26
		.50	12,13,29
	.30	.35	11,18,19
		.40	11,17,23
		.45	11,17,26
		.50	11,17,29

FOUR GRADES N=70

P1	P2	P3	ACCEPT. NOS.	P1	P2	P3	ACCEPT. NOS.	P1	P2	P3	ACCEPT. NOS.
.01	.05	.10	0,3,6	.05	.10	.15	3,6,9	.10	.20	.25	6,14,16
		.15	0,2,10			.20	3,6,13			.30	6,13,20
		.20	0,2,13			.25	3,6,17			.35	6,13,24
		.25	0,2,17			.30	3,6,20			.40	6,13,27
		.30	0,2,20			.35	3,6,24			.45	6,13,31
		.35	0,2,24			.40	3,6,27			.50	6,13,34
		.40	0,2,27			.45	3,6,31				
		.45	0,2,31			.50	3,6,34		.25	.30	6,18,19
		.50	0,2,34							.35	6,17,24
					.15	.20	2,10,12			.40	6,17,27
	.10	.15	0,6,9			.25	2,10,17			.45	6,17,31
		.20	0,6,17			.35	2,10,24				
		.30	0,6,20			.40	2,10,27		.30	.35	6,22,23
		.35	0,6,24			.45	2,10,31			.40	6,20,27
		.40	0,6,27			.50	2,10,34			.45	6,20,31
		.45	0,6,31							.50	6,20,34
		.50	0,6,34		.20	.25	2,14,16				
						.30	2,13,20	.15	.20	.25	10,13,16
	.15	.20	0,10,12			.35	2,13,24			.30	10,12,20
		.25	0,10,17			.40	2,13,27			.35	10,12,24
		.30	0,10,20			.45	2,13,31			.40	10,12,27
		.35	0,10,24			.50	2,13,34			.45	10,12,31
		.40	0,10,27							.50	10,12,34
		.45	0,10,31		.25	.30	2,18,19				
		.50	0,10,34			.35	2,17,24		.25	.30	10,18,19
						.40	2,17,27			.35	10,17,24
	.20	.25	0,14,16			.45	2,17,31			.40	10,17,27
		.30	0,13,20			.50	2,17,34			.45	10,17,31
		.35	0,13,24							.50	10,17,34
		.40	0,13,27		.30	.35	2,22,23				
		.45	0,13,31			.40	2,20,27		.30	.35	10,22,23
		.50	0,13,34			.45	2,20,31			.40	10,20,27
						.50	2,20,34			.45	10,20,31
	.25	.30	0,18,19							.50	10,20,34
		.35	0,17,24								
		.40	0,17,27					.20	.25	.30	14,17,19
		.45	0,17,31	.10	.15	.20	6,10,12			.35	14,16,24
		.50	0,17,34			.25	6,9,17			.40	14,16,27
						.30	6,9,20			.45	14,16,31
	.30	.35	0,22,23			.35	6,9,24			.50	14,16,34
		.40	0,20,27			.40	6,9,27				
		.45	0,20,31			.45	6,9,31		.30	.35	13,21,22
		.50	0,20,34			.50	6,9,34			.40	13,20,27
										.45	13,20,31
										.50	13,20,34

FOUR GRADES N=80

P1	P2	P3	ACCEPT. NOS.
.01	.05	.10	0,3,7
		.15	0,3,11
		.20	0,3,15
		.25	0,3,19
		.30	0,3,23
		.35	0,3,27
		.40	0,3,31
		.45	0,3,35
		.50	0,3,39
	.10	.15	0,7,11
		.20	0,7,15
		.25	0,7,19
		.30	0,7,23
		.35	0,7,27
		.40	0,7,31
		.45	0,7,35
		.50	0,7,39
	.15	.20	0,12,14
		.25	0,11,19
		.30	0,11,23
		.35	0,11,27
		.40	0,11,31
		.45	0,11,35
		.50	0,11,39
	.20	.25	0,16,18
		.30	0,15,23
		.35	0,15,27
		.40	0,15,31
		.45	0,15,35
		.50	0,15,39
	.25	.30	0,20,22
		.35	0,19,27
		.40	0,19,31
		.45	0,19,35
		.50	0,1939
	.30	.35	0,24,26
		.40	0,23,31
		.45	0,23,35
		.50	0,23,39

P1	P2	P3	ACCEPT. NOS.
.05	.10	.15	3,7,11
		.20	3,7,15
		.25	3,7,19
		.30	3,7,23
		.35	3,7,27
		.40	3,7,31
		.45	3,7,35
		.50	3,7,39
	.15	.20	3,12,14
		.25	3,11,19
		.30	3,11,23
		.35	3,11,27
		.40	3,11,31
		.45	3,11,35
		.50	3,11,39
	.20	.25	3,16,18
		.30	3,15,23
		.35	3,15,27
		.40	3,15,31
		.45	3,15,35
		.50	3,15,39
	.25	.30	3,20,22
		.35	3,19,27
		.40	3,19,31
		.45	3,19,35
		.50	3,19,39
	.30	.35	3,24,26
		.40	3,23,31
		.45	3,23,35
		.50	3,23,39
.10	.15	.20	7,12,14
		.25	7,11,19
		.30	7,11,23
		.35	7,11,27
		.40	7,11,31
		.45	7,11,35
		.50	7,11,39

P1	P2	P3	ACCEPT. NOS.
.10	.20	.25	7,16,18
		.30	7,15,23
		.35	7,15,27
		.40	7,15,31
		.45	7,15,35
		.50	7,15,39
	.25	.30	7,20,22
		.35	7,19,27
		.40	7,19,31
		.45	7,19,35
		.50	7,19,39
	.30	.35	7,24,26
		.40	7,23,31
		.45	7,23,35
		.50	7,23,39
.15	.20	.25	12,15,18
		.30	12,14,23
		.35	12,14,27
		.40	12,14,31
		.45	12,14,35
		.50	12,14,39
	.25	.30	11,20,22
		.35	11,19,27
		.40	11,19,31
		.45	11,19,35
		.50	11,19,39
	.30	.35	11,24,26
		.40	11,23,31
		.45	11,23,35
		.50	11,23,39
.20	.25	.30	16,19,22
		.35	16,18,27
		.40	16,18,31
		.45	16,18,35
		.50	16,18,39
	.30	.35	15,24,26
		.40	15,23,31
		.45	15,23,35
		.50	15,23,39

FOUR GRADES N=90

P1	P2	P3	ACCEPT. NOS.
.01	.05	.10	0,4,8
		.15	0,3,13
		.20	0,3,17
		.25	0,3,22
		.30	0,3,26
		.35	0,3,31
		.40	0,3,35
		.45	0,3,40
		.50	0,3,44
	.10	.15	0,8,12
		.20	0,8,17
		.25	0,8,22
		.30	0,8,26
		.35	0,8,31
		.40	0,8,35
		.45	0,8,40
		.50	0,8,44
	.15	.20	0,13,16
		.25	0,13,22
		.30	0,13,26
		.35	0,13,31
		.40	0,13,35
		.45	0,13,40
		.50	0,13,44
	.20	.25	0,18,21
		.30	0,17,26
		.35	0,17,31
		.40	0,17,35
		.45	0,17,40
		.50	0,17,44
	.25	.30	0,23,25
		.35	0,22,31
		.40	0,22,35
		.45	0,22,40
		.50	0,22,44
	.30	.35	0,27,29
		.40	0,26,35
		.45	0,26,40
		.50	0,26,44
.05	.10	.15	4,8,12
		.20	4,8,17
		.25	4,8,22
		.30	4,8,26
		.35	4,8,31
		.40	4,8,35
		.45	4,8,40
		.50	4,8,44
	.15	.20	3,13,16
		.25	3,13,22
		.30	3,13,26
		.35	3,13,31
		.40	3,13,35
		.45	3,13,40
		.50	3,13,44
	.20	.25	3,18,21
		.30	3,17,26
		.35	3,17,31
		.40	3,17,35
		.45	3,17,40
		.50	3,17,44
	.25	.30	3,23,25
		.35	3,22,31
		.40	3,22,35
		.45	3,22,40
		.50	3,22,44
	.30	.35	3,27,29
		.40	3,26,35
		.45	3,26,40
		.50	3,26,44
.10	.15	.20	8,13,16
		.25	8,12,22
		.30	8,12,26
		.35	8,12,31
		.40	8,12,35
		.45	8,12,40
		.50	8,12,44
.10	.20	.25	8,18,21
		.30	8,17,26
		.35	8,17,31
		.40	8,17,35
		.45	8,17,40
		.50	8,17,44
	.25	.30	8,23,25
		.35	8,22,31
		.40	8,22,35
		.45	8,22,40
		.50	8,22,44
	.30	.35	8,27,29
		.40	8,26,35
		.45	8,26,40
		.50	8,26,44
.15	.20	.25	13,17,21
		.30	13,17,26
		.35	13,16,31
		.40	13,16,35
		.45	13,16,40
		.50	13,16,44
	.25	.30	13,23,25
		.35	13,22,31
		.40	13,22,35
		.45	13,22,40
		.50	13,22,44
	.30	.35	13,27,29
		.40	13,26,35
		.45	13,26,40
		.50	13,26,44
.20	.25	.30	18,23,25
		.35	18,22,31
		.40	18,22,35
		.45	18,22,40
		.50	18,22,44
	.30	.35	17,27,29
		.40	17,26,35
		.45	17,26,40
		.50	17,26,44

FOUR GRADES N=100

P1	P2	P3	ACCEPT. NOS.
.01	.05	.10	0,4,9
		.15	0,4,14
		.20	0,4,19
		.25	0,4,24
		.30	0,4,29
		.35	0,4,34
		.40	0,4,39
		.45	0,4,44
		.50	0,4,49
	.10	.15	0,9,14
		.20	0,9,19
		.25	0,9,24
		.30	0,9,29
		.35	0,9,34
		.40	0,9,39
		.45	0,9,44
		.50	0,9,49
	.15	.20	0,15,19
		.25	0,14,24
		.30	0,14,29
		.35	0,14,34
		.40	0,14,39
		.45	0,14,44
		.50	0,14,49
	.20	.25	0,20,23
		.30	0,19,29
		.35	0,19,34
		.40	0,19,39
		.45	0,19,44
		.50	0,19,49
	.25	.30	0,25,28
		.35	0,24,34
		.40	0,24,39
		.45	0,24,44
		.50	0,24,49
	.30	.35	0,29,33
		.40	0,29,39
		.45	0,29,44
		.50	0,29,49

P1	P2	P3	ACCEPT. NOS.
.05	.10	.15	4,9,14
		.20	4,9,19
		.25	4,9,24
		.30	4,9,29
		.35	4,9,34
		.40	4,9,39
		.45	4,9,44
		.50	4,9,49
	.15	.20	4,15,19
		.25	4,14,24
		.30	4,14,29
		.35	4,14,34
		.40	4,14,39
		.45	4,14,44
		.50	4,14,49
	.20	.25	4,20,23
		.30	4,19,29
		.35	4,19,34
		.40	4,19,39
		.45	4,19,44
		.50	4,19,49
	.25	.30	4,25,28
		.35	4,24,34
		.40	4,24,39
		.45	4,24,44
		.50	4,24,49
	.30	.35	4,29,33
		.40	4,29,39
		.45	4,29,44
		.50	4,29,49
.10	.15	.20	9,15,19
		.25	9,14,24
		.30	9,14,29
		.35	9,14,34
		.40	9,14,39
		.45	9,14,44
		.50	9,14,49

P1	P2	P3	ACCEPT. NOS.
.10	.20	.25	9,20,23
		.30	9,19,29
		.35	9,19,34
		.40	9,19,39
		.45	9,19,44
		.50	9,19,49
	.25	.30	9,25,28
		.35	9,24,34
		.40	9,24,39
		.45	9,24,44
		.50	9,24,49
	.30	.35	9,29,33
		.40	9,29,39
		.45	9,29,44
		.50	9,29,49
.15	.20	.25	15,19,23
		.30	15,19,29
		.35	15,19,34
		.40	15,19,39
		.45	15,19,44
		.50	15,19,49
	.25	.30	15,25,28
		.35	15,24,34
		.40	15,24,39
		.45	15,24,44
		.50	15,24,49
	.30	.35	14,29,33
		.40	14,29,39
		.45	14,29,44
		.50	14,29,49
.20	.25	.30	19,24,28
		.35	19,23,34
		.40	19,23,39
		.45	19,23,44
		.50	19,23,49
	.30	.35	19,29,33
		.40	19,29,39
		.45	19,29,44
		.50	19,29,49

FOUR GRADES N=125

P1	P2	P3	ACCEPT. NOS.	P1	P2	P3	ACCEPT. NOS.	P1	P2	P3	ACCEPT. NOS.
.01	.05	.10	0,5,11	.05	.10	.15	5,12,17	.10	.20	.25	11,25,29
		.15	0,5,18			.20	5,11,24			.30	11,24,36
		.20	0,5,24			.25	5,11,30			.35	11,24,43
		.25	0,5,30			.30	5,11,36			.40	11,24,49
		.30	0,5,36			.35	5,11,43			.45	11,24,55
		.35	0,5,43			.40	5,11,49			.50	11,24,61
		.40	0,5,49			.45	5,11,55				
		.45	0,5,55			.50	5,11,61		.25	.30	11,31,36
		.50	0,5,61							.35	11,31,43
					.15	.20	5,18,23			.40	11,31,49
	.10	.15	0,11,18			.25	5,18,30			.45	11,31,55
		.20	0,11,24			.30	5,18,36			.50	11,31,61
		.25	0,11,30			.35	5,18,43				
		.30	0,11,36			.40	5,18,49		.30	.35	11,37,42
		.35	0,11,43			.45	5,18,55			.40	11,37,49
		.40	0,11,49			.50	5,18,61			.45	11,37,55
		.45	0,11,55							.50	11,37,61
		.50	0,11,61		.20	.25	5,25,29				
						.30	5,24,36	.15	.20	.25	18,24,29
	.15	.20	0,18,23			.35	5,24,43			.30	18,24,36
		.25	0,18,30			.40	5,24,49			.35	18,24,43
		.30	0,18,36			.45	5,24,55			.40	18,24,49
		.35	0,18,43			.50	5,24,61			.45	18,24,55
		.40	0,18,49							.50	18,24,61
		.45	0,18,55		.25	.30	5,31,36				
		.50	0,18,61			.35	5,31,43		.25	.30	18,31,36
						.40	5,31,49			.35	18,31,43
	.20	.25	0,25,29			.45	5,31,55			.40	18,31,49
		.30	0,24,36			.50	5,31,61			.45	18,31,55
		.35	0,24,43							.50	18,31,61
		.40	0,24,49		.30	.35	5,37,42				
		.45	0,24,55			.40	5,37,49		.30	.35	18,37,42
		.50	0,24,61			.45	5,37,55			.40	18,37,49
						.50	5,37,61			.45	18,37,55
	.25	.30	0,31,36							.50	18,37,61
		.35	0,31,43								
		.40	0,31,49					.20	.25	.30	24,30,36
		.45	0,31,55	.10	.15	.20	12,18,23			.35	24,30,43
		.50	0,31,61			.25	11,18,30			.40	24,30,49
						.30	11,18,36			.45	24,30,55
	.30	.35	0,37,42			.35	11,18,43			.50	24,30,61
		.40	0,37,49			.40	11,18,49				
		.45	0,37,55			.45	11,18,55		.30	.35	24,37,42
		.50	0,37,61			.50	11,18,61			.40	24,37,49
										.45	24,37,55
										.50	24,37,61

FOUR GRADES N=150

P1	P2	P3	ACCEPT. NOS.
.01	.05	.10	0,6,14
		.15	0,6,21
		.20	0,6,29
		.25	0,6,36
		.30	0,6,44
		.35	0,6,51
		.40	0,6,59
		.45	0,6,66
		.50	0,6,74
	.10	.15	0,14,21
		.20	0,14,29
		.25	0,14,36
		.30	0,14,44
		.35	0,14,51
		.40	0,14,59
		.45	0,14,66
		.50	0,14,74
	.15	.20	0,22,29
		.25	0,22,36
		.30	0,22,44
		.35	0,22,51
		.40	0,22,59
		.45	0,22,66
		.50	0,22,74
	.20	.25	0,29,36
		.30	0,29,44
		.35	0,29,51
		.40	0,29,59
		.45	0,29,66
		.50	0,29,74
	.25	.30	0,37,43
		.35	0,37,51
		.40	0,37,59
		.45	0,37,66
		.50	0,37,74
	.30	.35	0,44,51
		.40	0,44,59
		.45	0,44,66
		.50	0,44.74

P1	P2	P3	ACCEPT. NOS.
.05	.10	.15	6,14,21
		.20	6,14,29
		.25	6,14,36
		.30	6,14,44
		.35	6,14,51
		.40	6,14,59
		.45	6,14,66
		.50	6,14,74
	.15	.20	6,22,29
		.25	6,22,36
		.30	6,22,44
		.35	6,22,51
		.40	6,22,59
		.45	6,22,66
		.50	6,22,74
	.20	.25	6,29,36
		.30	6,29,44
		.35	6,29,51
		.40	6,29,59
		.45	6,29,66
		.50	6,29,74
	.25	.30	6,37,43
		.35	6,37,51
		.40	6,37,59
		.45	6,37,66
		.50	6,37,74
	.30	.35	6,44,51
		.40	6,44,59
		.45	6,44,66
		.50	6,44,74
.10	.15	.20	14,22,29
		.25	14,22,36
		.30	14,22,44
		.35	14,22,51
		.40	14,22,59
		.45	14,22,66
		.50	14,22,74

P1	P2	P3	ACCEPT. NOS.
.10	.20	.25	14,29,36
		.30	14,29,44
		.35	14,29,51
		.40	14,29,59
		.45	14,29,66
		.50	14,29,74
	.25	.30	14,37,43
		.35	14,37,51
		.40	14,37,59
		.45	14,37,66
		.50	14,37,74
	.30	.35	14,44,51
		.40	14,44,59
		.45	14,44,66
		.50	14,44,74
.15	.20	.25	21,29,36
		.30	21,29,44
		.35	21,29,51
		.40	21,29,59
		.45	21,29,66
		.50	21,29,74
	.25	.30	21,37,43
		.35	21,37,51
		.40	21,37,59
		.45	21,37,66
		.50	21,37,74
	.30	.35	21,44,51
		.40	21,44,59
		.45	21,44,66
		.50	21,44,74
.20	.25	.30	29,37,43
		.35	29,37,51
		.40	29,37,59
		.45	29,37,66
		.50	29,37,74
	.30	.35	29,44,51
		.40	29,44,59
		.45	29,44,66
		.50	29,44,74

FOUR GRADES N=200

P1	P2	P3	ACCEPT. NOS.
.01	.05	.10	1,9,19
		.15	1,9,29
		.20	1,9,39
		.25	1,9,49
		.30	1,9,59
		.35	1,9,69
		.40	1,9,79
		.45	1,9,89
		.50	1,9,99
	.10	.15	1,19,29
		.20	1,19,39
		.25	1,19,49
		.30	1,19,59
		.35	1,19,69
		.40	1,19,79
		.45	1,19,89
		.50	1,19,99
	.15	.20	1,29,39
		.25	1,29,49
		.30	1,29,59
		.35	1,29,69
		.40	1,29,79
		.45	1,29,89
		.50	1,29,99
	.20	.25	1,39,49
		.30	1,39,59
		.35	1,39,69
		.40	1,39,79
		.45	1,39,89
		.50	1,39,99
	.25	.30	1,49,59
		.35	1,49,69
		.40	1,49,79
		.45	1,49,89
		.50	1,49,99
	.30	.35	1,59,69
		.40	1,59,79
		.45	1,59,89
		.50	1,59,99
.05	.10	.15	9,19,29
		.20	9,19,39
		.25	9,19,49
		.30	9,19,59
		.35	9,19,69
		.40	9,19,79
		.45	9,19,89
		.50	9,19,99
	.15	.20	9,29,39
		.25	9,29,49
		.30	9,29,59
		.35	9,29,69
		.40	9,29,79
		.45	9,29,89
		.50	9,29,99
	.20	.25	9,39,49
		.30	9,39,59
		.35	9,39,69
		.40	9,39,79
		.45	9,39,89
		.50	9,39,99
	.25	.30	9,49,59
		.35	9,49,69
		.40	9,49,79
		.45	9,49,89
		.50	9,49,99
	.30	.35	9,59,69
		.40	9,59,79
		.45	9,59,89
		.50	9,59,99
.10	.15	.20	19,29,39
		.25	19,29,49
		.30	19,29,59
		.35	19,29,69
		.40	19,29,79
		.45	19,29,89
		.50	19,29,99
	.20	.25	19,39,49
		.30	19,39,59
		.35	19,39,69
		.40	19,39,79
		.45	19,39,89
		.50	19,39,89
	.25	.30	19,49,59
		.35	19,49,69
		.40	19,49,79
		.45	19,49,89
		.50	19,49,99
	.30	.35	19,59,69
		.40	19,59,79
		.45	19,59,89
		.50	19,59,99
.15	.20	.25	29,39,49
		.30	29,39,59
		.35	29,39,69
		.40	29,39,79
		.45	29,39,89
		.50	29,39,99
	.25	.30	29,49,59
		.35	29,49,69
		.40	29,49,79
		.45	29,49,89
		.50	29,49,99
	.30	.35	29,59,69
		.40	29,59,79
		.45	29,59,89
		.50	29,59,99
.20	.25	.30	39,49,59
		.35	39,49,69
		.40	39,49,79
		.45	39,49,89
		.50	39,49,99
	.30	.35	39,59,69
		.40	39,59,79
		.45	39,59,89
		.50	39,59,99

FOUR GRADES N=300

P1	P2	P3	ACCEPT. NOS.	P1	P2	P3	ACCEPT. NOS.	P1	P2	P3	ACCEPT. NOS.
.01	.05	.10	2,14,29	.05	.10	.15	14,29,44	.10	.20	.25	29,59,74
		.15	2,14,44			.20	14,29,59			.30	29,59,89
		.20	2,14,59			.25	14,29,74			.35	29,59,104
		.25	2,14,74			.30	14,29,89			.40	29,59,119
		.30	2,14,89			.35	14,29,104			.45	29,59,134
		.35	2,14,104			.40	14,29,119			.50	29,59,149
		.40	2,14,119			.45	14,29,134				
		.45	2,14,134			.50	14,29,149		.25	.30	29,74,89
		.50	2,14,149							.35	29,74,104
					.15	.20	14,44,59			.40	29,74,119
	.10	.15	2,29,44			.25	14,44,74			.45	29,74,134
		.20	2,29,59			.30	14,44,89			.50	29,74,149
		.25	2,29,74			.35	14,44,104				
		.30	2,29,89			.40	14,44,119		.30	.35	29,89,104
		.35	2,29,104			.45	14,44,134			.40	29,89,119
		.40	2,29,119			.50	14,44,149			.45	29,89,134
		.45	2,29,134							.50	29,89,149
		.50	2,29,149		.20	.25	14,59,74				
						.30	14,59,89	.15	.20	.25	44,59,74
	.15	.20	2,44,59			.35	14,59,104			.30	44,59,89
		.25	2,44,74			.40	14,59,119			.35	44,59,104
		.30	2,44,89			.45	14,59,134			.40	44,59,119
		.35	2,44,104			.50	14,59,149			.45	44,59,134
		.40	2,44,119							.50	44,59,149
		.45	2,44,134		.25	.30	14,74,89				
		.50	2,44,149			.35	14,74,104		.25	.30	44,74,89
						.40	14,74,119			.35	44,74,104
	.20	.25	2,59,74			.45	14,74,134			.40	44,74,119
		.30	2,59,89			.50	14,74,149			.45	44,74,134
		.35	2,59,104							.50	44,74,149
		.40	2,59,119		.30	.35	14,89,104				
		.45	2,59,134			.40	14,89,119		.30	.35	44,89,104
		.50	2,59,149			.45	14,89,134			.40	44,89,119
						.50	14,89,149			.45	44,89,134
	.25	.30	2,74,89							.50	44,89,149
		.35	2,74,104	.10	.15	.20	29,44,59				
		.40	2,74,119			.25	29,44,74	.20	.25	.30	59,74,89
		.45	2,74,134			.30	29,44,89			.35	59,74,104
		.50	2,74,149			.35	29,44,104			.40	59,74,119
						.40	29,44,119			.45	59,74,134
	.30	.35	2,89,104			.45	29,44,134			.50	59,74,149
		.40	2,89,119			.50	29,44,149				
		.45	2,89,134						.30	.35	59,89,104
		.50	2,89,149							.40	59,89,119
										.45	59,89,134
										.50	59,89,149

THE ABBA SEQUENCE:

A PROCEDURE FOR COMPARISON TESTING

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This paper introduces, if not a novel* concept, certainly a new acronym: ABBA, more precisely, A-B-B-A.

ABBA is an acronym and, as will be seen shortly, a mnemonic term. The ABBA sequence is discussed here as an alternative to the AB method. The latter term describes an accepted and effective comparison procedure - repeated alternation. It is the sequential procedure usually followed in comparing representative items of two batches, A and B. The two syllables formed by the letter sequence A-B-B-A, may be vocalized, although "ABBA" is not an English word. The letters, ABBA however, show the critical difference in the implied pattern. In contrast to the unidirectional A-B-A-B, etc., ABBA is an iterative doubling back. In a sequence of four operations, let two each be applied to two populations, A and B. Then the sequence looks like:

	A	B
	1	2
	4	3
SUMS:	5	5

(a)

(NOTE: The numbers in the columns, the k_i's are, strictly speaking, ordinal rather than cardinal numbers.)

*We learned at this Conference, at the lunch table, to be exact, that what we call the ABBA sequence was applied at the National Bureau of Standards many years ago.

The A-B comparison process is not periodic or cyclic. It starts at the left and moves right; then again at the left, thence to the right, viz.:

	A	B
	1	2
	3	4
SUMS:	4	6

(b)

The purpose of the ABBA sequence is to improve confidence limits. These confidence limits are not of the fiducial, or statistical variety. Rather, these limits are of the human variety, and refer to the confidence of three groups in the experimental data of mutual interest. These groups are: the experimenters, the technicians and, of course, the statisticians.

The procedure we propose will be recognized in its fundamental logic as related to the statistical principle of blocking. This blocking principle is exemplified in Latin and Graeco-Roman squares or similar planned arrays* of experimental data. It is our view that our procedure is a prior fundamental. It tells how to obtain the data which is later treated better, from the statistical viewpoint. The experiential basis of the proposal may be singularly our own, but we doubt this very much.

Our underlying postulates are these:

1. Measuring a physical property, injecting a chemical, or shooting a sample of ammunition is equivalent, sui generis, to experimental treatment. Hence, plural measurements (treatments) and population samples are combinations.

*See, for example, the Youden rectangle concept.

2. The uncertainties of temperament, temperature, and time, give rise to sources of error, bias, and sequential or cumulative effects.

3. Firing a gun, of any caliber, inserts heat into a dynamic system. Thermal energy transfer may cause changes in such kinetic parameters as velocity, yawing motion, or recoiling motion. These effects are known in ballistics. Hence, we view each shot as a treatment.

4. Planting seeds in each of several plots (≈ sampling each ground lot) is also a treatment. Let all its' seeds be planted first in one plot, and then its' seeds be planted in the second area of soil. Our view is that the seed of the first plot was "treated" differently. It might be in colder soil longer, or have more time to absorb initial moisture. Also, during the planting of Plot 1, the "planter" may have lost or increased its tension. The "planter" may be a human being, or a mechanical device incorporating control cables and springs. Tension is still tension.

5. Therefore, it is desirable that similar times shall have elapsed during the seeding (treatment) of all three plots of ground.

6. As a first approximation, the sums of the ordinal integers, in plain English - the step numbers, should be as nearly equal as we choose. The equalities may be required at any time during or after the experiment (see (a) on the first page of this article).

7. A first choice is that the sums of the ordinal numbers (the cumulative sum of the sequential positions) at the end of an experimental interval shall be equal. If columns are lots and rows are samples, then the sums of columns A, B, etc. (Σk_1) should be equal*.

*Epilogue: We were pleased to hear Dr. Youden recall how he had once worked on this equality of column sums and had found the attempt had been made for another purpose in an old math book.

8. More generally, we may prefer that the total time intervals of the preceding treatments, or sequence summations, are equal. This would mean that the sums of the columns across as many rows, from the second row to the last, should be equal. We shall see that we can have this half the time.

From the foregoing, a basic value judgment can be inferred. In an ABAB comparison, the time intervals within a column are equal. These are the time intervals between successive samples within the group or lot. The test samples of A:-1, 3, 5, and B:-2, 4, 6, have equal chances of something going wrong, intime, within the groups, A and B. But the environment, equipment and personnel are also subject to error-random or otherwise. We choose to equalize the error sources - time, temperature, and, psychologically, temperament. These may affect the sample behavior more than its standby time.

Finally, if we study the array in (d) below, we see a singular difference between ABBA and ABAB. Both are alike in that samples precede and follow the others, one treatment at a time (A-1 precedes B-2; B-2 follows A1; and precedes A-3, etc.). But in the ABBA sequence, equal members of treatments in <u>both</u> columns precede and follow another treatment (position or ordinal integer) <u>within the column</u>. In brief (cf (d)), there are <u>pairs</u> in the columns.

Our value judgment of vertical pairing for achieving better balance, i.e., less cumulative, sequential bias, is supported by C. C. Li (1):

"The criterion for balanced sequences is that every treatment is preceded or followed by all the other treatments, the same number of times."

The foregoing is cited as an advantage of alternate pairing with only two populations or lots (A and B). With three or more lots, (columns), the

pairing is found only in the extreme or "doubling-back" columns, i.e., under A and C only:

	A	B	C	
	1	2	3	
	6	5	4	(c)
	7	8	9	
	12	11	10	
Σk_i	26	26	26	

ON THE ASPECT OF SAMPLE SIZE

If the sample number per lot, r, is very large, the difference in the sums of the ordinal numbers under A and B (Σk_i) becomes relatively small. If the total number is forty, twenty samples per column, or lot, then dalet* 40 or $\lhd 40 = \frac{40(41)}{2} = 820$. Since every element in the B column is one greater than that in the same row in A, $\Sigma A_i + 20 = \Sigma B_i$, and $\Sigma A_i = 400$, $\Sigma B_i = 420$. The final difference, 20, is only 5 percent of the ΣA_i. For N = 60, the percentage difference is even less. Hence, for the final cumulative effect, an A-B-A-B may be just as good as an ABBA sequence. But, who has tested or compared thirty pairs of experimental Nike-Hercules motors in one day? - under "steady-state" conditions? - with a priori certainity that the experiment will be completed?

The advantage of ABBA comes when there are interruptions, either unforeseen or scheduled.

THE ABBA SEQUENCE AND THE DIAGONAL SEQUENCE

As indicated previously, the ABBA sequence has several features of interest. For two samples, usually a standard and an experimental sample,

*See Dalet N and the ABBA Sequence, below.

the ABBA sequence is an iteration of the staggered diagonal cycle. This is seen in the following array:

	A	B
	1	2
	4	3
	5	6
	8	7
	9	10
	12	11
Σk_1	39	39

(d)

In the two column array above, starting at A and ending at B is the first swing. Starting the next cycle at B, the 2nd column, and completing the cycle by advancing to the next vacancy in the row, terminates at A. If we have a series of columns, k, equal in number to the rows, r, we can have a staggered cycling sequence which provides a diagonal of starting points, e.g.:

CASE 5 X 5

	A	B	C	D	E
	1	2	3	4	5
	10	6	7	8	9
	14	15	11	12	13
	18	19	20	16	17
	22	23	24	25	21
Σk_1	65	65	65	65	65

(e)

The diagonal (1, 6, 11, 16, 21) gives us an r X k square.

Thus, if r = k, even if r is odd, we can attain the desired equality of final column sums. This staggered cycling, or diagonal inception of

each succeeding row, is found in Youden's rectangle.

For those who may insist on the repeated alternating cycle (A-B, A-B), for whatever reasons, the following is reassuring: A <u>two-population ABBA sequence</u> is nearly a repeated A-B. View the second and third (under B) as one sample (of two items) and the fourth and fifth steps (under A) as the other sample of two items. We are then testing <u>alternate pairs</u>. The difference is that we begin with singleton A (A_1), and end with singleton B (B_N), when the sample number for A equals that of B.

(Recall:

	A	B		
(singleton)	1	2)	"A"	
"B"	(4	3)		
	(5	6)	"A"	(f)
"B"	(8	7)		
	(9	10	(singleton)	

Note that the ABBA sequences of (d) and (f) above, give equal sums of the ordinal numbers at every even-numbered row. This equality holds for any number of columns.

If, as is often the case, r is much greater than k, we have another problem. We can form successive k by k squares, as a choice. Then, at the very best, we have a series of squares, at least 3 x 3. In these 3 x 3 squares the ordinal sums are equal only every third row. If such a series of odd-sided squares has a total number of rows which is even, it would of course be better to use ABBA. Then every second row is equal, including the last. If both r and k are odd and r doesn't divide into k, what do we do? Let k = 3, r = 5. A 3 by 3 diagonal, plus a doublet (ABC - CBA) gives equal sums at stage r = 3 and r = 5 (g) or at r = 2 and r = 5 (h).

CASE 3 X 5:

(g)	A	B	C	(h)	A	B	C
	1	2	3		1	2	3
	6	4	5		6	5	4
	8	9	7		7	8	9
Σk_1 -	15	15	15		14	15	16
	10	11	12		12	10	11
	15	14	13		14	15	13
Σk_1 -	40	40	40		40	40	40

Here we have the compromise or combination of diagonal and ABBA cycles.

DALET N AND THE ABBA SEQUENCE

A new symbol is appropriate for indicating the sum of the ordinal numbers of the total samples available for an experiment. The symbol we propose is ◅ , dalet*. Dalet is a triangle, like its Greek descendant, delta, but dalet points from right to left. It is applicable to both letter ◅N and to number ◅10 as a symbol of summation.

The sum of an arithmetic progression of the integers from zero to an indefinite integer, N, is obtained from the equation:

$$\sum_{i=1}^{i=N} i = \frac{N(N+1)}{2} = \blacktriangleleft N \qquad (1)$$

Hence, dalet N, or dalet any number is a "triangular number" in Pascals Triangle. The symbol dalet and equation (1) are useful for determining if the sums of the columns of ordinal numbers can be equalized.

* dalet (pronounced dah-let) is the name of the ancient Hebrew letter which is fourth in the alphabet. It is the precursor of the Greek "delta." Dalet means "a tent flap." Later, it came to mean "a door."

Let us distribute the N integers from one to N, in rows (r) and columns (k), with equal numbers of integers in each column. Then ◅N, or "dalet" N = $\frac{kr\ (kr + 1)}{2}$ (2)

In two columns of ten rows, the arithmetic progression, 1 to 20, sums to: $2 \cdot 10\ (\frac{(2 \cdot 10) + 1}{2})$ or 210. (◅20 = 210). (3)

To determine if there can be equality of sums of individual columns, of equal rows each, it is necessary to use formula (2) and let k be odd:

Then $2|k^{**} \cdot r\ (kr+1)$ implies $2|k$ or $2|r\ (kr+1)$. (4)

But 2 does not divide into k ($2 \nmid k$);

$\therefore 2|r\ (kr+1)$, and some integer, $S = \frac{[r\ (kr+1)]}{2}$. (5)

So that $\frac{kr\ (kr+1)}{2} = k \cdot S$ (6)

$\therefore k|\frac{kr\ (kr+1)}{2}$ (7)

An array of odd-numbered columns gives a sum that yields an integer quotient regardless of whether rows are odd or even.

$$k|k \cdot \frac{r}{2} \cdot (kr+1) \quad (8)$$

If the rows (r) are odd, and the columns (k) are even, then

$$k \nmid \frac{kr\ (kr+1)}{2} \quad \text{(k does not divide into ...etc.)} \quad . \quad (9)$$

If kr is even, then kr+1 is odd and $2 \nmid kr+1$, since $2|kr$ and $2 \nmid r$ (r is odd); then

$\frac{r\ (kr+1)}{2}$ is not an integer.

** a|b : a divides into b.

Therefore, odd rows do not automatically yield equal sums of columns, if the columns be even. Examples:

1	2
4	3
5	6
10	11

$\sum 20 = 210 = \frac{rk\,(rk+1)}{2}$

1	2	3	4
8	7	6	5
9	10	11	12
16	15	14	13
17	18	19	20

$\frac{210}{4}$ is not an integer.

The reader will note that the foregoing treatment applies to the case of equal sample numbers, or rows, however small, for each population sample or column. The same procedure, however, can be applied to a group of unequal samples, i.e.,

1. Determine N, the total number of samples.
2. Determine $\sum N$.
3. Divide by k, the number of columns or populations or lots. If the quotient is an integer, the sums of the ordinal numbers can be equal for all columns or lots.

ON CASES OF UNEQUAL SAMPLE SIZES

For statistical inferences based on application of Student's t and the t-like (t*) statistics, it has been shown that both statistics have the same value when two samples are of the same size. The mathematical expression for df* (degrees of freedom for the t-like or t* statistic) simplifies considerably when $n_1 = n_2$. Further, when the two groups are of equal size, the value of df* reduces to 2(n - 1), wholesomely large, if the variances[1] of the two groups S_1^2 and S_2^2 are equal[2]. This is taken to

(1) More precisely, this is the estimate of the variance with df = n - 1.
(2) These comments can be explored in Reference 1 (L1).

mean that when group variances are unequal it is even more desirable to have groups of equal size. In a planned experiment, therefore, hope for equal numbers of samples.

Let us suppose, however, that the samples are small in number, and of unequal size. This situation happens when the test items are expensive, experimental, or exotic. Another reason for unequal numbers of observations of samples may be the exigencies of time. What is the simplest rule for any number of plots, blocks, or columns, when the numbers of rows or samples per lot are unequal? A uniform procedure would be to start with the sample of largest number.

CASE 5 - 4 - 3 shows how the equality of final sums requires abandoning a partial square (a) with its diagonal 3 X 3 array:

(a)	A	B	C	(b)	A	B	C
	1				1		
	2	3			2	3	
	4	5	6		6	4	5
	9	7	8		7	8	9
	11	12	10		10	11	12
Σk_i	27	27	24		26	26	26

Note that ◅N is divisible by k.

Case b. illustrates that the use of dalet* or ◅N, here ◅12, is the first order of business. Second is the injunction evident in Case a. also, - use up the surplus first!

*Precursor of the Greek letter delta is this ancient Hebrew form of the letter "dalet" (modern type ד).

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ON EXPECTED PROBABILITIES OF MISCLASSIFICATION IN DISCRIMINANT ANALYSIS, NECESSARY SAMPLE SIZE, AND A RELATION WITH THE MULTIPLE CORRELATION COEFFICIENT

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INTRODUCTION AND SUMMARY. When a sample discriminant function is computed, it is desired to estimate the chance of misclassification using this discriminant function. This is often done by classifying the sample using the sample discriminant function or by computing $\Phi(-D/2)$ where Φ is the cumulative normal distribution, and D^2 is Mahalanobis' distance. When the sample discriminant function is applied to a new sample, the observed probabilities of misclassification are usually found to be greater than those computed from the initial sample.

The purposes of this paper are to show that this increase in the probabilities of misclassification are directly related to the "shrinkage" of R^2 in new samples and that these are related to the unbiased estimation of Mahalanobis' δ^2 using D^2.

DISCRIMINANT ANALYSIS. Discriminant analysis provides a method of obtaining a function of a set of p multivariate observations which provides maximum separation between groups. In this paper we shall be concerned only with the case of two groups. Let π_1 denote the first population, π_2 the second, $\underline{x} = (x_1, x_2, \ldots, x_p)'$ be a column vector of observations, μ_k the mean vector in the kth group ($k = 1,2$), Σ the common covariance matrix, and $\bar{\underline{x}}_k$ and S the sample means and covariances.

It is well known that the sample discriminant function for discriminating two groups is

$$D_s(x) = (\underline{x} - (1/2)(\bar{\underline{x}}_1 + \bar{\underline{x}}_2))' S^{-1}(\bar{\underline{x}}_1 - \bar{\underline{x}}_2) \tag{1}$$

which is conditionally (on $\bar{x}_1$, $\bar{x}_2$, and S) normally distributed and has mean (in the kth group),

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(2) $$D_s(\underline{\mu}_k) = (\underline{\mu}_k - \tfrac{1}{2}(\bar{\underline{x}}_1+\bar{\underline{x}}_2))' S^{-1}(\bar{\underline{x}}_1-\bar{\underline{x}}_2)$$

and variance (in either group),

(3) $$V_D = (\bar{\underline{x}}_1-\bar{\underline{x}}_2)' S^{-1} \Sigma S^{-1}(\bar{\underline{x}}_1-\bar{\underline{x}}_2).$$

If it is known that the chance of an individual, randomly selected from the population, has probability q of belonging to group 1 and 1-q of belonging to group 2, then the classification rule that is used is "classify $\underline{x}$ into π_1 if

(4) $$D_s(\underline{x}) + \log \frac{q}{1-q} > 0$$

and into π_2 otherwise."

In this paper we will assume $q = .5$ so $\log \frac{q}{1-q} = 0$.

If $\underline{x}$ is multivariate normally distributed, then the probability of misclassifying $\underline{x}$ conditional on $\bar{\underline{x}}_1$, $\bar{\underline{x}}_2$, and S is

(5) $$P_1 = P(D_s(\underline{x}) < 0 | \underline{x} \in \pi_1)$$

or $$P_2 = P(D_s(\underline{x}) > 0 \ \underline{x} \in \pi_2) .$$

P_1 is given by

(6) $$P_1 = \Phi(-D_s(\underline{\mu}_1)/\sqrt{V_D})$$

and a similar expression holds for P_2, where Φ is the cumulative normal distribution.

Estimating $D_s(\underline{\mu}_k)$ by $\bar{D}_k = \sum_{\alpha=1}^{n_k} D_s(\underline{x}_{k\alpha})/n_k$ and V_D by

$\hat{V}_D = \sum_{i=1}^{2} \sum_{\alpha}^{n_k} (D_s(\underline{x}_{k\alpha})-\bar{D}_k)^2 \Big/ (n_1+n_2-2)$ is equivalent to estimating $\underline{\mu}_k$ by $\bar{\underline{x}}_k$ and Σ by S.

Thus we obtain

$$(7) \qquad P_1^* = \Phi(-D^2/2 \,/\, D) = \Phi(-D/2) = P_2^*$$

where $D^2 = (\bar{\underline{x}}_1 - \bar{\underline{x}}_2)' S^{-1} (\bar{\underline{x}}_1 - \bar{\underline{x}}_2)$ is Mahalanobis' distance. This is a biased estimate of P_1 and gives too favorable an estimate of P_1.

Thus, in general, we should expect to find a higher rate of misclassification when applying the sample discriminant function to new data than indicated by $\Phi(-D/2)$.

It is of some interest to consider the expectation of $D_s(\underline{\mu}_k)$ and V_D over repeated samples of size n_1 and n_2. We shall need the expectations of $\underline{S}^{-1}$ and $\underline{S}^{-1}\underline{\Sigma}\underline{S}^{-1}$. Lachenbruch and Mickey [1965] have shown that

$$(8) \qquad E(\underline{S}^{-1}) = \frac{n_1+n_2-2}{n_1+n_2-p-3} \underline{\Sigma}^{-1} = C_1 \underline{\Sigma}^{-1}$$

and

$$E(\underline{S}^{-1}\underline{\Sigma}\underline{S}^{-1}) = \underline{\Sigma}^{-1} \frac{(n_1+n_2-3)(n_1+n_2-2)^2}{(n_1+n_2-p-2)(n_1+n_2-p-3)(n_1+n_2-p-5)} = C_2 \underline{\Sigma}^{-1}$$

Now,

$$(9) \qquad E(D_s(\underline{\mu}_k)) = \operatorname{tr} E(D_s(\underline{\mu}_k))$$

$$= \operatorname{tr} E((\bar{\underline{x}}_1 - \bar{\underline{x}}_2)(\underline{\mu}_k - \tfrac{1}{2}(\bar{\underline{x}}_1 + \bar{\underline{x}}_2))' \underline{S}^{-1})$$

$$= \operatorname{tr} E\{((\bar{\underline{x}}_1 - \bar{\underline{x}}_2)\underline{\mu}_k - \tfrac{1}{2}\bar{\underline{x}}_1\bar{\underline{x}}_1' + \tfrac{1}{2}\bar{\underline{x}}_2\bar{\underline{x}}_2')\underline{\Sigma}^{-1} C_1$$

$$= \operatorname{tr}\{(\underline{\mu}_1 - \underline{\mu}_2)\underline{\mu}_k' \underline{\Sigma}^{-1} - \tfrac{1}{2}\left(\frac{(n_2-n_1)}{n_1 n_2}\right)\underline{I} - \tfrac{1}{2}(\underline{\mu}_1 + \underline{\mu}_2)\ \underline{\Sigma}^{-1}(\underline{\mu}_1 - \underline{\mu}_2)\}C_1$$

$$= \left[\frac{(\underline{\mu}_1 - \underline{\mu}_2)'\underline{\Sigma}^{-1}(\underline{\mu}_1 - \underline{\mu}_2)}{2}(-1)^{k+1} - \frac{p(n_2-n_1)}{2n_1 n_2}\right] C_1$$

$$= \frac{C_1}{2}\left[\delta^2(-1)^{k+1} - \frac{p(n_2-n_1)}{n_1 n_2}\right].$$

Similarly, we have for

$$
\begin{aligned}
(10)\qquad E(V_D) &= E((\bar{\underline{x}}_1-\bar{\underline{x}}_2)'\underline{S}^{-1}\underline{\Sigma}\underline{S}^{-1}(\bar{\underline{x}}_1-\bar{\underline{x}}_2)) \\
&= \operatorname{tr} E((\bar{\underline{x}}_1-\bar{\underline{x}}_2)(\bar{\underline{x}}_1-\bar{\underline{x}}_2)'\underline{S}^{-1}\underline{\Sigma}\underline{S}^{-1}) \\
&= \operatorname{tr} E((\bar{\underline{x}}_1-\bar{\underline{x}}_2)(\bar{\underline{x}}_1-\bar{\underline{x}}_2)'\underline{\Sigma}^{-1})C_2 \\
&= \operatorname{tr}\left[(\underline{\mu}_1-\underline{\mu}_2)(\underline{\mu}_1-\underline{\mu}_2)' + \left(\frac{1}{n_1}+\frac{1}{n_2}\right)\underline{\Sigma}\right]\underline{\Sigma}^{-1}C_2 \\
&= C_2\left[(\underline{\mu}_1-\underline{\mu}_2)'\underline{\Sigma}^{-1}(\underline{\mu}_1-\underline{\mu}_2) + \frac{p(n_1+n_2)}{n_1n_2}\right].
\end{aligned}
$$

Thus,

$$
(11)\qquad E(D_s(\underline{\mu}_k)) = \frac{n_1+n_2-2}{2(n_1+n_2-p-3)}\left[\delta^2(-1)^{k+1} - \frac{p(n_2-n_1)}{n_1n_2}\right]
$$

$$
E(V_D) = \left\{\delta^2 + \frac{p(n_1+n_2)}{n_1n_2}\right\}\frac{(n_1+n_2-3)(n_1+n_2-2)^2}{(n_1+n_2-p-2)(n_1+n_2-p-3)(n_1+n_2-p-5)}.
$$

Although $D_s(x)$ is normally distributed conditionally on $\bar{\underline{x}}_1$, $\bar{\underline{x}}_2$ and S, it is not unconditionally normally distributed. For n_1 and n_2 sufficiently large, the unconditional distribution is very close to normal. Thus, considering the values of

$$
(12)\qquad \overline{P_1} = \Phi\left(E(-D_s(\underline{\mu}_1))\Big/\sqrt{E(V_D)}\right)
$$

$$
\text{and}\qquad \overline{P_2} = \Phi\left(E(D_s(\underline{\mu}_2))\Big/\sqrt{E(V_D)}\right)
$$

will supply approximate values of P_1 and P_2 for samples of sizes n_1 and n_2.

There are three error rates of interest:

(a) The error rate for the particular sample discriminant function. This is given by (6).

(b) The expected error rate over all samples of size n_1, n_2. This is given by (12).

(c) The error rate that would hold if we knew the parameters of the distribution. This is given by $\Phi(-\delta/2)$.

Some properties of equations (9), (10), and (12) are of interest. First, if $n_1 = n_2$, $|E(D_s(\mu_k))| > \delta^2/2$. For large n_1, n_2, $E(D_s(\mu_k)) \to \delta^2/2(-1)^{k+1}$. In general, the variance of the sample discriminant function is always greater than the variance of the population discriminant function. The properties of C_1 and C_2 imply that

a) If $n_1 = n_2$, $|E(D_s(\mu_k)|/\sqrt{E(V_D)} < \delta/2$.

b) If n_1/n_2 is large or n_2/n_1 is large and δ is small, then one of $|E(D_s(\mu_k))|/\sqrt{E(V_D)}$ will be $> \delta/2$ and the other will be $< \delta/2$.

c) In most circumstances we will have $|E(D_s(\mu_k))|/\sqrt{E(V_D)} < \delta/2$, so we may conclude that the probability of misclassification in either group is greater than the optimum, $\Phi(-\delta/2)$.

Table 1 gives examples of the ratio $E(D_s(\mu_k))/\sqrt{E(V_D)}$ for various values of δ, n_1, n_2 and p.

Table 1. Ratios Used in Calculating Error Rates

p	n_1	n_2	δ^2	$E(D_s(\mu_1))/\sqrt{E(V_D)}$	$E(D_s(\mu_2))/\sqrt{E(V_D)}$	$\delta/2$
2	6	6	1	.3086	-.3086	.5
2	6	6	4	.7377	-.7377	1.0
2	4	20	1	.2189	-.5108	.5
2	4	20	4	.7747	-.9469	1.0
4	12	12	1	.3368	-.3368	.5
4	12	12	4	.8051	-.8051	1.0
4	4	20	1	.0586	-.5277	.5
4	4	20	4	.6102	-.9153	1.0
10	30	30	1	.3478	-.3478	.5
10	30	30	4	.8313	-.8313	1.0
10	10	50	1	.0605	-.5448	.5
10	10	50	4	.6300	-.9450	1.0

Finally, we note that we may use the unbiased estimate of δ^2 based on D^2 to obtain

$$\widehat{E(D_s(\mu_1))} = D^2/2 - C_1\, p/n_1 \tag{13}$$

$$\widehat{E(D_s(\mu_2))} = -D^2/2 + C_1\, p/n_2$$

$$\widehat{V_D} = D^2 \frac{(n_1+n_2-3)(n_1+n_2-2)}{(n_1+n_2-p-2)(n_1+n_2-p-5)}$$

Thus we obtain an estimate of P_1 for the discriminant function based on samples of size n_1, n_2:

$$\hat{P}_1 = \Phi(-(D^2/2 - C_1 P/n_1) \,/\, \sqrt{\hat{V}_D})$$

which is always greater than $\Phi(-D/2)$.

Sample Size for Discriminant Functions

The above results may be used to determine the sample size required to obtain error rates within a given tolerance of the optimum. The question we ask is "How large should n_1 and n_2 be for the sample discriminant function to have an error rate within γ of the optimum value?" The answer depends on p, γ, and δ^2. For equal sample sizes the results are given in table 2.

From table 2, we see that $\gamma = .1$ yields very small sample sizes, while $\gamma = .01$ causes large samples to be taken. The larger the separation between the groups, the smaller the sample size needed. As p increases, the sample size also increases, but the ratio n/p decreases for fixed δ^2 and γ.

Because of the non-linear relation between n_1, n_2, p, δ^2 and $\bar{P}_1$, for $\gamma = .1$ we find that a larger sample is needed for $\delta^2 = 4$ than for $\delta^2 = 1$.

Table 2. Minimum Sample Size, $n(=n_1=n_2)$, in Each Group Required for Expected Error Rate, $\overline{P_1}$, to be Within γ of Optimum Error Rate, P_1, for Various Number of Parameters, p.

p	δ^2	P_1	γ	$\overline{P_1}$	n
2	1	.309	.1	.395	5
	4	.159	.1	.256	5
	9	.067	.1	.151	5
	1	.309	.05	.356	9
	4	.159	.05	.206	8
	9	.067	.05	.111	7
	1	.309	.01	.318	47
	4	.159	.01	.169	32
	9	.067	.01	.077	22
4	1	.309	.1	.403	7
	4	.159	.1	.245	8
	9	.067	.1	.154	7
	1	.309	.05	.358	15
	4	.159	.05	.206	13
	9	.067	.05	.116	10
	1	.309	.01	.318	89
	4	.159	.01	.169	56
	9	.067	.01	.077	37
6	1	.309	.1	.407	9
	4	.159	.1	.253	10
	9	.067	.1	.157	9
	1	.309	.05	.357	22
	4	.159	.05	.206	18
	9	.067	.05	.113	14
	1	.309	.01	.319	130
	4	.159	.01	.169	80
	9	.067	.01	.077	51
8	1	.309	.1	.403	12
	4	.159	.1	.258	12
	9	.067	.1	.159	11
	1	.309	.05	.358	28
	4	.159	.05	.208	22
	9	.067	.05	.115	17
	1	.309	.01	.319	172
	4	.159	.01	.169	104
	9	.067	.01	.077	66
10	1	.309	.1	.406	14
	4	.159	.1	.253	15
	9	.067	.1	.160	13
	1	.309	.05	.358	35
	4	.159	.05	.208	27
	9	.067	.05	.117	20
	1	.309	.01	.319	213
	4	.159	.01	.169	129
	9	.067	.01	.077	81
20	1	.309	.1	.406	26
	4	.159	.1	.256	27
	9	.067	.1	.163	23
	1	.309	.05	.358	67
	4	.159	.05	.209	51
	9	.067	.05	.116	38
	1	.309	.01	.319	421
	4	.159	.01	.169	250
	9	.067	.01	.077	154

The Regression Analogy

Fisher (1936) shows that by performing a regression analysis with the dependent variable equal to $n_2/(n_1+n_2)$ in the first group and $-n_1/(n_1+n_2)$ in the second, the regression coefficients obtained are proportional to the discriminant coefficients. In fact, this is true for any two distinct values of the dependent variable. See e.g., Cramer (1967). The analysis of variance of this regression yields the same F as the D^2 analysis does. Thus,

$$F = \frac{R^2}{1-R^2} \frac{n_1+n_2-p-1}{p} \tag{14}$$

and

$$F = D^2 \frac{(n_1+n_2-p-1)}{p} \frac{n_1 n_2}{(n_1+n_2)(n_1+n_2-2)} \tag{15}$$

are two ways of expressing the same F with p and n_1+n_2-p-1 degrees of freedom. Thus

$$D^2 \frac{n_1 n_2}{(n_1+n_2)(n_1+n_2-2)} = \frac{R^2}{1-R^2} \tag{16}$$

which is equivalent to

$$(17) \qquad D^2 = \frac{R^2}{1-R^2} \frac{(n_1+n_2)(n_1+n_2-2)}{n_1 n_2}$$

or

$$R^2 = \frac{D^2}{D^2 + \frac{(n_1+n_2)(n_1+n_2-2)}{n_1 n_2}}$$

These relations will be useful later.

Shrinkage

In using a set of regression coefficients computed from a sample for prediction purposes, it is found that the correlation between predicted and observed values in a new sample is less than R. This phenomenon is well-known as the "shrinkage" of the multiple correlation coefficient. A number of methods have been proposed to deal with the problem of obtaining estimates of the "shrunken" R^2. There are at least two correlations of interest. First, the population multiple correlation coefficient ρ^2 that would hold if we knew the parameters of the population. This value is the proportion of the variance that can be accounted for by the independent variables. The other quantity, which Lachenbruch and Mickey (1965) refer to as ρ_p^2, the Prediction Correlation Coefficient, is the correlation between the sample regression line and the dependent variable.

The following relation holds:

$$(18) \qquad \rho_p^2 \leq \rho^2 \leq E(R^2).$$

Approximate unbiased estimation of ρ^2 from R^2 can be done easily and methods of doing this will be discussed in the next section. Estimation of ρ_p^2 is a more difficult problem which can be handled fairly well by a technique described in Lachenbruch and Mickey (1965).

An exact formula for estimating ρ^2 is given by Olkin and Pratt (1958). Letting R_c^2 denote the estimate, they show that

$$(19) \qquad R_c^2 = 1 - \frac{n-3}{n-p-1}(1-R^2)F(1,1;\tfrac{1}{2}(n-p+1),1-R^2)$$

is an unbiased estimate of ρ^2 where $n = n_1 + n_2$, p = number of variables and $F(\cdot)$ is the confluent hypergeometric function.

A first order approximation to an unbiased estimate was given by Wherry (1931) and is easier to work with:

$$(20) \qquad R_c^2 = R^2 - \frac{(1-R^2)(p-1)}{n-p} .$$

We will use formula (20) in the ensuing work.

Estimation of P_1 and P_2

A number of methods for estimating P_1 and P_2 have been suggested [Lachenbruch and Mickey (1968)]. For this paper, we shall be concerned with methods based on D^2. Okamoto (1963) has given an approximation based on n_1, n_2 and δ^2, the theoretical distance between the populations.

Equations (17) suggest that one might estimate a "shrunk" D^2 by

$$(21) \qquad D_c^2 = \frac{R_c^2}{1-R_c^2}\,\frac{(n_1+n_2)(n_1+n_2-2)}{n_1 n_2} .$$

From (20) we obtain

$$(22) \qquad D_c^2 = D^2\,\frac{n_1+n_2-p}{n_1+n_2-1} - \frac{(p-1)(n_1+n_2)(n_1+n_2-2)}{(n_1+n_2-1)n_1 n_2} .$$

Table 3 gives values of the multiplier of D and the correction term for some combinations of $n_1 = n_2 = n$, and p.

Table 3. Multiplier and Constant Terms for "Shrunk" Estimate of D^2

n	p	multiplier	correction term
20	2	.97	.10
50	2	.99-	.02
100	2	.99+	.002
20	4	.92	.29
50	4	.97	.12
100	4	.98	.06
20	10	.77	.88
50	10	.91	.36
100	10	.95	.18

Thus, if $D^2 = 1.0$, $n = 20$, $p = 10$, $D_c^2 = .77 - .88 = -.11$ which illustrates one of the drawbacks of using unbiased estimation for D^2. If $n = 100$, but other values were the same, we would have $D_c^2 = .95 - .18 = .77$. When D^2 is small, and the number of parameters is large relative to the number of observations, the value of D_c^2 may be negative.

In Lachenbruch and Mickey (1968), it is noted that an unbiased estimate of δ^2 based on D^2 may be obtained from the non-central F distribution. This is another candidate for the value of D_c^2 and its value is given by

$$(23) \qquad D_c^2 = D^2 \frac{n_1+n_2-p-3}{n_1+n_2-2} - \frac{(n_1+n_2)p}{n_1 n_2}$$

Equations (22) and (23) agree asymptotically as they should. The difference between them is due to the approximation used to obtain equation (22), and to the fact that the R^2 computed from the discriminant analysis is based on only two possible values of the dependent variable.

DISCUSSION. When the population parameters are known, it is easy to show that the probabilities of misclassification are given by $\Phi(-\delta/2)$ where δ^2 is Mahalanobis' distance between populations. Thus, the probabilities of misclassification increase as δ^2 decreases. Okamoto's work indicates that this relation holds when estimates are used for the population parameters. Since D^2 is an overestimate of δ^2, $\Phi(-D/2)$ will always underestimate the true probabilities of misclassification. Similarly, the fact that R^2 is an overestimate for ρ^2 and ρ_p^2 and the correspondence with D^2 through the F statistic indicate the relationship between the shrinkage of R^2 and the increase of probabilities of misclassification.

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INTRA-PROFILE VARIANCE*
(INTRA-INDIVIDUAL VARIANCE)

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This discussion does not present a sophisticated new statistic, rather attention is called to an easily obtained but seldom used type of significant difference between specimen, individuals or groups. The title, intra-profile variance, should be meaningful to counselors, phychologists, and statisticians in the education and personnel fields. The comparable sub-title may be more meaningful to statisticians and researchers in other fields. The applications discussed are in the personnel areas, but the profile variance statistic could be made applicable whenever several characteristics or attributes of individual specimen, components, or other units are being measured.

Table 1 illustrates the individual differences which the proposed statistic reflects. Individuals "o" and "*" both have the same average standard score on the four characteristics measured by X_1, X_2, and X_3 and X_4, but are quite different individuals. The difference in consistency of relative level in the four distributions suggests that there may be a difference in the predictability of performance for the two and that the quantification of such intra-individual characteristics might prove useful.

My initial interest in this problem stemmed from some remarks made by Irving Lorge in 1947. He thought that, especially for some groups of personality factors, consistency in level might be indicative of adjustment. Dr. Lorge hypothesized that statistical representation of such intra-individual differences would, for some purposes at least, prove to contribute significantly to more valid predictions than those based solely upon inter-individual differences. In some types of situations high intra-individual variability might be more desirable, in others being at about the same level, "consistent across the board," could lead to greater predictability of performance.

However, the concept is not as new as was originally thought. In checking the literature this was found to be yet another area which had been investigated by Clark Hull in 1927. In an article entitled, "Variability in Amount of Different Traits Possessed by The Individual," he compared the variability among

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*This is a further analysis of a concept reported at the September 1966 conference of the Military Testing Association and at the March 1967 special session of the Psychometric Society.

**Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of the United States Military Academy or the Department of the Army.

TABLE 1

EXAMPLE OF DIFFERENCES IN INTRA-PROFILE VARIANCE

Individual	Mean	SDP
0	+1	2.37
*	+1	0.25

Standard Score $\left(\frac{X_n-\bar{X}_n}{SDX_n}\right)$	Measure X_1	X_2	X_3	X_4	Mean $\left(\frac{\Sigma X}{n}\right)$	Relative Level
3.0						
2.5						Very High
2.0						
1.5						
1.0						High
0.5						
0.0						Typical
-0.5						
-1.0						Low
-1.5						
-2.0						
-2.5						Very Low
-3.0						

different persons in trait measures with the variability in a single individual on these same traits. (Hull: 1927) He found the amount of variability within single individuals to be about 80 percent of the amount of variability among different individuals. Apparently a significant source of differences between individuals is being ignored when using one composite, or weighted average, of an individual's scores, for measurement, prediction, or decision making purposes.

Research by psychologists, educators and statisticians concerned with personnel problems has evidenced increased interest in intra-individual differences such as those indicated by variability among measures of different attributes of an individual and by the more complete profile or pattern analysis techniques. Much of this interest results from increased recognition that the interrelationships within an individual of a group of measures may be quite different from the interrelationships between these measures in the general population. Current moderator variable research has found in some situations a variable that successfully identifies subgroups within a population for which the interrelationships among variables differ significantly.

While complete pattern or profile analysis techniques entail several relatively complex problems, a statistic representing intra-profile, or intra-individual, variance is easily obtained.

When SD^2_p = intra-profile variance; n = number of tests, factors, subtests or other characteristics measured and reported on comparable scales; and Σx^2 = the sum of the squares of the deviations of an individual's scores on each variable from his mean score on all n variables, then:

$$SD^2_p = \frac{\Sigma x^2}{n} \qquad (1)$$

If ΣX = sum of one man's scores on all n variables, and ΣX^2 = the scores squared and added, the gross score formula would be:

$$SD^2_p = \frac{n\Sigma X^2 - (\Sigma X)^2}{n^2} \qquad (2)$$

When beta weights for comparably scaled scores on the different abilities or traits are available, these could be used to obtain a measure of intra-profile variance that should have more validity. The deviation formula for weighted intra-profile variance would be:

$$SD^2_{PW} = \frac{\Sigma Wx^2}{\Sigma W} \qquad (3)$$

The corresponding gross score formula would be:

$$SD^2_{PW} = \frac{\Sigma W \Sigma WX^2 - (\Sigma WX)^2}{(\Sigma W)^2} \qquad (4)$$

If the beta weights are not available, but other useful bases for weighting each scale in proportion to its importance in a given performance are available, such judgment derived weights might be used in this formula.

Although the United States Military Academy, West Point, currently compares favorably with the best universities in the effectiveness with which academic performance is predicted, this still means that only 50 percent of the factors that make for differences in level of academic performance are being measured. Though proud of our relative success, we would like to reduce the variance not predictable currently.

Appropriate basic data were available on the 843 cadets who entered USMA in July 1964 and completed enough of the first year at USMA to have academic grades; 816 remained one year and 789 remained one and one-half years. Table 2 shows, for the 843 cadets, the correlation coefficients between the following:

(1) the academic average earned at USMA--"Acad Av";

(2) the weighted average of Scholastic Aptitude Test-Verbal, SAT-Mathematics, College Entrance Examination Board Mathematics Achievement, CEEB English Composition, and High School Rank standard score,—the five components of the academic potential battery, "Acad Pot";

(3) the standard deviation of the weighted scores on these five components—the "SDPW" intra-profile statistic;

(4) the Academic Achievement Index, a statistic reflecting the academic average with measured academic potential held constant (partialed out) and thus identifying over-achievers, par-achievers, and under-achievers—"AAI."

TABLE 2. Selected Correlation Coefficients*

$r_{\text{Acad Av} \cdot \text{SDPW}} = -.06$	
$r_{\text{Acad Pot} \cdot \text{SDPW}} = -.23$	
$r_{\text{Acad Av} \cdot \text{Acad Pot}} = .68$	
$r_{\text{AAI} \cdot \text{SDPW}} = .12$	
$eta_{\text{SDPW on AAI}} = .21$	($f = 1.56$; $.10 > P > .05$)
$eta_{\text{AAI on SDPW}} = .17$	($f = 0.86$; $P > .10$)
$R_{\text{Acad Av} \cdot \text{Acad Pot, SDPW}} = .69$	

*Means and standard deviations on each variable are given in Table 3.

Even with a population of 843, simply adding the SDPW to the regression equation did not significantly increase the validity with which level of academic achievement was predicted. However, inspection of the AAI line-of-means revealed the marked tendency for the 241 cadets with a SDPW of 70 or more to be over-achievers. In fact, the 15 cadets with SDPW's greater than 160 had a mean Academic Average more than one standard deviation above that predicted from their measured academic potential. The 602 cadets with AAI's of 69 or less tended to be under-achievers relative to the academic achievement predicted from their composite academic potential score. Hence the possibility was explored that the SDPW would serve as a moderator variable (an executive variable) to identify two groups in which the interrelationships of the variables involved were sufficiently different that different equations for the two groups would yield more valid predictions.

For the total group of 843 cadets, the multiple correlation of the five regular academic potential components with the Academic Average was .694; for the 602 cadets with SDPW's of less than 70, the independently computed multiple correlation was .695; for the 241 whose SDPW's were 70 or more, the multiple correlation was .693. The beta weights in all three equations were almost identical. The hypothesis that the interrelationships among these variables were the same for two groups identified by a critical SDPW of 69.5 could not be rejected.

Although neither of the above approaches successfully utilized intra-profile variance, Table 3 shows clearly that the cadets who made the most of their objectively-measured academic potential had significantly higher intra-profile variance on the five component measures. Perhaps the individual differences model for multiple regression proposed by Dr. Cleary (1966) and discriminant function analyses will show how to use the statistic in this instance. At any rate, the ease with which this statistic can be obtained, along with other statistical data at no extra cost, would seem to warrant its incorporation into the model for validity studies. This would be especially true when there is reason to hypothesize that high intra-individual variance would be desirable or when across-the-board consistency in performance is desired.

Several other applications of this statistic may be useful. In the military personnel situation, intra-individual differences may be of considerable utility. In general, a man who is rather uniformly high in all areas of his military specialty might be considered to be more valuable to his service, in the series of successive assignments throughout his military career, than would a man who is very high in some areas and very low in others. One of the latter men might work out well in one assignment and be a complete failure in others. Thus, the utility a soldier's weighted intra-profile variance on pertinent measures of his abilities seems to warrant investigation. In the physical and biological sciences and technologies as well as in the behavioral sciences

TABLE 3
CHARACTERISTICS OF SELECTED GROUPS OF CADETS
ENTERING USMA IN 1964

Group	N	Academic Potential Score		Academic Average		Academic Achievement Index(AAI)		SDP		SDFW	
		M	SD	M	SD	M	SD	M	SD	M	SD
"Over-achievers" (Top 27% on AAI)	218	605	59	2.559	.113	622	48	70	33	73	40
"Par-achievers" (Middle 46%)	379	600	53	2.413	.107	501	35	66	27	67	31
"Under-achievers" (Bottom 27% on AAI)	219	601	55	2.272	.113	375	50	62	24	63	29
Total 1 year	816	602	55	2.415	.152	500	100	66	28	68	33
Total with grades	843	599	56	2.406	.158	500	100	66	28	67	33

and technologies an intra-profile variance type of statistic might be found useful. For example, reasonably accurate estimation of the probability of failure of a separate component or unit of equipment usually is possible. However, considerable difficulty often is encountered when using standard statistical techniques to estimate the composite failure probability of a complex assembly of a large number of these component units. An exploratory approach might begin by comparing the distributions of the weighted intra-profile variance, of each component unit's significant characteristics measured under standard conditions, for component units at different levels on the best available reliability statistic. Where adequately detailed records are available, data on the past success and failure of complex assemblies might be compared with distributions of the intra-profile variance of the characteristics of all of its component units, including intra-profile variance of the component units' intra-profile variances. An appropriate model for such an investigation could be developed by quality control researchers for a specific type of equipment.

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A STATISTICAL TEST OF TWO HYPOTHETICAL RELIABILITY GROWTH CURVES OF THE LOGISTIC FORM IN THE DISCRETE CASE

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ABSTRACT. This paper demonstrates a mathematical method by which curves can be developed useable as a tool for aid in solving the problem of monitoring reliability growth; and also illustrates how statistical tests of hypotheses may be conducted in conjunction with these growth curves.

The growth curves discussed are applicable for use where units are undergoing development phases; specifically where it is desired to periodically assess the actual reliability growth of these units for comparison with hypothetical reliability growth curves.

The unique facet of these growth curves as presented herein lies in their use. Since their application is directed towards the improved development of a unit type, this development is dependent upon the reliability achieved as a result of improvements made on previously tested units. The reliability, or probability of success, at each stage of development is independent and varies from stage to stage.

A curve embodying the assumptions necessary for the measurement of reliability growth during development is termed the Logistic curve. Two such curves are plotted, representing two alternative hypothetical growth patterns based on specified values of a unit's inherent reliability. From the observed sample of proportion of successes (Reliability) accumulated at some trial of the development program, a selection is made of the true curve of the unit or group of units that just finished the test. If the upper growth curve is actually true of the population from which the sample of units is randomly drawn, a small risk, α, is desired that the sample would be so poor as to bring rejection of this curve. Likewise if the lower growth curve is true, a very small risk, β, is desired that the sample will be so good as to bring erroneous acceptance of the upper curve.

The subject curves have been found useful in the past to study population growth, learning and developmental processes. The application of the Logistic Growth Curve concept in assessing reliability has only recently been directed toward the engineering development of expensive electronic components. Prior to this use, extensive literature search had not revealed its application for this purpose.

The concept of reliability growth during the development stages is one which should be emphasized throughout governmental and industrial circles. The growth pattern concept, saving time and money, can also assist in creating a better understanding between the consumer and the producer regarding their mutual problems, through the joint visual monitoring of a statistically sound method of reliability assessment.

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Introduction

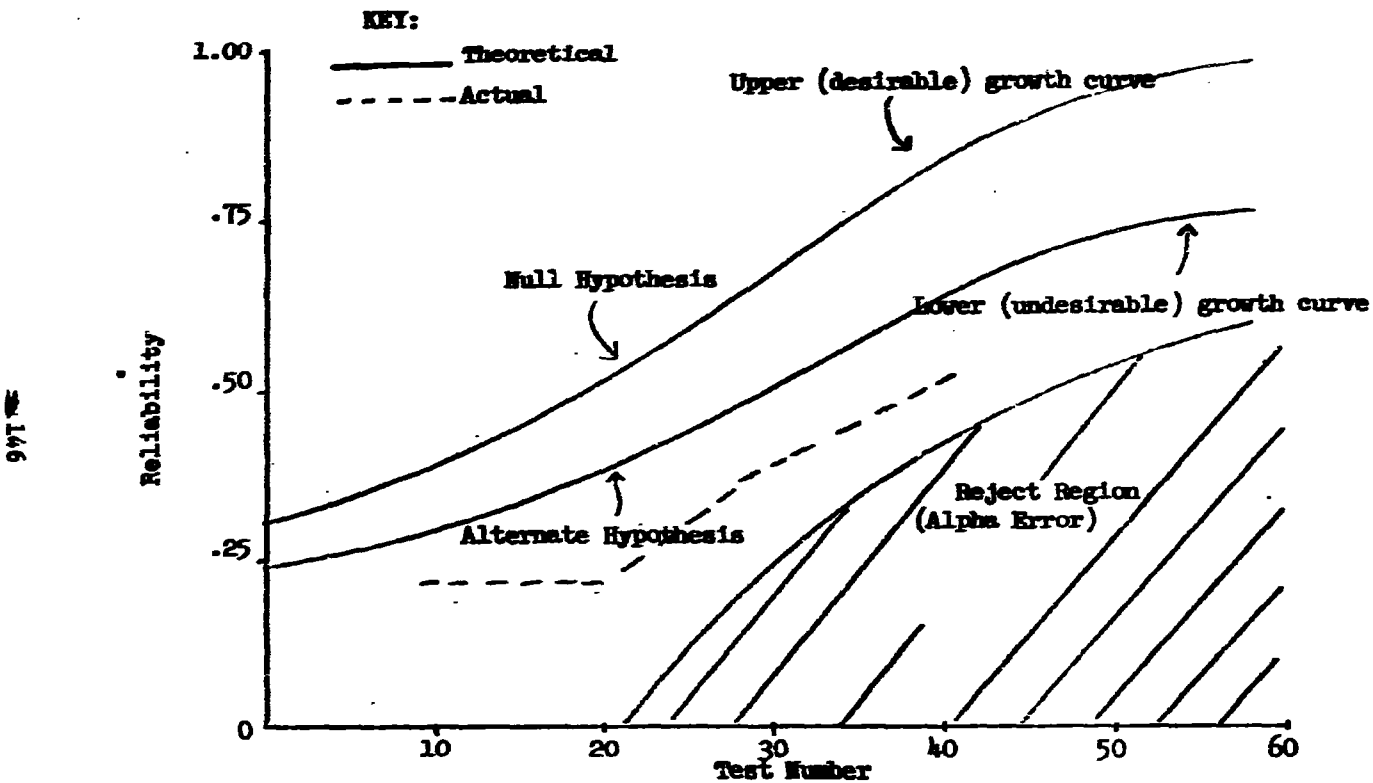
Reliability evaluation is as essential a task during the development of a unit as it is during the production of the unit. During development, reliability is directly affected by a necessary and thorough knowledge of the use and capabilities of the proposed unit. Continuous reliability design analyses and engineering changes on the unit cause a developmental growth pattern which must be identified. This identification is necessary in order that a trend can be predicted and the reliability requirement can be quantitatively specified for use in the evaluation of the unit during the production. This developmental growth pattern is dependent upon the reliability achieved as a result of improvements made on previously tested units. A mathematical function which has been found useful in the past to describe population growth, learning and developmental processes, and more recently to fit the engineering developmental growth pattern of mechanical and electronic components is the S-shaped growth function presented herein as the reliability growth model.

During the reliability development phase, the first unit is put to test. Its performance is judged a failure or success. Subsequently, the unit's reliability is assessed, an analysis is made of its performance and design improvements are made. These improvements are built into another unit (or the same unit if no damage was done on the first test) which then undergoes the same process; that is, testing for failure or success. Again improvements are made and the cycle is repeated. By such a procedure, it is intended that reliability will grow from some low initial value (state-of-the-art) to a higher target value at the end of the program.

The Problem

It is not unusual that although the inherent reliability of a unit is growing properly, the "sample values of tested units" may vary enough to present a poor reliability picture. "Sample values of units" here means that the one unit tested was only one out of many (of a population) that could have been tested. Thus even if reliability is high, a rash of failures in a sample can occur and cast doubts upon the inherent reliability. Of course, it can also happen that a unit with low inherent reliability will by chance produce a high number of successes in a sample, possibly resulting in wrongful acceptance of the unit as being satisfactory. It is against these possibilities of error that reliability statisticians direct themselves when designing meaningful test programs.

Two reliability growth curves are plotted on Figure 1, representing two alternative specified values of a unit's inherent reliability. From the observed sample of proportion of successes (Reliability) accumulated at some trial of the development program, we wish to choose between which curve is true of the unit or group of units that just finished the test. If the upper growth curve is actually true of the population from which the sample of units is randomly drawn, we only want a small risk, α, that the sample would be so poor as to bring rejection of this curve. Likewise if the lower growth curve is true, we want a very small risk, β, that the sample will be so good as to bring erroneous acceptance of the upper curve. These two risks are usually specified by the experienced engineer or manager who must also consider such things as delivery time, cost, availability of test equipment and



Reliability Growth Curve Testing

Figure 1

many other facets which contribute towards the profit picture of an industry.

Mathematical Derivations

The Growth Function

S-shaped curves of growth functions have been arrived at by many learned people as cited in the references on growth. However, Herbert K. Weiss [1] in a reliability sense by the method of maximum likelihood, arrived at S-shaped growth curves by starting with the assumptions that each failure source in a system has a parameter failure rate and that a constant probability exists that each failure source will be properly discovered and corrected by way of development engineering. A. Hald [2] arrived at the same form of the curve through the use of differential equations. This section will concern itself with characterizing processes by differential equations from which reliability growth will be derived.

Let x denote time or the magnitude of a growth factor which influences the size of y of the observed phenomenon. Then the differential coefficient dy/dx denotes the rate of growth; i.e., the increase per unit of time. At this point, the growth process can be characterized by:

$$\frac{dy}{dx} = f(x,y),$$

which indicates the growth rate depends both on time (x) and of the size obtained (y). We shall only deal with special cases of the type:

$$\frac{dy}{dx} = f(y)g(x), \tag{1}$$

which may be written as:

$$\frac{dy}{f(y)} = g(x)dx \ ,$$

in differential notation.

Integration yields

$$F(y) = G(x) \tag{2}$$

Thus by means of (2) y is determined as a function of x.

To apply (1) to a specific case, we take the situation whereby the growth rate is proportional to the achieved reliability R, and to a function of time, $g(t)$ as:

$$\frac{dR}{dt} = Rg(t) \tag{3}$$

rearranging terms and using differential notation we obtain:

$$\frac{dR}{R} = g(t)\,dt \Longrightarrow g(t) = \frac{1}{R}\frac{dR}{dt} \tag{4}$$

Since

$\frac{dR}{R} = d \ln R$, we have from (4)

$$\frac{d \ln R}{dt} = \frac{dR}{dt}\frac{1}{R} \tag{5}$$

which is called the logarithmic differential coefficient to be used further.

Introducing $R = 1$, R, $\lambda - R$, and $R(\lambda - R)$ in (3), we obtain the following four differential equations:

$$\frac{dR}{dt} = \begin{cases} 1)\ g(t) \\ 2)\ Rg(t) \\ 3)\ (\lambda - R)\,g(t), & (0 \le R \le \lambda), \\ 4)\ R(\lambda - R)\,g(t), & (0 \le R \le \lambda), \end{cases} \tag{6}$$

where λ denotes the maximum value of R.

The four equations respectively indicate that at a given time, the reliability growth rate (1) depends on time but is independent of the size reached, (2) is proportional to the size reached and to a function of the time, (3) is proportional to the "remaining size;" that is, the

maximum size minus the size reached, and a function of the time, and (4) is proportional to both the size reached and the remaining size as well as a function of the time.

We must study the character, or form, of the fourth differential equation from equation (6) which is:

$$\frac{dR}{dt} = R(\lambda - R)\, g\,(t), \quad (0 \leq R \leq \lambda),\ \lambda = \text{constant}, \tag{7}$$

which is set up under the important assumptions which are worthy of repeating: At a given number of trials, the reliability growth rate, dR/dt is a function of:

1) the number of trials, t,

2) the growth, R, reached at a number of trials t, and

3) the remaining growth $(\lambda - R)$ to the maximum possible reliability value λ.

By introduction of the logarithmic differential coefficient as in equation (5) we shall derive the reliability growth function.

Dividing (7) by R and substituting in (5), we get:

$$\frac{d\ \ln R}{dt} + Rg(t) = \lambda g(t). \tag{8}$$

Solving (7) for R:

$$R = \frac{1}{(\lambda - R)g(t)} \frac{dR}{dt} \tag{9}$$

Substituting (9) into (8)

$$\frac{d\ \ln R}{dt} + \frac{1}{(\lambda - R)g(t)} \frac{dR}{dt}\ g(t) = \lambda g(t),$$

or

$$\frac{d\ \ln R}{dt} - \frac{d\ \ln (\lambda - R)}{dt} = \lambda g(t), \tag{10}$$

since

$$\frac{d \ln (\lambda - R)}{dt} = \frac{1}{(\lambda - R)} \cdot d(\lambda - R) \frac{1}{dt} = - \frac{1}{(\lambda - R)} \frac{dR}{dt} .$$

Multiplying (10) by dt and integrating yields:

$$\ln R - \ln(\lambda - R) = \lambda G(t).$$

Solving for R:

$$R = \frac{\lambda}{1 + e^{-\lambda G(t)}} \qquad (11)$$

which includes a constant of integration, and may be determined from a given value of (t, R).

By introducing special functions for g(t) such as:

$$g_1(t) = B_o + B_1 t + B_2 t^2$$

and

$$g_2(t) = B_o + B_1/t + B_2/t^2,$$

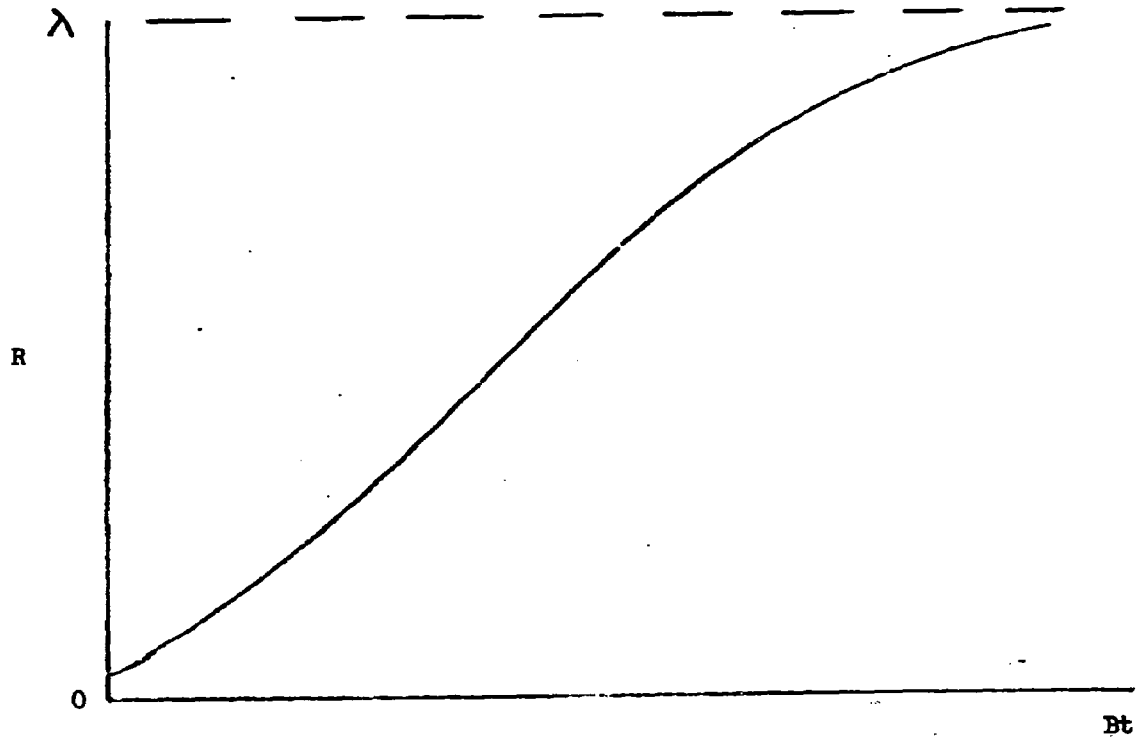
one can obtain a number of examples of frequently applied growth curves. For the reliability growth tests of the discrete case presented in this thesis, it is sufficient to assume g(t) = B. Therefore, equation (11) can be written as:

$$R = \frac{\lambda}{1 + Ae^{-\lambda Bt}} \qquad (12)$$

where $A = e^{-\lambda \alpha}$; α is a constant of integration.

For B > 0, equation (12) is an increasing function of t having the asymptote λ. Since $0 \leq R \leq \lambda$ and $0 \leq \lambda \leq 1$, the desired S-shaped reliability growth function is obtained:

$$R = \frac{1}{1 + Ae^{-Bt}} \qquad (13)$$



S - Shaped Growth Curve

Figure 2

Figure 2 shows the growth function given in (13) as the S-shaped curve with horizontal asymptotes at R = 0, and R = λ.

The Critical Region

In designing statistical tests of hypotheses, it is necessary to specify the size of the critical or rejection region α. α is defined as the Type I error of the test and it is the probability that the null hypothesis will be rejected when it is actually true. The procedure of calculating the critical region when applying the reliability growth function is to find the acceptance number of successes "a" such that

$$\alpha \leq \sum_{i=0}^{a} R_{i,n}$$

where $R_{i,n}$ is the reliability, or probability of getting i successes in n trials. Thus we must develop a probability for each of the 2^n permutations and sum these to some minimum acceptance number "a" of successes, which equals or just exceeds α. The acceptance number of successes, obtained from the above summation of probabilities when divided by its corresponding n, gives the proportion successful (a/n), which when plotted on the same graph as the reliability growth curves, outlines the critical region.

The probabilities of each of the permutations can be computed by the powerful device of generating functions as outlined by Uspensky [3] The generating function, $\phi(\xi)$, for this problem is:

$$\phi(\xi) = \prod_{i=1}^{n} (R_i \xi + Q_i) = \sum_{i=1}^{n} R_i \xi^i,$$

where the coefficients R_i will give both the permutations and the

probabilities involving i successes indicated by the exponent of the dummy variable ξ. This equation can easily be computed recursively, giving the permutations and their probabilities at any K^{th} stage of the development program:

$$\phi_K(\xi) = \phi_{K-1}(\xi) \cdot (R_K \xi + Q_K)$$

$$= (P_{0,\,K-1}\,\xi^0 + P_{1,\,K-1}\,\xi^1 + P_{2,\,K-1}\,\xi^2 + \cdots + P_{K-1,\,K-1}\,\xi^{K-1})(R_K \xi + Q_K)$$

$$= P_{0,\,K}\,\xi^0 + P_{1,\,K}\,\xi^1 + P_{2,\,K}\,\xi^2 + \cdots P_{K,\,K}\,\xi^K,$$

where $P_{S,\,M}$ is the probability of S successes in M trials. Then the critical region is found by finding the minimum acceptance number, "a," which satisfies:

$$\alpha \leq \sum_{i=0}^{a} R_{i,n},$$

and dividing this number by its corresponding n, to get the probabilities which, when plotted, outline the critical region.

For example, if n = 3 the recursion becomes:

$$\prod_{i=1}^{3} (R_i \xi + Q_i) = (R_1 \xi + Q_1)(R_2 \xi + Q_2)(R_3 \xi + Q_3)$$

$$= R_{0,3}\,\xi^0 + R_{1,3}\,\xi^1 + R_{2,3}\,\xi^2 + R_{3,3}\,\xi^3$$

$$= (Q_1 Q_2 Q_3)\,\xi^0 + (R_1Q_2Q_3 + Q_1R_2Q_3 + Q_1Q_2R_3)\,\xi^1$$

$$+ (R_1R_2Q_3 + R_1Q_2R_3 + Q_1R_2R_3)\,\xi^2 + (R_1R_2R_3)\,\xi^3,$$

where $Q_i = 1.00 - R_i$. These terms are now individually evaluated and the coefficients of the dummy variable ξ^i are summed from the ξ^0 term through the ξ^3 term. When the summation equals or just exceeds the preassigned α value, the exponent of ξ is divided by 3. This proportion is then plotted to outline the critical region for n = 3.

Testing One Growth Curve Against an Alternative

The mathematical model for this problem is based on the assumption that at each ith (discrete) trial in a development program the reliability, R_i, of the unit is given by the growth curve:

$$R_i = \frac{1}{1 + Ae^{-Bi}}, \text{ where A and B are constants.}$$

Two curves are considered; the upper (desired growth) and the lower (undesired growth).

The upper growth curve, our hypothesis, is determined by the initial current state-of-the-art reliability which is given to be R_0, the desired or target reliability which is a specified value R_N at program's end, and the total number of trials for the entire development program, i = N.

The lower growth curve, the alternative, is determined in the same manner as the upper curve, with R_0' being the minimum permissible level and R_N' being the minimum target level at i = N trials.

A single unit is to be tested at each i^{th} stage of the development program, registering either a failure or success. Even if the upper specified R_i curve is a true characteristic of the unit, the random variations of sampling units will produce observed proportions of suc-

cesses to each i^{th} test which will deviate quite widely from the trend of the basic growth curve. Each "path" or "random walk" of the observed proportion successful depends on the permutation of successes, S, and failures, F, that can result in sampling units when the specified R_i is the probability of success and $Q_i = 1.00 - R_i$ the probability of failure at each i^{th} trial. The total possible random walks or permutations of failures and successes is 2^N.

Types of Error

In designing statistical tests of hypotheses, it is necessary to specify the size of the critical or rejection region as α; also called the producer's risk. Thus we must develop a probability for each of the above permutations and sum these to some minimum acceptance number, "a," of successes which just exceeds probability α. When "a" is divided by its corresponding n, the proportions obtained outline the critical region.

However, if some lower, undesirable growth curve, which does not reach target R_N, is actually true of our system there is some risk or chance β, that the observed proportion will not fall in the rejection region, resulting in erroneous acceptance of the system.

Since the α error (Type I) is predetermined, we will show the derivation of the β error (Type II).

1. Use the upper growth curve R_i's for a given α value to calculate the acceptance number of successes "a" such that:

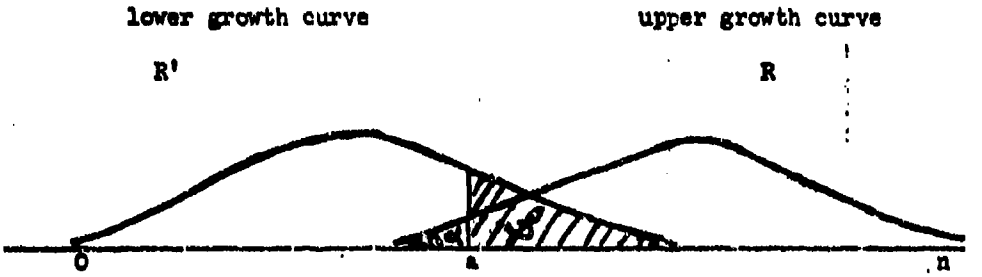
$$\alpha \leq \sum_{i=0}^{a} R_{i,n}$$

where $R_{i,n}$ is the total probability of "i" successes in n trials consisting all possible permutations.

2. Use the number of successes "a" obtained from the first step, applying to the lower growth curve to calculate β values such that

$$\beta = \sum_{i=a}^{n} R'_{i,n} .$$

The calculation can be conceptually diagrammed as below (although in actuality we are dealing with discrete distributions):



Number of Successes

Relationship of α and β error calculations

Figure 3

It should be noted that the errors of α and β pertain only to <u>each</u> value of n. No attempt has been made to evaluate the overall error for the decision procedure, namely, we do not know what is the probability of accepting or rejecting R or R', independent of the number of items tested.

<u>Construction of Growth Curves</u>

Calculation Instruction: For upper growth curves

1. At each i^{th} (discrete) trial in a development program the reliability R_i, of the unit is given by the growth curve:

$$R_i = \frac{1}{1 + Ae^{-Bi}} \tag{14}$$

The constants A and B can be obtained in the following manner:

For A, let i = 0, we have

$$R_0 = \frac{1}{1 + Ae^{-B0}} \Longrightarrow A = \frac{1}{R_0} - 1 \tag{15}$$

For B, let i = N, we have

$$R_N = \frac{1}{1 + Ae^{-BN}} \quad . \tag{16}$$

Substituting the value of A in (15) into (16) we have

$$R_N = \frac{1}{1 + \left(\frac{1-R_0}{R_0}\right) e^{-BN}}$$

that is

$$e^{-BN} = \frac{R_0}{1-R_0} \left(\frac{1-R_N}{R_N}\right), \text{ and}$$

$$B = -\frac{1}{N} \log_e \frac{R_0(1-R_N)}{R_N(1-R_0)} \quad .$$

2. Assign a number N (=5, 10, 15, ...N) to i in formula (14) to calculate its corresponding R_N which will make up the body of the table of upper-growth-curve-values.

Similarly we can obtain the values of lower growth curves.

AN EXAMPLE

Due to ever increasing passenger and freight air travel and resultant need for more rapid turn around of equipment, a contract was let to a bonding material manufacturer to develop a considerably more effective, but more expensive metallic brake lining, evaluate it, and measure its effect on the braking system of the aircraft. At a given braking horsepower in ft. lbs/sec., the criteria for determining success S or failure F of the material are two-fold:

1. The maximum wear of the lining is not to exceed --- inch[1/], and
2. The maximum wear of the bell (brake drum) is not to exceed --- inch [1/]

Any brake lining which could not meet these two criteria were classified as rejects (failure), since these criteria are considered to be critical defects, if exceeded. No previously tested linings can be retested.

A pre-design meeting was held with attendees representing management, the customer, engineering, purchasing and reliability. Since the reliability of this lining was of prime importance, reliability chaired the meeting. The most significant points made in the meeting were that the cost of the metallic material required is extremely high, and the required reliability was .993; that is, on the average the customer was willing to live with seven lining failures in 1000. It was also mutually agreed that a six percent probability of rejecting the material was allowed when the sample

---

1/ Classified information, in thousands of an inch, with braking applied for x hours.

showed poor material coming from a good lot. This is known as the producer's risk (α error). The initial current state-of-the-art reliability of the brake lining was given to be 20 percent. The undesired or alternative initial reliability was given to be 19 percent, with the alternative final reliability value being .935. This value of .935 was chosen since a review of the β values for this combination of .935 and .993 indicated that the power of the test (1.00 - β) was at a desirable level, considering the cost of the material and the alpha error. The beta error was an eight percent probability of accepting the material when the sample showed good material coming from a bad lot.

There were four critical environmental tests which the braking material was required to pass. These were:

1. Humidity
2. Temperature
3. Shock
4. Vibration

A success in one particular environment does not mean that the specific lining would have passed in another environment. It was decided that since two brake linings were required to simulate a braking system, the linings for two shoes at a time would be manufactured, tested two at a time on each of the four environmental tests and their reliability evaluated and growth structure monitored. The total manufactured sample size due to cost, was allowed to be 72 pairs of shoes.

The test results up to and including test number 40 are shown in Table 1 and are plotted on Figure 4.

Table 1

BRAKE BONDING MATERIAL
TEST DATA SHEET

Number of Tests (Pair)	Number of Successes	Cumulative Number of Successes	Cumulative Proportion of Successes (Reliability)
4	1	1	.25
8	0	1	.12
12	3	4	.33
16	3	7	.44
20	2	9	.45
24	3	12	.50
28	2	14	.50
32	1	15	.47
36	2	17	.47
40	2	19	.48

Figure 4 illustrates the test results given in Table 1 plotted in increments of four pairs of linings, the upper desired reliability growth curve, R_N, the alternative lower undesirable reliability growth curve, R'_N, and the critical or reject region. As can be seen, the reliability was not growing as desired, so the manufacturing and testing were halted after test number 40. A very strict analysis was ordered of the design before authorization was given to proceed further.

TESTING TWO HYPOTHETICAL RELIABILITY GROWTH CURVES

$R_i = \frac{1}{1+Ae^{-Bi}}$

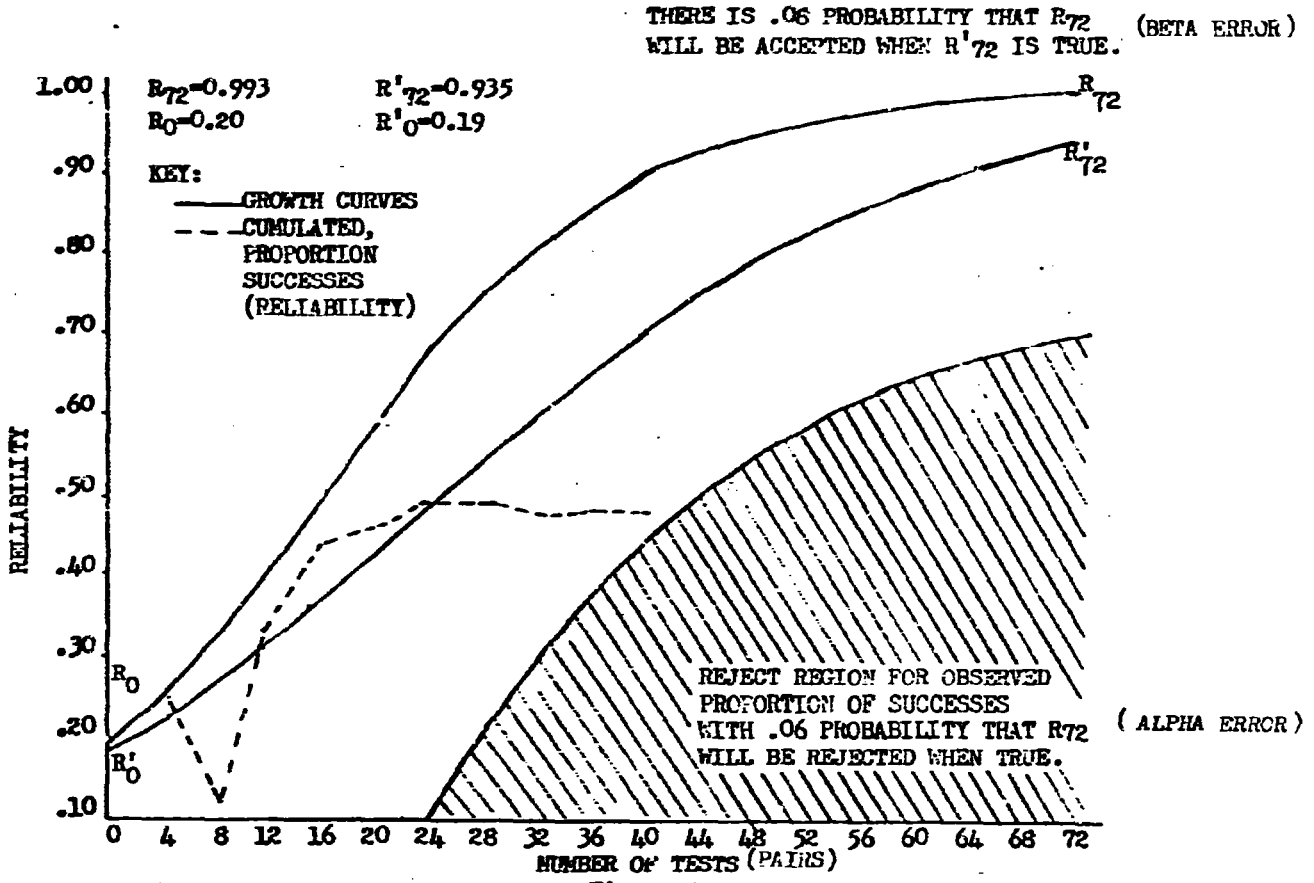


Figure 4

Conclusions:

The concept of reliability growth during the development stages is one which should be emphasized throughout governmental and industrial circles. The growth pattern, saving time and money, can also create a better understanding by the consumer and the producer of their problems through the visual monitoring of statistically sound methods of assessment. The producer and consumer should get together before development to understand and agree on the following items related to the monitoring of the to-be-developed unit's reliability:

1. The current state-of-the-art reliability.
2. The desired final reliability value at program's end.
3. The alternative, or undesired, reliability values corresponding to steps 1 and 2.
4. The inspection size and final inspection size.
5. Null hypothesis and alternative(s).
6. Alpha error, beta error and power of the test.

Thus a thorough knowledge of the ability and use of the subject items will be overlapped with a sound statistical technique for use in assessing the proposed item during development.

When choosing reliability curves of the type presented herein for use in describing the growth pattern of a particular item in development, care must be taken in selecting proper combinations of sample size and pre-assigned alpha values. Small alpha values will tend to be equalled or exceeded rather quickly when the sample size is quite small, say 10 or less. For larger sample sizes, the values of alpha are not as quickly equalled or exceeded, and when exceeded, the cumulative probabilities closely approximate the preassigned alpha values. Of course, the power of the test (1.00 - Beta) will assist the choice of the proper combinations of alpha and the sample size.

PROGRAMMING THE GROWTH MODEL

Introduction

The reliability growth model program was written in Fortran IV language and was run on the IBM 7040 computer. The program is flexible in the sense that positive or negative step sizes are permitted in choosing sequences of upper or lower curves. It is also possible to skip certain curves in a sequence of upper or lower curves. The program will run approximately twenty minutes on the 7040 for 60 combinations of upper and lower reliability growth curves and 50 different values for sample size. The program listing is included for use by those wanting to generate curves, critical regions and β errors. Please note the program statement numbers are included in brackets to the right of the appropriate statements.

Description

The reliability growth program calculates the quantities Probability_i, $i = 1, \ldots, n$ outlining the critical regions corresponding to different values of the Type I or α error, α_i, $i = 1, \ldots, n$, and the corresponding Type II or β error represented by the quantities S_i, $i = 1, \ldots n$, when given the following:

1. various inspection sizes of i components ranging from 0 to a total of N of components,

2. a reliability $R_{i_{upper}}$ which represents the expected reliability

level for the inspection size of i components; i.e. the observed reliability R_i (ratio of number of accepted components to the accumulated total number i of components which have been inspected at a given stage of time) is not to fall within the critical region determined by the value of the Type I error, α, and the value of $R_{i_{upper}}$ and,

3. a reliability $R_{i_{lower}} < R_{i_{upper}}$ which represents an alternative reliability level for inspection sizes of i components such that; if the observed reliability R_i for the inspection size of i components is less than $R_{i_{lower}}$, we wish to calculate the probability β of committing a Type II or β error, where β is defined as the probability that the observed reliability R_i does not fall within the critical region determined by $R_{i_{upper}}$ and the specified value of α, but in actuality the expected reliability at the given state is given by $R_{i_{lower}}$

The quantities $Prob_1 = a_1/i$, $Prob_2 = a_2/i$, $Prob_3 = a_3/i$, $Prob_4 = a_4/i$ computed for the four choices of $\alpha_1, \alpha_2, \ldots, \alpha_n$ of α for each inspection size i represent the proportion of the number of successes a[1] or reliable components to the number i of components which have been inspected at the given stage. Therefore, $Prob_1, Prob_2, \ldots, Prob_n$ at any given stage of inspection of lots of i components will be functions of $R_{i_{upper}}$ and of $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively. The quantities $Beta_1$, $Beta_2, \ldots, Beta_n$ represent the Type II or β errors which are functions respectively of the values $a_1, a_2, \ldots, a_n$ of number of successes computed previously in the determination of $Prob_1, Prob_2, \ldots, Prob_n$ and functions also of $R_{i_{lower}}$.

---

1. (where $a_1, a_2, \ldots, a_n$ are the different values of "a" corresponding to the four choices $\alpha_1, \alpha_2, \ldots, \alpha_n$ of α.)

The input to the program is specified by various combinations of R_0, R_N, R'_0, R'_N and various inspection sizes i of components where,

R_0, R_N are initial and target reliabilities respectively which are used to compute the ordinate points $R_{i_{upper}}$ on the upper growth curve corresponding to the abscissa points i representing inspection size and

R'_0, R'_N, are initial and target reliabilities respectively which are used to compute the ordinate points $R_{i_{lower}}$ on the lower growth curve corresponding to the abscissa points which represent inspection size i.

The initial inspection size, the step between inspection sizes, and the largest inspection size or total N of components may be varied without altering the program. Also the values of α may be varied where the notation convention $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_n$ is to be observed. In choosing various combinations of R_0, R_N, R'_0, R'_N any initial values of R_N, R'_N may be chosen, a step for simultaneously increasing R_N, R'_N may be chosen, and the number of steps desired may be chosen. Similarly the initial values for R_0, R'_0, the step size, and the number of steps desired may be specified.

List of Symbols

Fortran Notation	Statistical Notation	Description
HRO	h_{R_O}	Step size between succeeding values of R_O (must be the same as step size between succeeding values of R'_O).
HRN	h_{R_N}	Step size between succeeding values of R_N (must be the same as step size between succeeding values of R'_N).
ROUI	$R_{O_{initial}} - h_{R_O}$	if $R_{O_{initial}}$ is the smallest value of R_O which is used in the specified combinations of values R_O, R_N, R'_O, R'_N, then ROUI is equivalent to $R_{O_{initial}} - h_{R_O}$.
ROLI	$R'_{O_{initial}} - h_{R_O}$	if $R'_{O_{initial}}$ is the smallest value of R'_O which is used in the specified combinations of values R_O, R_N, R'_O, R'_N, then ROLI is equivalent to $R'_{O_{initial}} - h_{R_O}$.
RNUI	$R_{N_{initial}} - h_{R_N}$	if $R_{N_{initial}}$ is the smallest value of R_N which is used in the specified combinations of values R_O, R_N, R'_O, R'_N, then RNUI is equivalent to $R_{N_{initial}} - h_{R_N}$.
RNLI	$R'_{N_{initial}} - h_{R_N}$	if $R'_{N_{initial}}$ is the smallest value of R'_N which is used in the specified combinations of values R_O, R_N, R'_O, R'_N, then RNLI is equivalent to $R_{N_{initial}} - h_{R_N}$.

Fortran Notation	Statistical Notation	Description
IRO		total number of given values for R_O (must be the same as total number of given values for R'_O).
IRN		total number of given values for R_N (must be the same as total number of given values for R'_N).
MAXTRI, XMAX	N	largest number of components (inspection size) considered
HTRI	h_i	step size between succeeding inspection lots.
INTRI	$i_{initial}$	$i_{initial}$ is the initial or smallest lot which is to be sampled.
ITRI		number of inspection sizes to be sampled.
ALPHA (1) ALPHA (2) ALPHA (3) ALPHA (n)	α 1 α 2 α 3 α n	specified values of α such that $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_n$
AU	A	$A = (1-R_O)/R_O$
AL	A'	$A' = (1-R'_O)/R'_O$
BU	B	$B = (-1/N) \log_e \frac{R_O (1-R_N)}{R_N (1-R_O)}$
BL	B'	$B' = (-1/N) \log_e \frac{R'_O (1-R'_N)}{R'_N (1-R'_O)}$
TRIALS, NTRILS	i	number of components i in inspection size being considered at given stage in sequential sampling procedure.

Fortran Notation	Statistical Notation	Description
RIU	$R_{i_{upper}}$	$R_{i_{upper}} = 1/(1 + Ae^{-Bi})$
RIL	$R_{i_{lower}}$	$R_{i_{lower}} = 1/(1 + A'e^{-B'i})$
QIU	Q_i	$Q_i = 1 - R_i$
QIL	Q_i'	$Q_i' = 1 - R_i'$
BETA (1)	$Beta_1$	Type II or β errors corresponding to α_1, α_2, α_3, α_n
BETA (2)	$Beta_2$	
BETA (3)	$Beta_3$	
BETA (n)	$Beta_n$	
PROB (1)	$Prob_1 = a_1/i$	Answers printed out for values α_1, α_2, α_3, α_n of α denoting ratio of critical region to number inspected.
PROB (2)	$Prob_2 = a_2/i$	
PROB (3)	$Prob_3 = a_3/i$	
PROB (n)	$Prob_n = a_n/i$	

Procedure

The procedure is as follows:

1. Choose a particular combination of values R_0, R_0', R_N, R_N'.

2. Choose a particular inspection size i and final size N.

3. Choose a particular combination of α values.

4. Find the numbers a_1, a_2, a_3, a_n for the given values of α_1, α_2, α_3, α_n.

5. Divide the a's in step 4 by n to get the probabilities which outline the critical region.

6. Find the corresponding β values for each combination of R_0', R_N', α, and i.

Fortran Program Listing
(Statement Numbers in Brackets)

```
                DLOGIC,MAP,FILES
 GROWTH  FULIST,REF,DD
DIMENSION  RUI(501),RIL(501).QIU(501),QIL(501),TERMU(501),TERML(50
1),ALPHA(4),PROB(4),BETA(5),TRIALS(501)
READ (5,1001)ROUI,ROLI,RNUI,RNLI,HROU,HROL,HRNU,HRNL,IRO,IRN,MAXTR     [10]
I,INTRI,HTRI,(ALPHA(I),I=1,4)
FORMAT(8F5.3,2I3,I6,I5,F4.0/4F5.3)                                     [1001]

INITIALIZE RELIABILITY VALUES

ROU=ROUI
ROL=ROLI
DO 2001 I1=1,IRO
ROU=ROU+HROU
ROL=ROL+HROL

CALCULATE CONSTANTS A AND B FOR USE IN SOLVING FOR R(I)

AU=(1.0-ROU)/ROU
AL=(1.0-ROL)/ROL
RNU=RNUI
RNL=RNLI
DO 2002 I2=1,IRN
RNU=RNU+HRNU
RNL=RNL+HRNL
PRINT 4015,ROU,ROL,RNU,RNL
FORMAT (10x4HROU=,F5.3,10x4HROL=,F5.3,10x4HRNU=,F5.3,10x4HRNL=,F5.    [4015]
3//)
XMAX=MAXTRI
BU=(-1.0/XMAX)*ALOG((ROU*(1.0-RNU))/(RNU*(1.0-ROU)))
BL=(-1.0/XMAX)*ALOG((ROL*(1.0-RNL))/(RNL*(1.0-ROL)))
WRITE (6,1002)ROU,RNU,ROL,RNL,(ALPHA(I),I=1,4),(ALPHA(J),J=1,4)
FORMAT(1H1  53X24HCRITICAL  REGION  CURVES///35X18HUPPER GROWTH CU     [1002]
RVE,32X18HLOWER GROWTH CURVE//30X4HRO =,F6.3,8X4HRN =,F6.3,22X4HRO
 =,F6.3,8X4HRN =,F6.3//9X9HNUMBER OF,15X22HPROBABILITY OF SUCCESS,
29X19HTYPE II(BETA) ERROR/11X6HTRIALS,4(2X6HALPHA=,F5.3),1X,4(2X6H
ALPHA=,F5.3))
PTRIAL=INTRI
DO 2003 I3=1,MAXTRI
TRIALS(I3)=I3

CALCULATE UPPER AND LOWER RELIABILITY VALUES, R(I)

RIU(I3)=1.0/(1.0+(AU*(2.71822**(-BU*TRIALS(I3)))))
RIL(I3)=1.0/(1.0+(AL*(2.71822**(-BL*TRIALS(I3)))))

CALCULATE Q VALUES, Q(I)

QIU(I3)=1.0-RIU(13)
QIL(I3)=1.0-RIL(13)
PI3=I3
IF(PI3.NE.PTRIAL) GO TO 2003
PTRIAL=PTRIAL+HTRI
SUMU=1.0
SUML=1.0
BETA(1)=1.0
```

```
IALPHA=1
DO 5000 I=1,I3

SUM PROBS OF ZERO SUCCESSES

SUMU=SUMU*QIU(I)
SUML=SUML*QIL(I)                                                    [5000]

COMPARE SUM OF PROBS OF ZERO SUCCESSES WITH ALPHA VALUES

IF(SUMU-ALPHA(IALPHA))5025,5020,5020                                [5010]
PROB(IALPHA)=0.0                                                    [5020]
BETA(IALPHA+1)=BETA(IALPHA)
IALPHA=IALPHA+1
IF(IALPHA-4)5010,5010,132

CALCULATE BETA VALUES

BETA(IALPHA)=BETA(IALPHA)-SUML                                      [5025]
DO 5040 I=1,I3                                                      [5030]
TERMU(I)=SUMU
TERML(I)=SUML                                                       [5040]
K3=I3-1
IF (K3.EQ.0) GO TO 5120
DO 5080 K=1,K3
J=K
DO 5050 I=1,I3
TERMU(I)=TERMU(I)*RIU(J)/QIU(J)

SUM PROBS OF (ZERO SUCCESSES AND MIDDLE TERM SUCCESSES)

SUMU=SUMU+TERMU(I)
TERML(I)=TERML(I)*RIL(J)/QIL(J)
J=J+1
IF(J.GT.I3)J=1
CONTINUE                                                            [5050]

COMPARE SUM OF MIDDLE TERM PROBS OF SUCCESSES PLUS SUM OF ZERO
SUCCESSES (PROBS OF) WITH ALPHA VALUES

IF(SUMU-ALPHA(IALPHA))5075,5070,5070                                [5060]
PK=K                                                                [5070]
PROB(IALPHA)=PK/TRIALS(I3)
BETA(IALPHA+1)=BETA(IALPHA)
IALPHA=IALPHA+1
IF(IALPHA-4)5060,5060,132
DO 5078 L=1,I3                                                      [5075]

CALCULATE BETA VALUES

BETA(IALPHA)=BETA(IALPHA)-TERML(L)                                  [5078]
CONTINUE                                                            [5080]
RTERMU=1.0                                                          [5120]
RTERML=1.0
DO 5090 I=1,I3
RTERMU=RTERMU*RIU(I3)                                               [5090]
```

```
SUM PROBS OF ALL SUCCESSES

      SUMU=SUMU+RTERMU

COMPARE SUM OF PROBS OF (ZERO,MIDDLE TERMS, AND ALL SUCCESSES)
WITH ALPHA VALUES

      IF(SUMU-ALPHA(IALPHA))3105,5110,5110                              [5100]
 5110 PK=I3                                                             [5110]
      PROB(IALPHA)=PK/TRIALS(I3)
      BETA(IALPHA+1)=BETA(IALPHA)
      IALPHA=IALPHA+1
      IF(IALPHA-4)5100,5100,132
  132 WRITE (6,1003)TRIALS(I),(PROB(I),I=1,4),(BETA(J),J=1,4)           [132]
 1003 FORMAT( F16.0,F12.5,3F13.5,1X,4F13.5)                             [1003]
      GO TO 2003
 3105 WRITE (6,1004)TRIALS(I),IALPHA                                    [3105]
 1004 FORMAT(/F16.0,5X,20HCONDITIONS ON ALPHA(,I1,38H) NOT SATISFIED AFT [1004]
     ER SUMMING OF TERMS)

CHECK TO SEE WHICH ALPHA VALUE WAS NOT EXCEEDED

      IF(IALPHA-1)2003,2003,141
      IF(IALPHA-2)151,151,152                                           [141]
  151 WRITE (6,1005)PROB(1),BETA(1)                                     [151]
 1005 FORMAT(16X,F12.5,40X,F13.5)                                       [1005]
      GO TO 2003
  152 IF(IALPHA-3) 161,161,162                                          [152]
  161 WRITE (6,1006)PROB(1),PROB(2),BETA(1),BETA(2)                     [161]
 1006 FORMAT(16X,F12.5,F13.5,27X,2F13.5)                                [1006]
      GO TO 2003
  162 WRITE (6,1007)(PROB(I),I=1,3),(BETA(I),I=1,3)                     [162]
 1007 FORMAT(16X,F12.5,2F13.5,14X,3F13.5)                               [1007]
 2003 CONTINUE                                                          [2003]
      WRITE (6,1009)
 1009 FORMAT(1H1 50X25HRELIABILITY GROWTH CURVES///35X18HUPPER GROWTH CU [1009]
     RVE,32X18HLOWER GROWTH CURVE//30X2HRI16X2HQI30X2HRI16X2HQI//
     9X9HNUMBER OF /11X6HTRIALS)
      WRITE (6,1008)(TRIALS(I), RIU(I),QIU(I),RIL(I),QIL(I),I=1,MAXTRI)
 1008 FORMAT(F16.0,10X,F7.5,12X,F7.5,25X,F7.5,10X,F7.5)                 [1008]
 2002 CONTINUE                                                          [2002]
 2001 CONTINUE                                                          [2001]
      GO TO 10
      END
```

Hand Calculations

In order to ascertain the validity of the program logic, hand calculations were performed and compared with the computer run as given in Table 2. Due to the high alpha values, the example used is not recommended for other than comparing with hand calculations. As will be seen, the hand calculation ends at n = 3 due to cumbersome calculations for n > 3.

The hand calculations proceed as follows:

1. Calculate R_i and Q_i values for the upper growth curve.

2. For a particular n, calculate probabilities of successes from zero successes through all successes.

3. Compare the probabilities calculated in step two above, with the preassigned alpha values.

4. If the probabilities in step 3 are equal to, or exceed alpha, determine the number of successes of the term which determined if the alpha was met or exceeded, and divide this number by n.

Table - 2

EXACT COPY OF A COMPUTER RUN

	UPPER GROWTH CURVE RO = 0.250 RN = 0.900				LOWER GROWTH CURVE RO = 0.200 RN = 0.800			
Number of Trials	PROBABILITY OF SUCCESS ALPHA=0.60	ALPHA=0.70	ALPHA=0.80	ALPHA=0.90	TYPE II (BETA) ERROR ALPHA=0.60	ALPHA=0.70	ALPHA=0.80	ALPHA=0.90
1.	0.00000	1.00000	1.00000	1.00000	1.00000	0.24805	0.24805	0.24805
2.	0.50000	0.50000	0.50000	1.00000	0.47609	0.47609	0.47609	0.07523
3.	0.33333	0.66667	0.66667	0.66667	0.66722	0.22147	0.22147	0.22147
4.	0.50000	0.75000	0.75000	1.00000	0.41364	0.20940	0.20940	0.11015
5.	CONDITIONS ON ALPHA (3) NOT SATISFIED AFTER SUMMING OF TERMS							
	0.80000	1.00000			0.32656	0.27101		
6.	CONDITIONS ON ALPHA (2) NOT SATISFIED AFTER SUMMING OF TERMS							
	1.00000				0.45392			
7.	CONDITIONS ON ALPHA (1) NOT SATISFIED AFTER SUMMING OF TERMS							
8.	CONDITIONS ON ALPHA (1) NOT SATISFIED AFTER SUMMING OF TERMS							
9.	CONDITIONS ON ALPHA (1) NOT SATISFIED AFTER SUMMING OF TERMS							
10.	CONDITIONS ON ALPHA (1) NOT SATISFIED AFTER SUMMING OF TERMS							

RELIABILITY GROWTH CURVES

Number of Trials	UPPER GROWTH CURVE R(I)	UPPER GROWTH CURVE Q(I)	LOWER GROWTH CURVE R(I)	LOWER GROWTH CURVE Q(I)
1.	0.31669	0.68331	0.24805	0.75195
2.	0.39187	0.60813	0.30327	0.69673
3.	0.47256	0.52744	0.36481	0.63519
4.	0.55470	0.44530	0.43112	0.56888
5.	0.63397	0.36603	0.49999	0.50001
6.	0.70658	0.29342	0.56886	0.43114
7.	0.77002	0.22998	0.63517	0.36483
8.	0.82317	0.17683	0.69672	0.30328
9.	0.86618	0.13382	0.75194	0.24806
10.	0.89999	0.10001	0.79999	0.20001

The actual hand calculations follow:

UPPER GROWTH CURVE

$R_0 = .25,\ R_N = .90,\ N = 10, \qquad R_i = \frac{1}{1 + Ae^{-Bi}}$

$A = \frac{1.00}{R_0} - 1.00 = 3.0$

$B = \frac{1}{N} \log_e \frac{R_0\ (1.00 - R_N)}{R_N\ (1.00 - R_0)} = .3297$

i	B	Bi	e^{-Bi}	A	$1 + Ae^{-Bi}$	R_i	Q_i
0	.3297	0	1	3.0	4.0	.250	.750
1	"	.3297	.71913	"	3.15739	.31672	.68328
2	"	.6594	.51716	"	2.55148	.39193	.60807
3	"	.9891	.37191	"	2.11157	.47358	.52642

LOWER GROWTH CURVE

$R_0 = .2,\ R_N = .8,\ N = 10, \qquad R_i = \frac{1}{1 + Ae^{-Bi}}$

$A = 4$

$B = .2773$

i	B	Bi	e^{-Bi}	A	$1 + Ae^{-Bi}$	R_i	Q_i
0	.2773	0	1	4	5	.200	.800
1	"	.2773	.75782	"	4.03128	.24806	.75194
2	"	.5545	.57435	"	3.29740	.30327	.69673
3	"	.8318	.43526	"	2.74104	.36483	.63517

Comparison with Table 2 shows the hand calculated values of R_i and Q_i to closely approximate the values printed out by the computer; the difference being the computer retains more decimal places than used in the hand calculations.

The calculation of the "Probabilities of Success" as outlined in the main body of Table 2 follows, using probabilities from the upper growth curve:

i	Expansion: $\prod_{i=1}^{n} (R_i \xi + Q_i)$	Alpha .60	.70	.80	.90
1	$R_1 \xi^1 + Q_1 \xi^0$				
	.31672 + .68328	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$
2	$R_1R_2 \xi^2 + (R_2Q_1 + R_1Q_2) \xi^1 + Q_1Q_2 \xi^0$				
	.124 + (.268 + .193) + .415	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{2}$
3	$R_1R_2R_3 \xi^3 + [(R_2R_3Q_1)+(R_1R_3Q_2)+(R_1R_2Q_3)]\xi^2$				
	$+ [(R_3Q_1Q_2)+(R_2Q_1Q_3)+(R_1Q_2Q_3)]\xi^1$				
	$+ Q_1Q_2Q_3 \xi^0$				
	.059 + [.1268 + .0912 + .0653]				
	+ [.1968 + .1410 + .1014] + .2187	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$

It must be remembered that the summation of probabilities begins with zero successes to the number of successes which determines that the alpha of interest has been equalled or exceeded.

The Beta values proceed as follows, working with the lower growth curve values of R_i and Q_i beginning with the "a" value determined from the probabilities of success calculations:

i	$\sum_{i=a}^{n} R_{i,n} = \beta$	Alpha .60	.70	.80	.90
1	Sum terms from 0 successes to 1	1.000			
	Sum terms from 1 success to 1		.24805	.24805	.24805

i		.60	.70	.80	.90
2	Sum terms from 1 success to 2				
	$(R_2Q_1 + R_1Q_2)\,\xi^1 + R_1R_2\,\xi^2$				
	$(.228 + .173)\,\xi^1 + .075\,\xi^2$	.476	.476	.476	
	Sum terms from 2 successes to 2				.075
3	Sum terms from 1 success to 3				
	$[(R_3Q_1Q_2)+(R_2Q_1Q_3)+(R_1Q_2Q_3)]\,\xi^1$				
	$+[(R_2R_3Q_1)+(R_1R_3Q_2)+(R_1R_2Q_3)]\,\xi^2+R_1R_2R_3\xi^3$				
	$[.191 + .145 + .110]\,\xi^1 +$				
	$[.083 + .063 + .048]\,\xi^2 + .027\,\xi^3 =$	.667			
	Sum terms from 2 successes to 3				
	$[(R_2R_3Q_1)+(R_1R_3Q_2)+(R_1R_2Q_3)]\,\xi^2 + R_1R_2R_3\,\xi^3$				
	$[.083 + .063 + .048]\,\xi^2 + .027\,\xi^3 =$		.221	.221	.221

Thus the logic of the computer program listing is proven to be valid since the hand calculations agree with the results shown in Table 2 and Table 2 is an exact copy of a computer run.

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ON FITTING OF THE WEIBULL DISTRIBUTION WITH NON-ZERO LOCATION PARAMETER AND SOME APPLICATIONS

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ABSTRACT. The Weibull distribution is difficult to fit when the location parameter is different from zero.

Although for engineering problems a graphical method for determination of the parameters exist, an application to numerous data samples is very time consuming and elaborate, moreover when the location parameter is different from zero.

Two methods are presented, applicable to computer usage. One method is based upon the moments of the distribution and the second upon a curve fitting procedure. Although neither method utilizes the maximum likelihood principle, application in practical engineering problems may be quite adequate.

Examples of application are given, and the analytical curves from the two methods are compared with observed distributions. Emphasis is placed on close approximation of the 90, 95 and 99% value of wind speed and wind shear distributions.

I. INTRODUCTION. The Weibull distribution (1) has become very popular for many statistical problems in recent times. This is understandable if one considers that this distribution offers several conveniences.

The distribution form

$$F(x) = 1 - e^{-\left(\frac{x-\gamma}{\theta}\right)^{\beta}} \qquad (1)$$

shows 3 parameters, β determining the shape, θ defining the scale and γ establishing the location of reference. The popularity of this distribution is based upon a number of attractive features. The distribution is versatile and can assume various types of other distributions. The application does not necessarily require a specific statistical model, although in life testing a typical case of utilization arises. It is a cumulative distribution, where the threshold can be readily computed directly rather than by an elaborate process of integration as in most other types of distributions. Its three parameters make it more adaptable to many empirical frequency distributions in comparison with two parameter fittings. Difficulties arise, however, if all 3 parameters must be determined.

Usually it is assumed that $\gamma = 0$ and then no problems exist for adequate fitting of the distribution. Maximum likelihood (2, 3) or other

methods (4) are readily available. Limitation to a 2 parameter fit restricts the utilization of the distribution and does not render its full capacity. Curve fitting is rather difficult, however, if $\gamma \neq 0$. Two methods are therefore presented in the following, by which the γ can be determined in objective ways, although the methods are not based upon the maximum likelihood principle. For many engineering applications, however, the two methods, which can also be adapted for computer use, may be quite satisfactory.

One method is derived for the moments fit and does not need the frequency distribution. The second method requires a frequency distribution, although not equal class intervals, as usually assumed by maximum likelihood methods. This second method is based upon a curve fitting procedure.

II. THE MOMENTS FIT. A moments fit of a distribution is in most cases very convenient. The moments of a distribution can be easily computed, and it is not necessary that the total frequency distribution is known for a moments fit. Usually an analytical solution for the parameter computation can be derived. Unfortunately this form of explicit solution for the Weibull distribution with $\gamma \neq 0$ is not trivial, as the β in the moments fit appears implicit in the $\Gamma(n)$. One finds for the Weibull distribution

$$E_{(x)} = \bar{x} = \theta \cdot a + \gamma \tag{2}$$

$$\sigma^2 = \theta^2 \, (b - a^2) \tag{3}$$

$$\varepsilon_3 = \theta^3 \, (c - 3ab + 2a^3) \tag{4}$$

where

$$a = \Gamma \left(1 + \frac{1}{\beta}\right) \tag{5}$$

$$b = \Gamma \left(1 + \frac{2}{\beta}\right) \tag{6}$$

$$c = \Gamma \left(1 + \frac{3}{\beta}\right) \tag{7}$$

and ε_3 denotes the third moment with reference of the mean. This leads to the equation

$$A_1 = \frac{\varepsilon_3}{\sigma^3} = \frac{c - 3ab + 2a^3}{(b - a^2)^{3/2}} \tag{8}$$

In equation (8) the β is the only unknown, although it appears in implicit form. Tables for determining β can be found in a recent report by the author (5). After the β has been obtained,

$$\theta^2 = \frac{\sigma^2}{b - a^2} \tag{9}$$

Tables for the denominator with reference to β are given in the above mentioned report (5). Finally

$$\gamma = \bar{x} - \theta \cdot a \tag{10}$$

The respective numerical value of "a" has also been included in above referenced tables (5).

Thus the moments method is relatively simple. Equation (8) can also be adapted for solutions by high speed electronic computers with subsequent calculations of θ and γ.

The moments fit may have practical value in engineering application.

III. THE "STRAIGHT LINE" FIT. Reservations against the moments fit are largely based upon two objections. First, the moments fit is not always a maximum likelihood fit, which is the modern trend in statistics. Those who oppose the moments fit for that reason will not use this type of solution, although utilization may provide similar results for practical purposes. Therefore no further discussion of this argument is necessary here. The second objection is based upon the fact that 3 pieces of information from the data is employed only, while more information may be available. This is true especially when the frequency distribution is given or known.

Thus the engineers sometimes prefer graphical methods as demonstrated e.g. by Plait (6) or Berrettoni (7). The graphical method as introduced by Berrettoni (7) attracts because of its simplicity for 2 parameters, when $\gamma = 0$. If $\gamma \neq 0$, then the distribution becomes a curved line in log/log paper instead of an easily determined straight line (see Figure at the end of this article). As Berrettoni suggests, one must determine γ by trial. With γ known, the Weibull distribution appears as a straight line in log/log paper, and β and θ can be obtained readily. The cumbersome procedure is to determine γ by this graphical method and make a judgment when the transformed curve is considered a straight line.

By this method γ can only be determined to a certain degree of accuracy, and arguments about differences between moments and maximum likelihood fit become the more irrelevant. The idea behind Berrettoni's method is certainly to employ more information on the distribution than given by the moments. It must therefore be possible to derive an objective way of trial to determine γ and at the same time bring the inaccuracy under a certain limit, which can be arbitrarily selected.

In order to derive the equations for this procedure, start with the transformed equation (1) with $F_y = 1 - F(x)$. Then

$$\ln \left[\ln \frac{1}{F_y}\right] = Y = \beta \ln (x-\gamma) - \beta \ln \theta \tag{11}$$

The goal is a straight line. Thus

$$Y = a_1 z - a_o \tag{12}$$

and

$$z = \ln (x-\gamma) \tag{13}$$

if $\gamma = 0$, then $z = \ln x$. (13a)

$\gamma \neq 0$ produces a curved line (see Figure at end of article) and

$$y = a_1 z - a_o + a_2 z^2 + \ldots a_n z^n \tag{14}$$

or in orthogonal functions

$$y_i = A_o + A_1 \phi_{1i} + A_2 \phi_{2i} + \ldots A_n \phi_{ni} \tag{15}$$

Then the $Y \rightarrow y$, if $A_j \rightarrow 0$ for $j \geq 2$. To meet this condition, a test is necessary for A_2 only since all higher order coefficients must be zero at the same time. Otherwise one would not find a straight line. One can therefore restrict the computations to

$$A_2 = \frac{1}{\phi^2_j} \sum_1^n Y_i \phi_{2i} = 0 \tag{16}$$

where z_i in this orthogonalized system at equidistant intervals corresponds to

$$z_i = \ln (x_i - \gamma) = \ln z'_i \tag{17}$$

More details can be found in a separate report by the author (5), where examples for the solution are given.

IV. COMPARISON OF METHODS. Before applications are presented it may be adequate to discuss some technical details and limitations of the two methods.

It has been previously stated that both methods are not derived from the maximum likelihood principle and may therefore be of no interest to the theoretical statistician or may be considered as substitute methods. The moments fit attracts as being straight forward with a relatively simple way of computing the parameters. Only three moments need to be known. From the engineering point of view the "straight line method" comes closer to a graphical type of solution and renders the better curve fitting. There is no necessity of the frequency distribution being given in equal class intervals. This is quite convenient, but the frequency distribution is required in contrast to the moments fit.

One limitation can be found, however, in the exclusion of the $F_{(x_1)} = 0$. This leads to $\ln(\ln 1)$ in equation (11), which being infinity must be eliminated. The question arises therefore, how close to $F_{(x_1)}$ can one go,

or should the first x be omitted. In Table I a survey is given for 3 data samples, where the Weibull distribution has been established as being appropriate. The first column in Table I lists the parameters, which Berrettoni (7) has derived by his graphical method. The second column represents the moments solution. The subsequent columns reflect the parameters for the straight line method under various conditions.

It is self-explanatory that in the column "without $F_{(x_1)}$" the origin point has been omitted. The other columns show how the parameters change, if $F_{(x_1)} = 0.001$; 0.0005 or 0.0001. It can be concluded that the γ responds somewhat to the change of the origin and with γ the other parameters will vary (see especially case 3). It can be noted, too, that the solutions without the $F_{(x_1)}$ agree well with Berrettoni's results. The small differences can easily be explained by inaccuracies between graphical and computational methods. Under the aspect that the graphical solution and the straight line method are not maximum likelihood solutions these small differences become even more insignificant.

In order to test the differences of the methods for significance, one can apply the Kolmogorov-Smirnov Test (8). None of the deviations proved to be statistically significant. More details can be found in a forthcoming article by the author (9).

Since case 3 displayed the largest differences, the cumulative distribution was computed for various postulations and is summarized in Table II. The first column (after the variable x) contains the observed distribution. Berrettoni's solution (7) follows next. Subsequently the computed frequency for the "moments" and the "straight line" method are listed. The underlining of numbers indicates the maximum deviation for all presented curves in that particular line. This example is quite typical. Although none of the differences to the observed value reaches statistical significance at the 95% level, it can be seen that the 3 methods approximate the observed distribution in specific ways. The moments method reveals closer fitting towards the maximum values, while the straight line procedure deviates less at the minimum values. The graphical solution (Berrettoni) provides the maximum deviation in the center.

The other 6 columns, experimenting with varying x_1 and the related cumulative frequency as outlined in the heading lie somewhat in between except for $x_1 = 1.1$ with $F_{(x_1)} = 0.001$. This condition exhibits the largest deviation. It proves that the frequency of $F_{(x_1)}$ should be kept as close to zero as possible, although no specific value can be established.

V. APPLICATION TO WIND SPEED AND WIND SHEAR DATA. In an earlier article the author has introduced the negative binomial distribution which functions quite satisfactorily for frequency distributions of wind speed (10). Cumulative threshold values, however, are very cumbersome to compute for the negative

binomial frequency distribution. The question was thus raised whether the Weibull distribution may be an adequate replacement. Cumulative thresholds can be easily obtained from the Weibull distribution.

Table III displays typical results of fitting the Weibull distribution to wind data. It can again be recognized that the moments fit approximates closer the maximum wind speeds, while the straight line method adjusts better to lower wind speeds. In general, the Kolmogorov-Smirnov test shows statistically significant differences between observed and analytical values computed from the Weibull distribution (see details in 11). This proves that the Weibull distribution is not the best suitable form to fit wind speed or shear data. A limited application, however, turned out to be quite valuable.

The engineer is often faced with the problem to determine 90 - 99% values when no detailed distribution is given. Since the moments fit of the Weibull distribution has given good results for the maximum wind speeds, an attempt was made to analytically determine the 90, 95 and 99% wind speed and wind shear value and compare it with the observed. The results are presented in Tables IV thru VII.

In Table IV three methods are compared for computing 90, 95 and 99% thresholds for wind speed and wind shear values. Montgomery was selected, as it illustrates typical results. The three threshold values were analytically computed, employing the negative binomial, bivariate and Weibull distribution (moments fit). Analytical distributions for negative binomial and bivariate distribution are described in detail in a recent report by the author (10).

The thresholds were computed at 1 km altitude intervals up to 31 km for all months. The (linear) correlation between observed and analytical value was thus computed, as exhibited in the top part of Table IV. This gives evidence that the Weibull distribution is equivalent to the negative binomial except for the wind shear and 99% threshold. The Weibull distribution is even better than the bivariate distribution, which is generally agreed to be the proper distribution form for wind speed and shear.

The central part of Table IV lists the mean of Δ, the difference between analytical and observed wind speed or shear. Although the observed values for the wind speed appear to be systematically higher than the analytical values, the bias is smallest for the Weibull distribution. No bias is exposed for the wind shear.

The bottom part of Table IV deals with the standard deviation of the difference Δ. Again, the results are very favorable for the Weibull distribution except for the 99% wind shear estimate.

Since correlation, mean values or standard deviations can sometimes be misleading, the frequency distributions of the Δ are presented in Tables V, VI, and VII. This also gives a survey on the maximum deviations to be expected for the various analytical approaches. Table V contains the frequency distribution of Δ for the 90% threshold, where the Weibull distribution looks

very good. Less than 5% of the data exceed ± 2 m/sec for the wind and ± 1 m/sec per km for the wind shear. This is entirely in the range of measurement accuracies.

The differences are higher for the 95% threshold of the wind speed, but still under 10% of the data fell outside the above cited range. The amount is far higher for the negative binomial or the bivariate distribution. The wind shear differences are equivalent for all three types of analytical forms in the 95% thresholds.

Finally the frequency distribution of the differences Δ for the 99% threshold is given. Although the range is extended compared with the previous thresholds, the Weibull distribution displays still the smallest scatter of all three methods for the wind and could be considered equivalent to the negative binomial for the wind shear. This may be proof enough that the Weibull distribution could be adequately used for practical purpose in the analytical approximation of 90 to 99% thresholds.

VI. CONCLUSIONS. Two methods for fitting the Weibull distribution with non-zero location parameter have been discussed. One method, based upon the first 3 moments of the distribution, provides a simple way of obtaining the basic input for determining the parameters of the Weibull distribution, although the solution necessitates a computer or table as derived by the author (5).

A further method is based upon a curve fitting procedure. The property of the Weibull distribution to delineate a straight line in log/log scale for known location parameters is the fundamental principle employed in solving for the parameters. The latter method requires that the frequency distribution is known, although not at equidistant intervals. In turn, more information (namely all known frequency points) is utilized by this curve fitting procedure in contrast to the 3 moments only for the moments fit.

Both methods are suitable for determining the Weibull parameters without electronic computers, while the iterative procedure for a maximum likelihood solution cannot be processed without computer help. The moments fit, however, needs the tables derived by the author (5) if no computer is available. Thus both methods may prove beneficial to the engineer for quick solution in limited number of samples, although it is not restricted to a small number.

It has been shown that the moments fit in most cases represents the better fit towards the end of the maximum values, while the curve fitting procedure puts more weight on the proper approximation of the minimum values. This could be changed, however, by weighting the frequency points of the distribution for the curve fitting method.

Finally an application of the Weibull distribution for wind and wind shear data is shown. Although the Weibull distribution has limited application for wind and wind shear, the moments fit proved to be satisfactory to represent 90 to 99% thresholds.

TABLE I

Comparison of Parameters Estimation for the Weibull Distribution

Case 1	Berrettoni	Moments	Straight line method with $F(x_1)$ equal to: Without $F(x_1)$	.001	.0005	.0001*
γ	3.0	2.74	2.71	3.03	3.06	3.07
θ	3.67	3.97	3.99	3.59	3.55	3.51
β	1.8	2.00	2.00	1.76	1.74	1.72
α	10.4					
Case 2						
γ	3.6	1.95	3.63	2.82	2.92	3.03
θ	7.39	9.04	7.13	8.06	7.96	7.82
β	1.7	2.25	1.66	2.06	2.02	1.98
α	29.96					
Case 3				(x_1 = 1.1)		
γ	2.0	1.67	2.10	.71	.83	.97
θ	4.485	4.87	4.39	6.23	6.09	5.93
β	1.8	1.85	1.69	2.52	2.46	2.40
α	14.9					
				(x_1 = 2.0)		
γ				1.89	1.93	1.95
θ				4.71	4.66	4.64
β				1.83	1.80	1.79

*) The limit was .0002 for case 1.

TABLE II

Comparison of Weibull Distribution for Various Methods for Data of Berrettoni's Table III (CFD in %)

					Straight Line Method					
					$x_1 = 1.1$			$x_1 = 2.0$		
X	Obs	Berr.	Mom	Without	.001	.0005	.0001	.001	x .0005	.0001
1.0	-	-	-	-	.04	.01	.00	-	-	-
2.0	-	.00	.68	-	1.87	1.70	1.47	.09	.04	.02
3	6.6	6.49	8.66	6.54	7.71	7.56	7.30	6.85	6.77	6.74
5	38.4	38.42	39.04	39.01	32.32	32.49	32.57	37.37	37.51	37.56
7	71.4	70.36	69.32	70.00	64.09	64.36	64.55	68.65	68.72	68.72
9	87.9	89.23	88.12	88.37	87.17	87.26	87.29	88.01	87.96	87.90
11	96.7	96.98	96.41	96.35	97.09	97.08	97.03	96.44	96.37	96.32
13	98.9	99.34	99.15	99.06	99.60	99.59	99.56	99.17	99.13	99.11
15	100.0	99.88	99.84	99.80	99.96	99.96	99.96	99.84	99.83	99.82
17	-	-	99.97	99.96	-	-	-	-	99.97	99.97

The maximum deviations are underlined.

TABLE III

Wind Speed (m/sec) Comparison of Weibull Distribution

CFD*)	Observed	Moments fit	Straight Line Method
.0001	3.0	11.9	3.4
.0100	20.4	19.4	17.9
.0228	21.7	22.4	22.3
.0500	25.6	26.1	27.3
.1000	31.6	30.3	32.6
.1590	35.3	33.9	36.7
.5000	47.8	47.5	50.7
.8410	61.9	61.5	63.2
.9000	65.8	65.4	66.5
.9500	69.8	70.3	70.5
.9772	76.9	75.0	74.3
.9900	80.6	79.3	77.6
.9995	91.0	91.5	86.6

*) CFD = cumulative frequency distribution

TABLE IV

Montgomery (June 1956 - May 1964)

(All Months Combined)

Average Correlation Between Observed and Analytical Thresholds

	Wind			Wind Shear		
	Neg. Bin.	Biv.	Wei.	Neg. Bin.	Biv.	Wei.
90%	.982	.996	.998	.977	.986	.991
95%	.984	.995	.997	.968	.977	.976
99%	.985	.985	.994	.940	.904	.914
			Mean of Δ			
90%	- .50	1.36	- .44	.16	.93	.05
95%	- .72	.46	- .66	.03	.08	.10
99%	-1.14	-2.09	-1.01	-.63	-2.59	-.02
			Standard Deviation of Δ			
90%	1.51	1.57	.80	.53	.50	.33
95%	1.65	1.93	1.07	.73	.71	.65
99%	2.20	3.08	1.80	1.37	2.07	1.60

Δ = analytical - observed

TABLE V

Montgomery (June 1956 - May 1964)

90%

Frequency Distribution of Δ

	Wind			Wind Shear		
	Neg. Bin.	Biv.	Wei.	Neg. Bin.	Biv.	Wei.
< -10	1					
-9.99 to -9.0	0					
-8.99 to -8.0	0					
-7.99 to -7.0	2					
-6.99 to -6.0	1	1				
-5.99 to -5.0	2					
-4.99 to -4.0	6		1			
-3.99 to -3.0	6		3			
-2.99 to -2.0	14	2	7			
-1.99 to -1.0	54	4	32	13		4
-0.99 to 0.0	174	37	252	116	5	132
.01 to 1.0	83	132	68	204	203	209
1.01 to 2.0	22	94	8	13	128	3
2.01 to 3.0	4	39	1	2	11	
3.01 to 4.0	3	22			1	
4.01 to 5.0		8				
5.01 to 6.0		6				
6.01 to 7.0		5				
7.01 to 8.0		2				
8.01 to 9.0						
9.01 to 10.0						
> 10.0						
N	372	372	372	348	348	348

Δ = analytical - observed

classes in m/sec for wind and m/sec per 1 km for wind shear

TABLE VI

Montgomery (June 1956 - May 1964)

95%

Frequency Distribution of Δ

	Wind			Wind Shear		
	Neg. Bin.	Biv.	Wei.	Neg. Bin.	Biv.	Wei.
< -10						
-9.99 to -9.0	1	1				
-8.99 to -8.0	1					
-7.99 to -7.0	2		1			
-6.99 to -6.0	2	3	1			
-5.99 to -5.0	4	1	2			
-4.99 to -4.0	6	2	1			
-3.99 to -3.0	12	5	7		2	
-2.99 to -2.0	25	5	18	4	3	3
-1.99 to -1.0	58	27	60	22	15	11
-0.99 to 0.0	146	97	207	148	114	134
.01 to 1.0	86	123	64	148	197	174
1.01 to 2.0	21	57	11	23	17	24
2.01 to 3.0	6	25		3		2
3.01 to 4.0	2	10				
4.01 to 5.0		9				
5.01 to 6.0		2				
6.01 to 7.0		2				
7.01 to 8.0						
8.01 to 9.0		3				
9.01 to 10.0						
> 10.0						
N	372	372	372	348	348	348

Δ = analytical - observed

class interval in m/sec for wind and m/sec per km for wind shear

TABLE VII

Montgomery (June 1956 - May 1964)

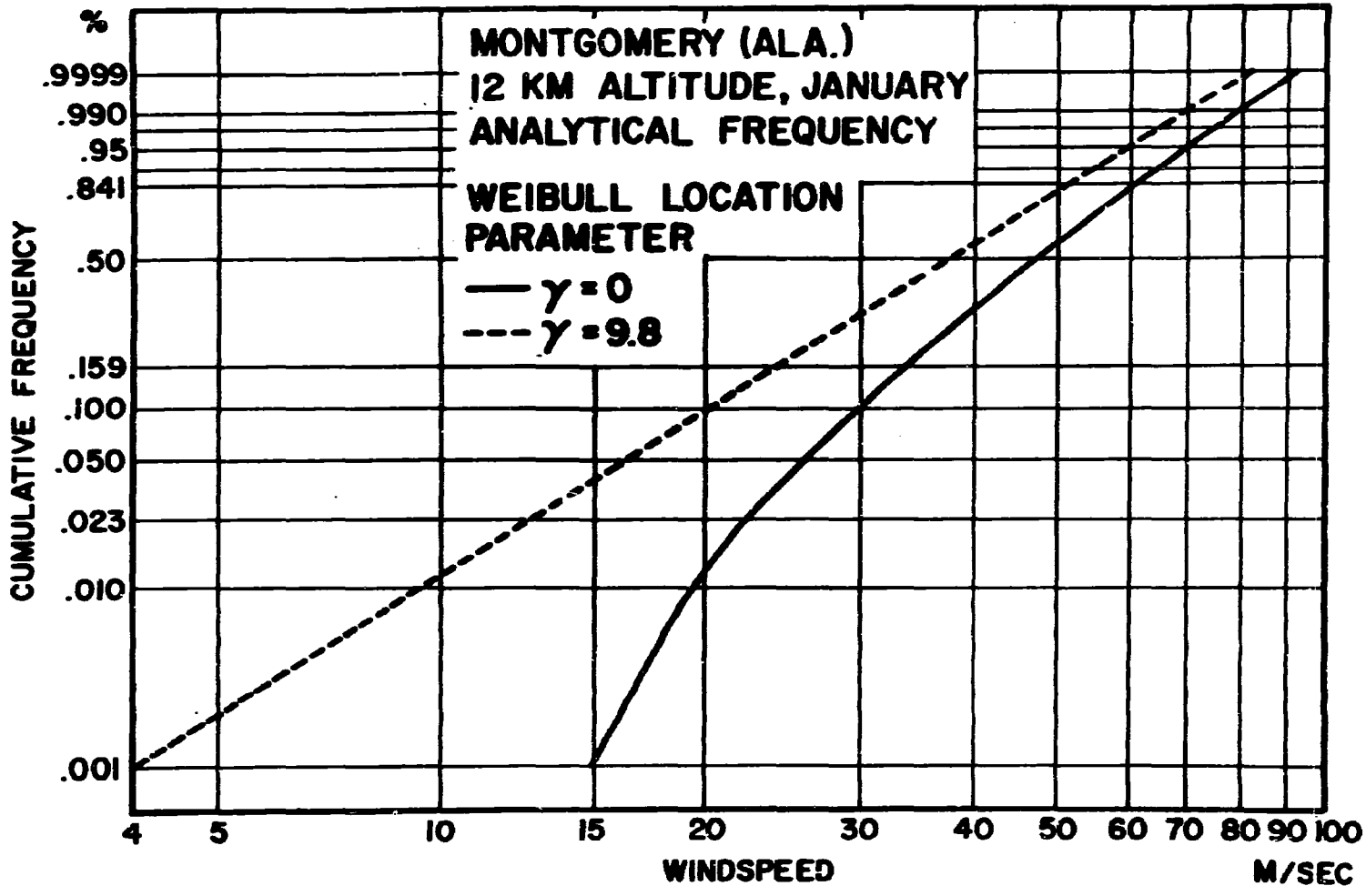
99%

Frequency Distribution of Δ

	Wind			Wind Shear		
	Neg. Bin.	Biv.	Wei.	Neg. Bin.	Biv.	Wei.
< -10	1	8			3	
-9.99 to -9.0	1	4	2		2	
-8.99 to -8.0	3	4	0		4	
-7.99 to -7.0	6	10	2	1	6	
-6.99 to -6.0	0	17	5	2	6	1
-5.99 to -5.0	8	21	1	2	20	0
-4.99 to -4.0	10	24	5	3	29	2
-3.99 to -3.0	12	26	21	9	49	7
-2.99 to -2.0	59	35	42	32	75	7
-1.99 to -1.0	90	70	105	61	66	49
-0.99 to 0.0	100	70	113	131	70	119
.01 to 1.0	43	37	42	88	18	84
1.01 to 2.0	19	24	21	14		31
2.01 to 3.0	13	12	5	4		20
3.01 to 4.0	3	6	2	1		9
4.01 to 5.0	1	2	5			6
5.01 to 6.0	1	2	0			2
6.01 to 7.0	0		1			1
7.01 to 8.0	0					
8.01 to 9.0	1					
9.01 to 10.0	0					
> 10.0	1					
N	372	372	372	348	348	348

Δ = analytical - observed

class interval in m/sec for wind and m/sec per 1 km for wind shear


MONTGOMERY (ALA.)
12 KM ALTITUDE, JANUARY
ANALYTICAL FREQUENCY
WEIBULL LOCATION PARAMETER
γ = 0
γ = 9.8
%
.9999
.990
.95
.841
.50
.159
.100
.050
.023
.010
.001
CUMULATIVE FREQUENCY
4
5
10
15
20
30
40
50
60
70
80
90
100
WINDSPEED
M/SEC

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THIRD SAMUEL S. WILKS AWARD

Presented by

Dr. Frank E. Grubbs

Left to right: Frank E. Grubbs, William Cochran, Frank Robertson, Walter Foster, Frank Proschan

COCHRAN AWARDED 1967 WILKS MEMORIAL MEDAL

Professor William G. Cochran of Harvard University received the 1967 Samuel S. Wilks Memorial Medal during the 13th Annual Conference on the Design of Experiments in Army Research, Development and Testing, which was held at Fort Belvoir, Virginia, 1-3 November 1967. Professor Cochran has long been recognized as an international authority for his outstanding contributions to experimental statistics, mathematical statistics, the design and analysis of scientific experiments, teaching activities, stimulation of research workers and personal leadership in the world statistical community.

The Annual Design of Experiments Conferences are sponsored by the Army Mathematics Steering Committee on behalf of the Office of the Chief of Research and Development, Department of the Army.

The Wilks Award is given each year to a statistician and is based primarily on his contributions, either recent or past, to the advancement of scientific or technical knowledge in Army statistics, ingenious application of such knowledge, or successful activity in the fostering of cooperative scientific matters which coincidentally benefit the Army, the DOD, and the Government, as did Samuel S. Wilks himself.

The Award consists of a medal, with a profile of Professor Wilks and the name of the Award on one side, and the seal of the American Statistical Association and the name of the recipient on the other side; an honorarium related to the magnitude of the award funds donated by Mr. Rust; and a citation.

With the approval of President Frederick Mosteller of the American Statistical Association (ASA), the Wilks Award Committee for 1967 consisted of:

Professor Robert E. Bechhofer, Cornell University
Dr. Francis G. Dressel, Duke University and the Army Research Office-Durham
Dr. Churchill Eisenhart, National Bureau of Standards
Professor Oscar Kempthorne, Iowa State University
Dr. Alexander M. Mood, U.S. Office of Education
Major General Leslie E. Simon (Ret.), Winter Park, Florida
Dr. Frank E. Grubbs, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland - Chairman

Professor Cochran was born in Rutherglen, Scotland, and received MA degrees from Glasgow University and Cambridge University. He was a statistician working with the eminent R.A. Fisher at the Rothamsted Experimental Station (England), 1934-1939; Professor of Mathematical Statistics, Iowa University, 1939-1946; Associate Director of the Institute of Statistics, University of North Carolina, 1946-1948; Professor of Biostatistics, School of Hygiene and Public Health, The Johns Hopkins University, 1948-1957; and has been Professor of Statistics at Harvard University since 1957.

In connection with professional societies and committee activities, Professor Cochran has served as follows:

Fellow, American Statistical Association:
President, 1953
Editor of the Journal of the American Statistical Association, 1945-1950

Fellow, Institute of Mathematical Statistics:
President, 1946

Fellow, American Public Health Association

Member, Biometric Society:
President, 1954,1955

Honorary Fellow, Royal Statistical Society

Member, International Statistical Institute:
Vice-President, 1963-1967

Fellow, American Association for the Advancement of Science:
Vice-President, 1966.

Committee activities:

Chairman, Panel of Advisors on Sampling, U.S. Bureau of the Census

Chairman, Committee on Training in Epidemiology and Biometry, N.I.H.

Member, Advisory Committee to Atomic Bomb Casualty Commission

Member, Committee on Statistical Education, Inter-American Statistical Institute.

Professor Cochran has published books as follows:

E.J. Russell, J.A. Voelcker, and W.G. Cochran, Fifty years of field experiments at the Woburn Experimental Station. Longmans, Green and Co., London, 1936.

W.G. Cochran and Gertrude M. Cox, Experimental designs. John Wiley and Sons, New York, 1950. Second edition, 1957. Japanese translation, 1954. Spanish translation, 1965.

W.G. Cochran, Sampling techniques. John Wiley and Sons, New York, 1953. Second edition, 1963. Portuguese translation, 1965.

W.G. Cochran, F. Mosteller and J.W. Tukey, Statistical problems of the Kinsey Report. American Statistical Association, Washington, D.C., 1954.

G.W. Snedecor and W.G. Cochran, Statistical methods, 6th edition. Iowa State University Press, 1967.

Professor Cochran is the author of some eighty-five papers which form the very basis for much of the wide-spread use of statistical techniques, and

otherwise represent some of the more significant and widely employed methodology in the entire field of theoretical and experimental statistics.

Indeed, statisticians throughout the world regard Professor Cochran as a "giant" in the field due to his numerous and wide-spread basic contributions.

Professor Cochran's most recent honor is the presidency of the International Institute. His work in design of experiments recently has dealt with the efficient sequential determination of levels, and more recently he has also been working on the design and analysis of observational studies. His books have been translated into several languages.

The citation to Professor Cochran reads as follows:

> "To Professor William G. Cochran - for continued research on the statistical treatment of data, for his highly fertile research on the design and analysis of experiments and surveys, for his excellent books on the theory and practice of statistical methodology, for his efforts in the training of statisticians at all levels, and for his contributions to national and international statistical societies."

Professor Cochran received the third Wilks Memorial Medal at the banquet held in connection with the Thirteenth Conference on the Design of Experiments. Dr. Frank E. Grubbs made the presentation. The acceptance remarks of Professor Cochran are printed below.

Chairman Grubbs, Ladies and Gentlemen;

I greatly appreciate this high honor. It is especially pleasing because Sam Wilks was the first American statistician whom I ever met. This was in 1933, when I was a graduate student at Cambridge University. Sam came there as a postdoctoral International Fellow, so that I enjoyed over 30 years of his friendship, including working under Sam in 1944 in the Princeton Statistical Research Group of the Office of Scientific Research and Development.

An occasion like this naturally stimulates reflection about one's past work. I might mention one habit, common among statisticians, that has helped me. In consulting, there are always times when I cannot answer the question, and times when I give an answer, but realize after the investigator has left that I distorted his question in order to make it fit into some standard statistical mold. I like to make a note of the difficulty and, so far as my ability permits, to see if anything constructive can be done about it.

This habit also protects effectively against any tendency to develop a swelled head. From time to time I see a new paper that presents the first competent handling of a problem of which I have been ineffectively aware for many years. Now my subconscious self is one of the most unpleasant characters

I have ever had to deal with. On such occasions it always surfaces and says, "See, if you had any brains, or had paid a little attention to the advice that I keep giving you subconsciously, you might have cleaned up this problem 20 years ago."

In thinking about the present state of work in statistics from the viewpoint of allocation of resources, we seem now to be well provided with research manpower in mathematical statistics. In fact, I have sometimes tried to argue that there is too much research in mathematical statistics, though when I do this, everybody jumps on me. In academic circles, the idea that one can have too much research on any subject is heresy of the worst kind.

As an illustration, consider a problem that has arisen in the last 15 years. In the sampling of institutions like businesses, schools, hospitals, and counties, that vary in size, there is need for a method of selecting a sample without replacement and with probabilities proportional to measures of the sizes of the units. There are two main difficulties. With the simplest methods of selection, it is impossible to compute from the sample an unbiased estimate of the variance; with other methods, the estimate of variance is so unstable that negative estimates of variance can turn up. Secondly, as the sample size increases, it becomes harder to keep the probabilities proportional to size.

The problem is important enough so that under a system of planned resource allocation one could justify assigning three or four good men in different places and preferably in different countries, to work on it independently. Now Mr. Kenneth Brewer, Director of Methodology, Commonwealth Bureau of Census and Statistics, Canberra, Australia, is currently spending some time with us at Harvard. One of his tasks is to prepare what will be a highly useful comparative and critical review of the methods that have been produced for sampling with probabilities proportional to size without replacement. To date he has found in the literature not 3 or 4 methods, but 34. Indeed, when the latest issue of any journal reaches my desk these days, I hesitate to open it, in case it contains yet another method which Mr. Brewer will have to compare with the current crop of 34. It almost sounds like too much of a good thing.

In two other aspects of the health of our profession, however, the situation seems less favorable. One aspect concerns mechanisms for ensuring that new and useful statistical techniques are regularly explained to the potential users in language that they can understand; the other, mechanisms by which statisticians are regularly kept informed of unsolved statistical problems encountered by users. The difficulties in this kind of communication are well known. Users have little time to devote to learning statistical techniques and often very limited knowledge of mathematics and probability; work by statisticians on exposition carries little prestige; and the efforts of the statistical profession in this area have been most sporadic and voluntary. In this connection, I think that Sam Wilks, after his early years of brilliant and productive research, deliberately chose to sacrifice his future research interests in order to concentrate on organizational problems in the new field of statistics, including problems of communication with

users, because he judged them so important. For the same reason I regard these regular annual conferences on the Design of Experiments in Army Research, Development and Testing as one of the most useful activities in statistics in this country. These conferences help to meet both needs that I have mentioned. They provide an annual opportunity for describing new techniques on a wide range of topics by speakers skilled in this kind of exposition, and also an occasion on which statisticians can learn about questions raised by the users to which current research does not supply an answer. Long may they continue.

DETERMINATION OF TBO BY WEIBULL DISTRIBUTION USING REPAIRABLE COMPONENTS

John L. Mundy
U.S. Army Aviation Materiel Command
St. Louis, Missouri

ABSTRACT.

1. The Army Aviation Command has found that a serious discrepancy exists between the figures set by contractors for the life time of critical components and for the Time Between Overhaul (TBO) for noncritical components and the figures actually achieved in practice. For many components only 8% ever reach their rated life time, and only 5% reach their TBO time.

2. In addition, it is necessary to determine the time required to break-in systems, if any. This break-in time is sometimes referred to as burning-in.

3. To determine the statistical TBO, life times, and break-in periods, as set by actual field usage, the Weibull probability distribution was applied. The work of Mr. J.H.K. Kao was extended from non-replaceable items to repairable items. Three-phase life was used consisting of Infant Mortality, Catastrophic or Random Failure period, and the Wearout period. The graphical trial and error method of Mr. Kao was replaced by a Fortran computer program. In addition, the iterative method was streamlined into a deterministic method. This represents a major contribution which reduced the computer time by 85%.

4. Flow charts of the operation have been prepared. The source of data is the DA Form 2410 and DA Form 2408-3.

ACKNOWLEDGEMENTS. Many AVCOM engineers contributed technical assistance to this report. Gratitude is due to Mr. Wm. Brabson, R&D Division, Mr. J.K. Gerdel, Special Studies Office, Mr. D. Burchfield, Quality Assurance Office and Mr. D. Fleming, Quality Assurance Office for Supervisory support.

The lengthy Computer Flow Chart was prepared by Mr. M. Ploudre, R&D Division.

Valuable programming assistance is being supplied by Mr. F. Blackshear, Special Studies Office.

The intricacies of the TAERS operations were thoroughly explained by Mr. R. Jesse and Mr. F. Grueninger, Directorate of Maintenance and Mr. M. Christianer, General Engineering.

The essential 'hammer and nails' work of hand checking of calculations was done by Mr. M. Ploudre, R&D Division, with the help of Mr. D. Carter, R&D Division.

CONTRIBUTIONS.

1. Use of TAERS to:

   a. Break down Failure History to No. of Overhauls.

   b. Break down Failure History to No. of Repairs after overhaul.

   c. Break down Failure History to Age of item after "N" repairs after "M" overhauls.

   d. Use of TAERS to report Unfailed as well as Failed Items.

2. Method for Identification of Pblty Distribution Composite, under assumptions that composite consists of not more than 3 other Pblty Distributions.

3. Determination of Burn-In, TBO or Finite Life for components, as determined by field usage, instead of engineers prior to fielding.

4. Two methods for Non-Graphical Determination of Weibull Shift Parameters (Gamma).

5. Complete Elimination of need of Kao Plotting Paper.

THE AUXILIARY WORK TAPE LAYOUT. The Army Aviation Command of St. Louis, Mo. maintains a Validated Tape File of DA 2410 forms received from the field. This form is completed by Repair-Personnel, and contains data concerning the removal and repair of components. This form is one of the class of forms, known as TAERS. Presently about 12 million 2410 records which have been validated, are on file.

From this tape, certain items were extracted. These items were combined with other items from other tapes to create an auxiliary work tape, which contains all the items needed in this program. This work tape layout is shown in Figure 1. This figure 1 shows one record on the tape. One record will exist for each 2410 report.

This program will determine the "Burn-In" time, and the "Time Between Overhaul", (TBO) from field data. The field value of TBO will be compared with the Established TBO in columns 39-42; and it will also be compared with values of competitor's interchangeable parts. The interchangeable part numbers are obtained from another program, and printed in columns 168-297.

Since this program analyzes the failure times of each component, the four dates in columns 73-88 are very important.

The first date in 73-76 is the Date of First Installation of a new serial number, (FID). This is obtained from Copy 6 of a 2410 record under the condition that Copy 1 is missing, for the same Document Control No. in Copy 6. Copy 6 is used for installation reporting, and Copy 1 is used for removal reporting. Therefore, when an item is new, and originally installed, a Copy 6 will be made out before Copy 1, under one document number. Later,

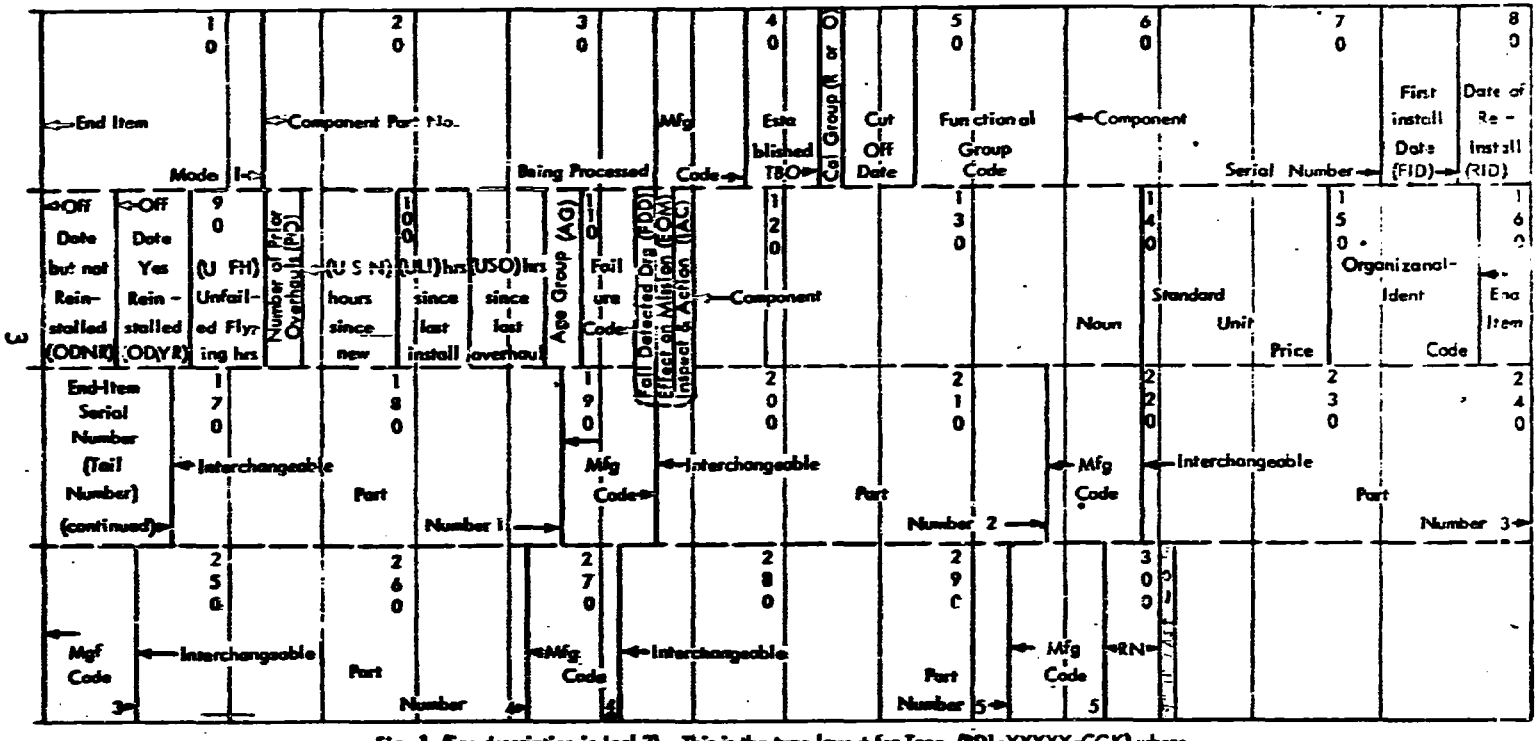


Fig. 1 (See description in Incl 2). This is the type layout for Tape (RDI-XXXXX-CGK) where X's mean End Item Model such as CH-47, and "K" may be either "R" which means "recent" or "K" may be "O" which means "older".

when the component is removed, a Copy 6 will be found under a different control number.

The second date, in 77-80, is the Date of Re-Installation (RID).

This Julian date (RID) is taken from Copy 6, when a Copy 1, is present, under the same document number.

The third Julian Date of importance is (ODNR) which is Off-Date, Not Reinstalled. It is taken from Copy 1, if Copy 6 is absent.

The fourth and last Julian date needed is (ODYR) which is Off-Date Yes - Reinstalled. It is taken from Copy 1, if Copy 6 is present.

After this auxiliary tape is created, two other data tapes are made from it, by various sortings and re-arrangements to facilitate programming.

One tape will contain records within 25 days to 390 days of the most recent record. Data within the first 25 days is deleted, to allow for delays in the mail.

Each tape is then sorted as shown in Fig. 2.

2. SORTING OPERATIONS ON THE WORK TAPE. Fig. 2 shows that the major grouping is by Part Number, followed by 3 minor groupings - Overhaul Group, Age Group, and Equal Number of Repair Times. This is followed by sorting according to the time of failure.

This program is the first that categorizes the parts according to the number of overhauls. This will determine the failure history of new parts, such as engines, compared to engines which have been overhauled X-number of times.

Another first, within AVCOM, is the grouping of parts of the same age, within the same set. This method will reveal whether failure rates are constant, for components of different ages. This assumption of constant failure rate is made many times, for no other reason but that it is simple to use. This analysis will check the validity of this assumption.

The lower right bar shows that unfailed items are also considered. This is the first time that TAERS has been used to report unfailed as well as failed items. This method is presented next.

3. HANDLING OF UNFAILED ITEMS. The first step is to identify those items still in the aircraft.

This is done by analysis of the 4 critical Julian Dates previously discussed. A sample set of these 4 dates for one serial numbered component is shown below in Figure 3.

Sorting Operations

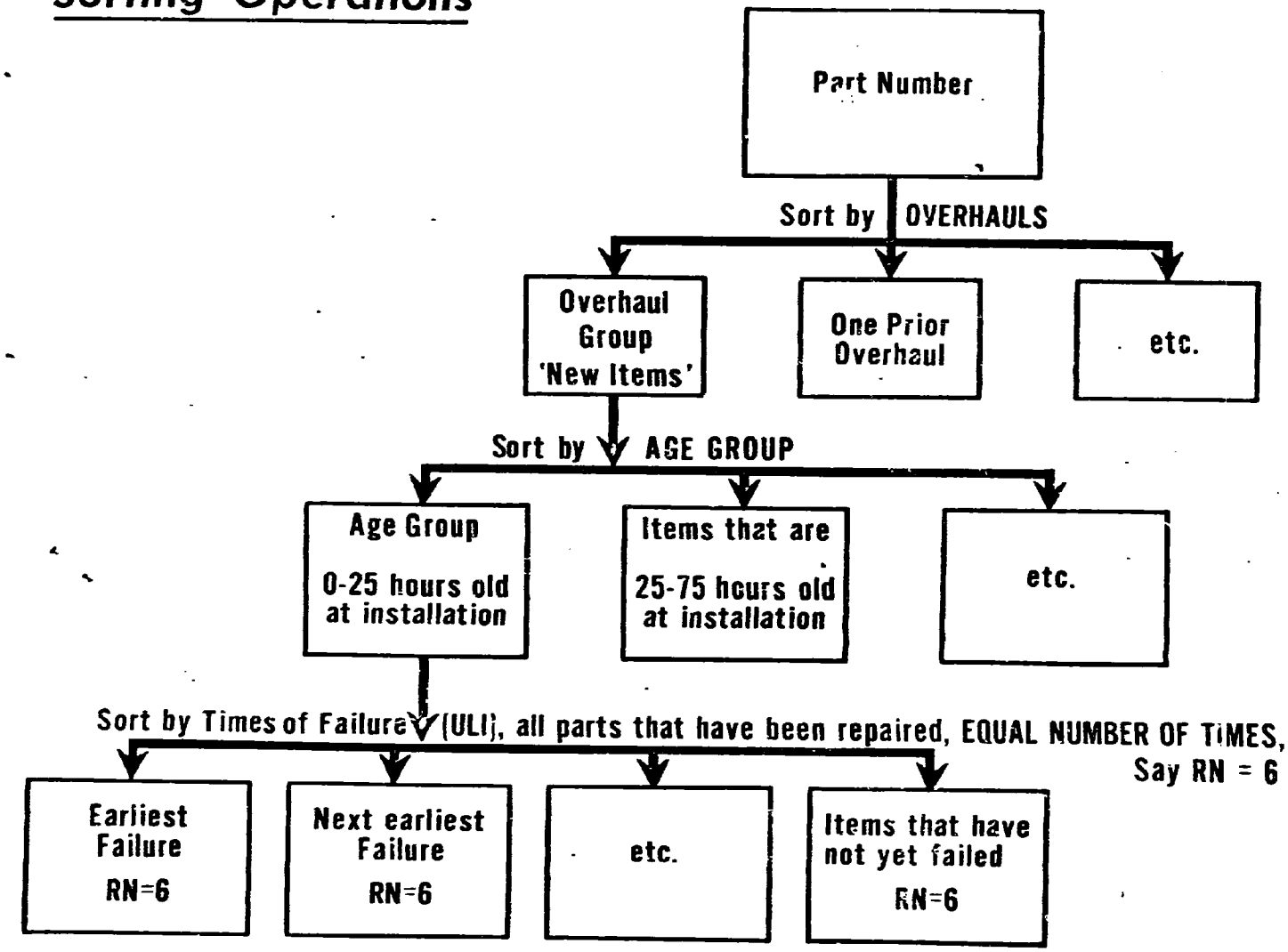


Fig 2

Doc. Control Number	FID (Copy 6, date, when no 1)	ODYR Copy 1, date, when have 6	RID Copy 6 date when have 1	ODNR (Copy 1 date when no 6)
1	4(365)	0	0	0
2	0	5(002)	5(003)	0
3	0	6(205)	6(206)	0
4				6(115)

Look at the FID or RID - item installation date. Then look at the ODRN, to see if the item has been removed at a later date.

In this set, the highest installation date is RID = 6(206).

This item has not been removed at any date later than 6(206), as shown in the ODNR column, because the highest ODNR is 6(115).

Therefore, the test consists of 2 steps:

Step 1) Find the highest installation date, whether it is FID, or RID. In our sample, it is RID = 6(206).

Step 2) Test whether this RID is greater than the highest value of "Off Date - Not Reinstalled," (ODNR). Here it is 6(115), so the answer is "Yes." This item is then treated as an "Unfailed Item."

Note that this ODNR column would be all zeroes, for an Unfailed Item, if all were complete. However, if a Copy 6 were deleted by the Validation Tests, a number such as 6(115) would appear here.

After identification of an unfailed item, the number of hours logged in this unfailed item (UFH) by the end of the calendar period must be found. This UFH is given by the following formula:

$$UFH = FFH - OFH$$

where OFH = Original Flying Hours, and FFH = Final Flying Hours. OFH is found by searching the 2408-3 data tape. The file is entered with Tail Number of the Aircraft that the component was installed on, and searching for the record whose date is nearest to the date of component installation (RID). From this record, is read the Number of Flying Hours on the aircraft at installation time.

FFH is found by again searching the 2408-3 data tape in a similar manner. But this time a search is made for the record whose date is nearest to the

data at the end of the test period, (DEP). One test period will be for the past year, and the other will go back to the oldest record in the file.

For each unfailed item a new record is generated, resembling the record that indicated an unfailed item. However, the number of unfailed hours is entered on this record.

It must be noted, however, that in the analysis part of the program, there is a condition on the acceptance of an unfailed item into the group of failed items. The item is included, if and only if, the number of unfailed hours is equal to, or greater than the largest number of hours logged in a component that failed.

So, if one Serial Number item operated for 200 hours, and then failed, another serial number, with the same part number, cannot be called an unfailed item until it has run at least 200 hours.

This completes the discussion of the data tape.

The next item to be discussed will be the Format which is the output of the program.

4. OUTPUT FORMAT OF RESULTS. The output, Fig. 4 shows the desired values that the program determines.

The header lists the interchangeable parts and information on the Prime Part Number (i.e., the part number being analyzed.)

The objectives of the program are:

(1) The data TBO or Finite Life, and the Burn-In Time, which will be compared with the Established Value by the Contractor.

(2) The next objective is the Composite Probability Density Distribution, for all 3 Life Phases: 1) Burn-In Time, 2) Random Failure Phase and 3) Wearout Phase.

It is necessary to find if the failure rate follows an exponential distribution (B=1) or a normal distribution (B=2.6) or some other of the Weibull family; Fig. 5. This distribution is a function of 3 parameters, Gamma (g), which is the shift parameter, Beta (B), which is the shape parameter, and Eta (N), which is the characteristic Life parameter.

(3) Life characteristics are then found from the Weibull Distributions. These are: Reliability, Hazard Rate, Variance, Expected Value, etc. An example of Hazard Rate as a function of B is shown in Fig. 6. Fig. 7 shows an Average Hazard Rate for an actual item.

5. MATHEMATICAL DEVELOPMENT. The determination of the first objective will be presented. We want to determine the TBO, or Finite Life, and the Burn-In time. This will be presented graphically, first. See Fig. 8.

FORMAT FOR INDIVIDUAL PART NUMBER

HEADER:

1	2	3	4	5	6	7	8	9	10
Aircraft Type	Calender Group	Functional Group	Price Part Number	Manf Code	Interchangeable Part No. A	Manf Code A	Interchangeable Part No. B	Manf Code B	Repeat Prior 2 columns if necessary

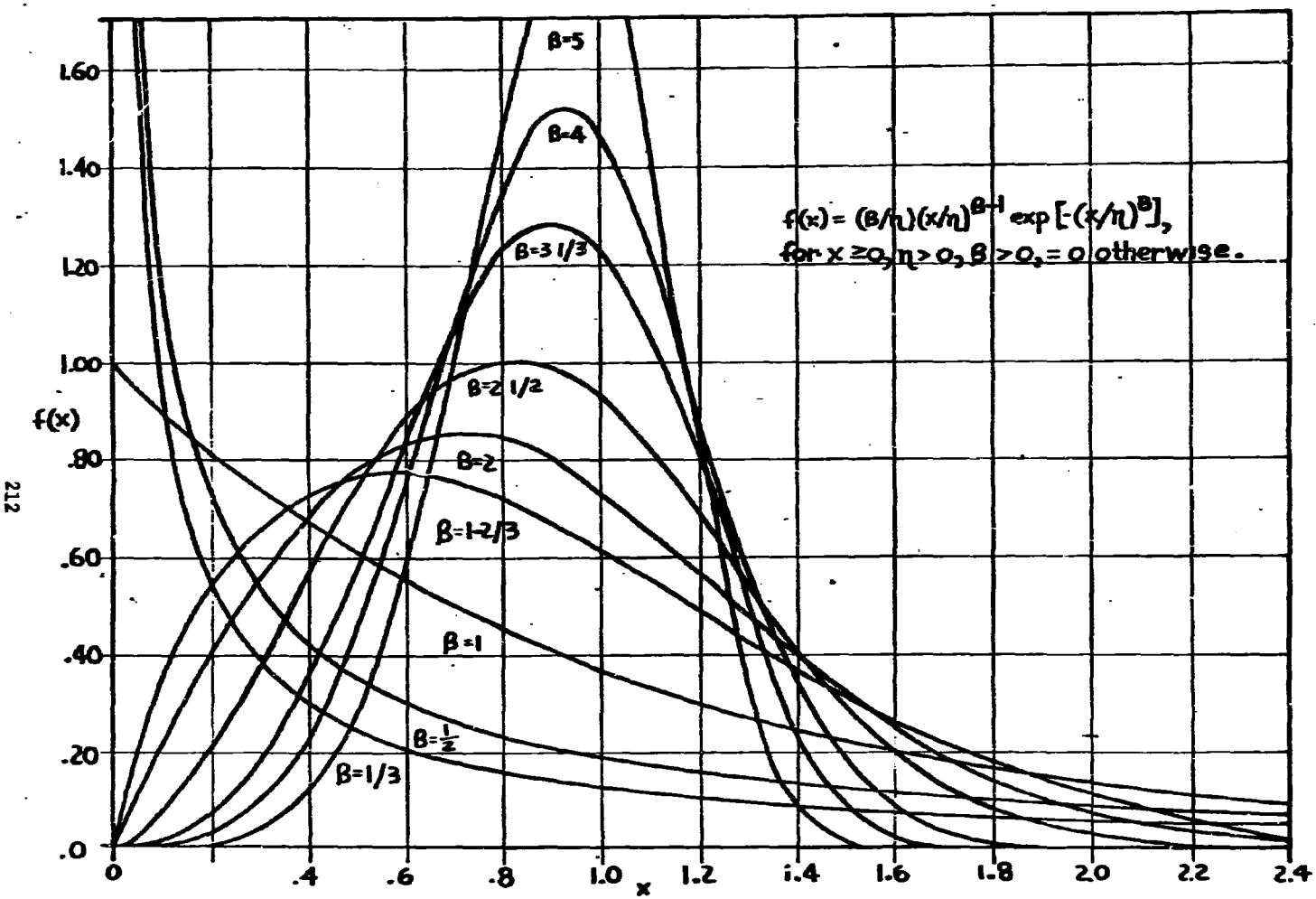
ESTIMATION OF PRICE PART NUMBER: Number of Prior Overhauls = ; Established TBO or Finite Life hrs Standard Unit Price

Age of Item (hrs)	Life Phase	R(25)	F(25)	R(50)	F(50)	R(75)	F(75)	R(100)	F(100)	R(500)	F(500)	R(1000)	F(1000)	R(4000)	F(4000)	Weibull Parameters Gamma	Beta	ETBR
0 to 25	Burn in																	
25 to 50	Burn in																	
50 to 75	Burn in																	
Max for Burn in	Burn in																	
0 to 25	Random																	
"	"																	
"	"																	
Max for Random	"																	
0 to 25	WearOut																	
25 to 50	WearOut																	
"	"																	
Max for WearOut	"																	

Fig 4.

	Weibull Parameters									DBURN		DWEAR
	Gamma	Beta	EFFA	Expected Value	Variance	MTBFA	MTBFU all failures	No. failures (I)	No. Tail #'s Reporting (TN)	Burn In Time (BIT)	Repair Number (RN)	Data [illegible]

FIGURE 4 (Continued)



PLOT OF THE WEIBULL PROBABILITY DENSITY FUNCTION FOR VARIOUS VALUES OF β; $(\eta=1)$, $(\epsilon=0)$

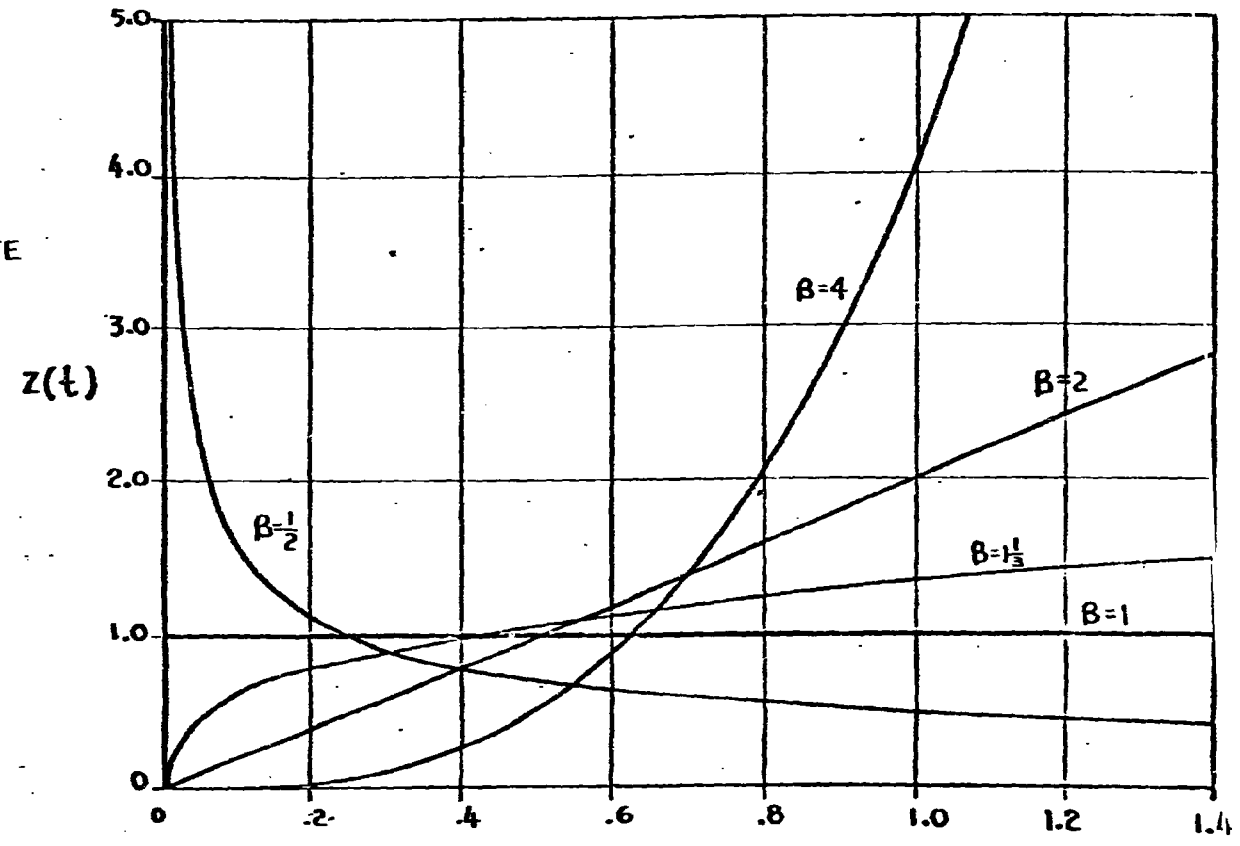
Fig 5

E-656

HAZARD RATE

$$Z(t) = \frac{\text{p.d.f.}}{\text{RELIABILITY}}$$

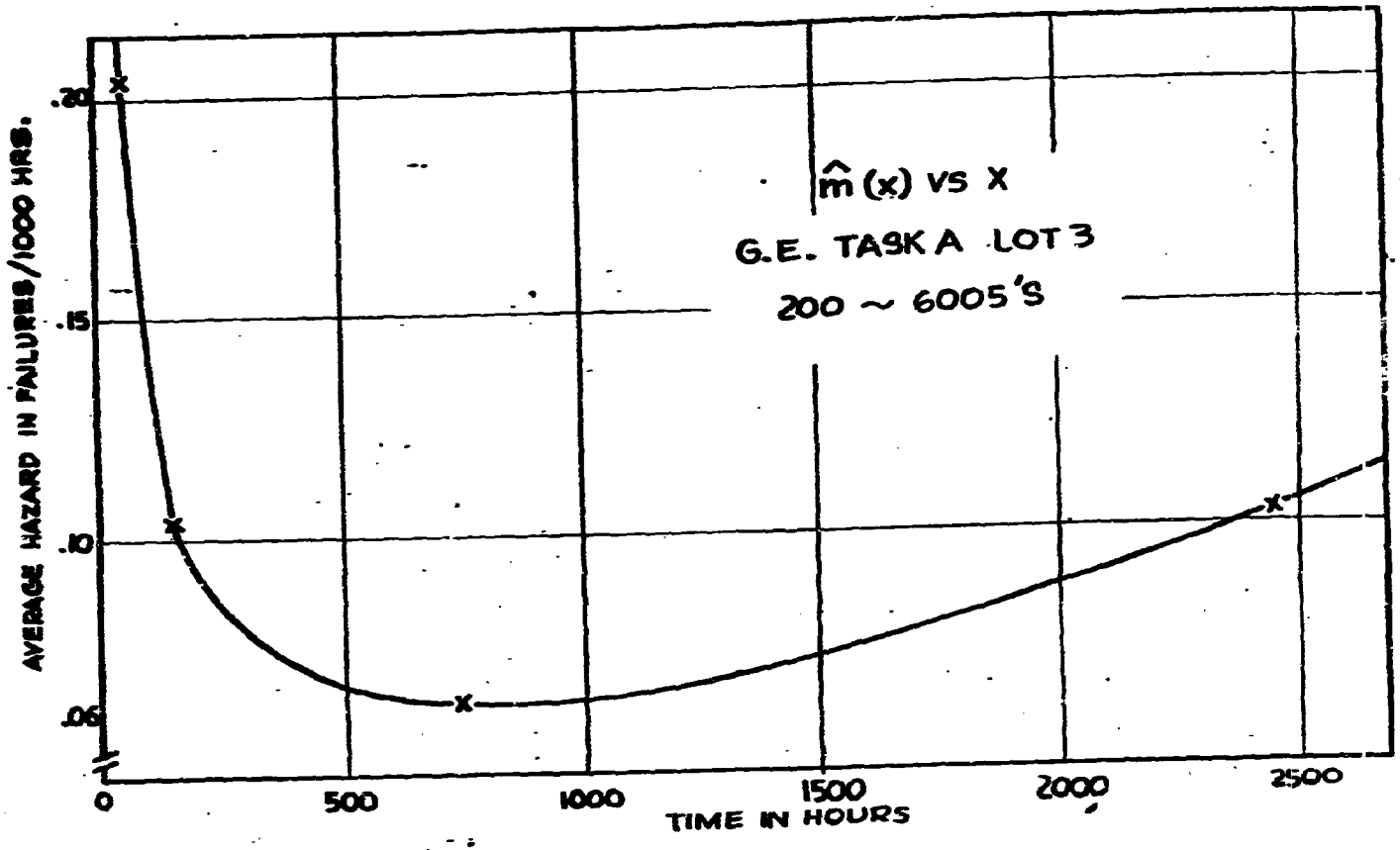
$$Z(t) = \left(\frac{\beta}{\eta}\right)\left(\frac{t}{\eta}\right)^{\beta-1}$$



INSTANTANEOUS FAILURE RATE AS A FUNCTION OF TIME FOR VARIOUS VALUES FOR β (η=1)

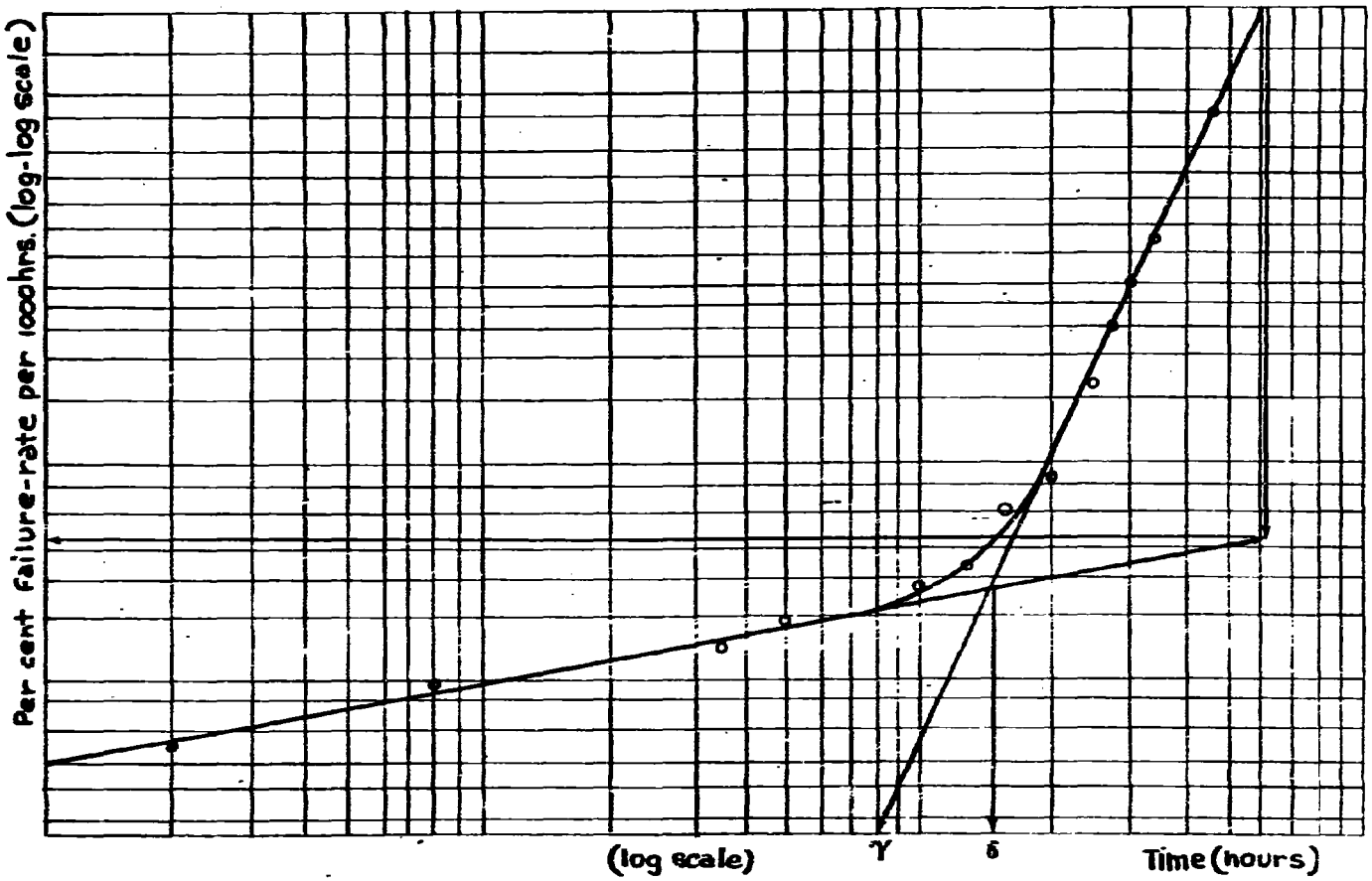
E-6154

Fig 6


$\hat{m}(x)$ VS X
G.E. TASK A LOT 3
200 ~ 6005'S
AVERAGE HAZARD IN FAILURES/1000 HRS.
.20
.15
.10
.06
0
500
1000
1500
2000
2500
TIME IN HOURS

E-652

Fig 7



THE WEIBULL PLOT OF FAILURE-RATE VERSUS TEST DURATION

Fig 8

E-653

An ordered plot is made of all times of failure on the so-called Kao Weibull Paper.

This graph paper has the scales chosen such that a plot of a straight line on this paper, yields the shaping parameter "B" of the Weibull p.d.f. However, the original plot of the raw data will not yield straight line segments. The linearization process will be discussed later.

Now, assuming a plot, as shown on Fig. 8, the TBO, or Finite Life is defined as the intersection of the two tangent lines. The value is read from the horizontal scale, indicated by the symbol Delta (S). This point is indicated by the maximum positive, change of slope between any two data points.*

Next, the Burn-In time will be determined. (Turn the figure upside down). Suppose that the failure points fell along this plot. This plot shows that this component has a Burn-In time at Delta. Beyond the Burn-In time, the failure rate gets much better, and becomes significantly less. This point is indicated by the Maximum NEGATIVE Rate of Change of Slope, between any two points.

Now, that our 1st objective has been reached, the 2nd objective will be determined. This is the identification of the type of the Weibull 3-Phase Composite Cumulative Probability Density Distribution.

$$F(t) = 1 - \sum_{i=1}^{3} A_i \exp\left[-\left(\frac{t-g_i}{N_i}\right)^{B_i}\right].$$

The Weibull distribution is identified by the 3 parameters, Beta, Eta, and Gamma. The coefficients A_i represent the proportion of failures in each phase.

For a "Kao Plot" consisting of only one straight line segment: (Fig. 9)

$$f(t) = \frac{\text{Beta}\ (t-\text{Gamma})^{(\text{Beta}-1)}}{\text{Eta}}, \quad \exp\left\{\frac{-(t-\text{Gamma})^{\text{Beta}}}{(\text{Eta})}\right\}$$

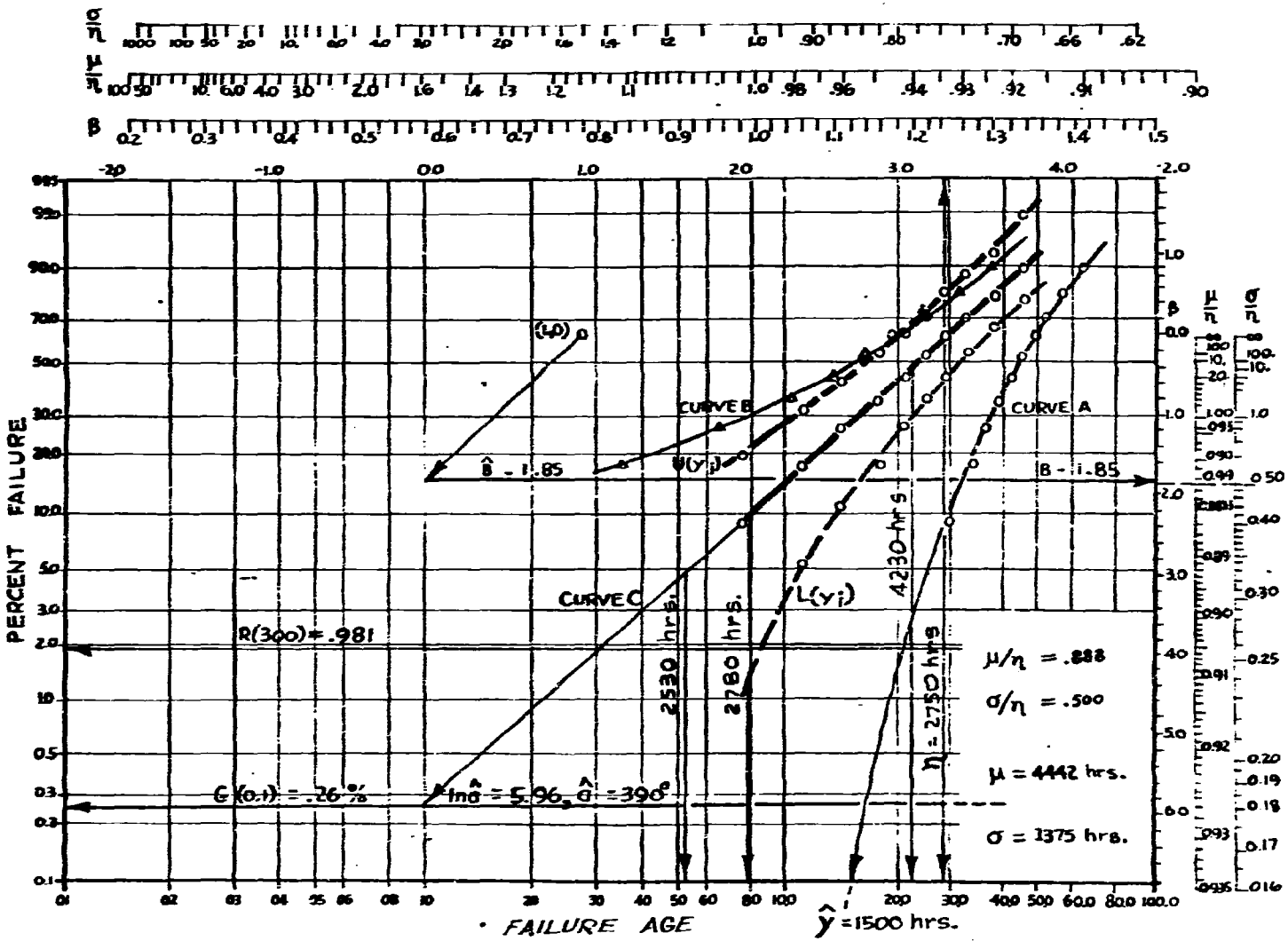
A set of 3 parameters is needed for each straight segment of the plot. There may be a maximum of 3 straight line segments.

---

*Delta (S) may be found mathematically. It is the time at which the proportion of random failures, indicated by the lower straight section, is equal to the proportion wearout failures, indicated by the upper straight section. It is equated below:

$$1 - \exp\left[-(S \mid N_2)^{B2}\right] = 1 - \exp\left[-(S \mid N_3)^{B3}\right]$$

$$S = \exp\left[(B_2 \ln N_2 - B_3 \ln N_3)/(B_2 - B_3)\right].$$



WEIBULL PROBABILITY, KAO PLANE

E-655

Fig 9

Beta (the Shape Parameter) is found from the slope of each straight line segment, (or Sub-population).

Eta, the characteristic Life Parameter, denotes the failure age of 63.2% of the items. From the graph, Eta can be read directly by entering the Vertical Scale at 63.2, and progressing to the intersection of the straight line segment. Then read Eta on the horizontal scale, directly under this intersection. For computer purposes, the equations of the two lines involved must be formulated. The equation for ETA turns out to be:

$$\text{Eta} = \exp\left[\frac{-\overline{YY}}{\text{Beta}} + \overline{XX}\right]$$

where $\overline{XX}$ refers to the mean data point on the horizontal axis, and $\overline{YY}$ refers to the mean data point on the vertical axis.

The exponential function is involved because the $\overline{XX}$ and $\overline{YY}$ values are actually logarithmic values to the base "e". Therefore, the anti-log must be taken.

Gamma, the Shift-Parameter, is found by determination of the data-shift used to convert non-linear data into linear data. This determination will be discussed later. Fig. 9 shows that the final value of Gamma should check with the value of intercept on the horizontal scale. The physical meaning of Gamma is the time, before which, no failures had a chance of occurring.

The method for determination of Gamma follows:

$$F(x) = 1 - \exp\left[-\left(\frac{x-g}{N}\right)^B\right] \qquad X,B,N > 0$$

where F(x) is the Cumulative Probability Distribution.

$$1 - F(x) = \exp\left[-\left(\frac{x-g}{N}\right)^B\right]$$

$$\frac{1}{1-F(x)} = \exp\left(\frac{x-g}{N}\right)^B$$

$$\ln[1/(1-F(x))] = [(x-g)/N]^B$$

$$\ln \ln [1/(1-F(x))] = B \ln [(x-g)/N]$$

$$= B \ln (x-g) - B \ln N.$$

This equation shows that a plot of the left hand versus $\ln(x-g)$ will yield a straight line, with a slope equal to B.

[F(x)] (100) = Percent Failure, or, in other words; - this is the percent of failures that we can expect in the time "X".

The y-intercept is [-B ln(N)] is a measure of the Goodness-of-Fit of the

Weibull plot. It can be compared with a geometrical determination of [-n ℓn(N)], at the x=1 vertical line, which is the ℓn x = 0 line. See Fig. 9, that shows that sample data for Gamma = 20.

The original data yielded Curve A. Since this is not linear, the slope, Beta, cannot be determined yet.

The first guess of Gamma may be taken graphically by extending the curve down to the horizontal axis. This gives Gamma = 1500 hrs. Since a computer program does not have access to the graph, the first guess of Gamma is taken to be (2/3)(Time of First Failure), i.e. (2/3)(Time of 1st Data).

To apply this first guess of Gamma, the value of 1500 hrs. is subtracted from each time of failure.

Curve B is an example of a plot of data, which was adjusted for a value of Gamma = 27.5, Hecto-hours. The fact that these two curves, (A and B) have opposite curvatures indicate that the true value of Gamma lies somewhere between 15. and 27.5. Further trials showed that Gamma = 20 is the value that linearizes the curve. This is shown on Curve C.

SUCCESSIVE DETERMINATION OF GAMMA.

The equation that gives us the next value of Gamma to try is developed next.

Refer to Curve A of Figure 9. It is known, in this case, that if the correct value of g = 20 would be subtracted from each X-value of each data point, the result is the linear Curve C. The problem is to set up an equation which will first be developed for the simple case of only 3 data points.

DETERMINATION OF GAMMA FOR 3 IDEAL POINTS.

As seen on Figure 10, the point (Y_2) is ideally located equidistant between Y_1 and Y_3.

In addition, there are only 3 points, which is the simplest case possible.

The analysis is shown in Figure 10.

The difficulty with this equation is its sensitivity to X_1 or X_3.

For example: If X_1 = 33.3, g = 8. and if X_1 = 35.3, g = 28.

Also: 1) Always subtract off (2/3)(1st data pt) to get first guess for g, to reduce sensitivity. 2) Reiterate. Also use "g" equation in the form

$$g_h = \left[\frac{X_d^2}{X_3} - X_1\right] \div \left[\frac{2X_d}{X_3} - \frac{X_1}{X_3} - 1\right] .$$

DETERMINATION OF GAMMA (g) FOR 3 IDEAL POINTS

Phase 1

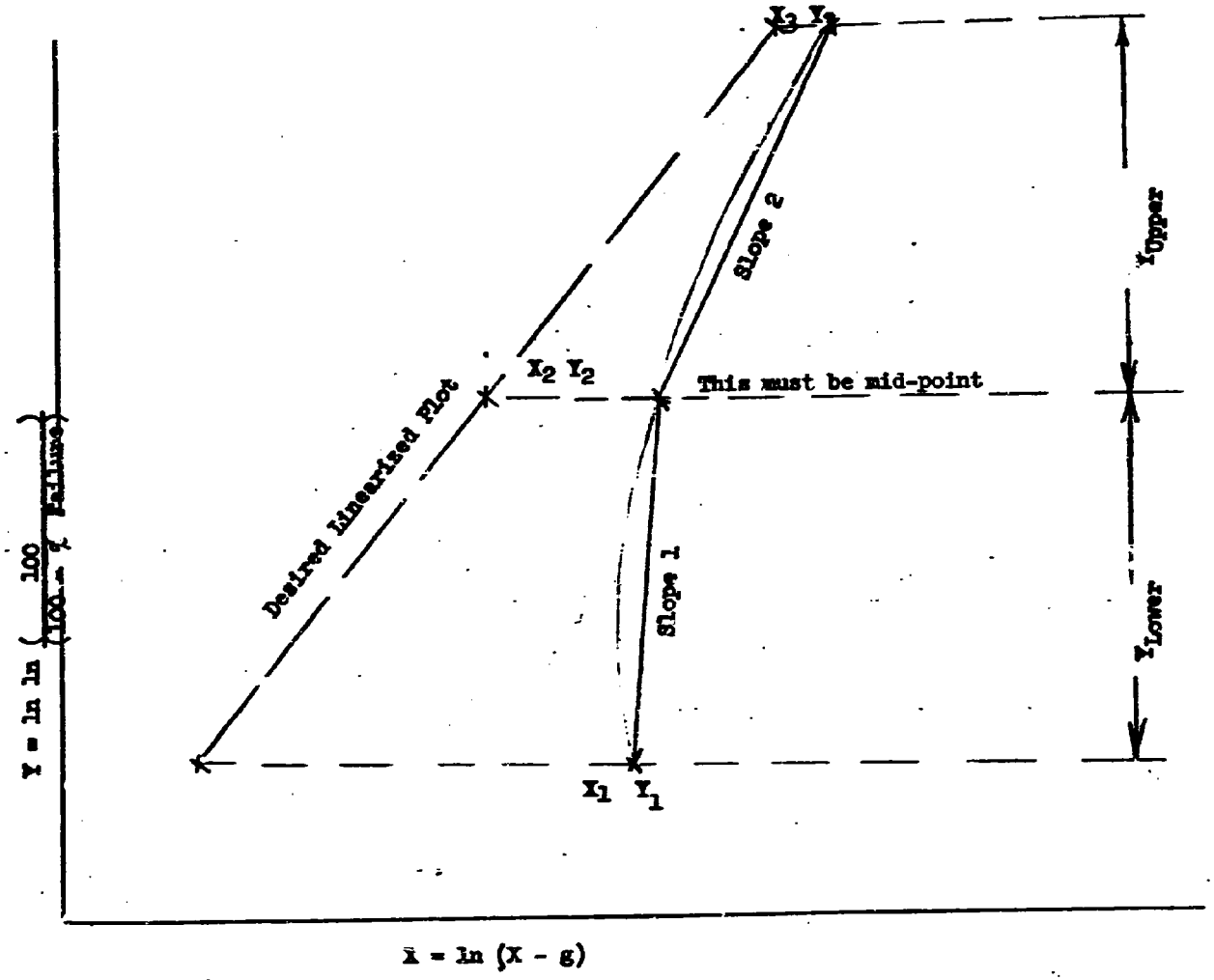


Fig 10

DETERMINATION OF GAMMA (g) FOR 3 IDEAL POINTS - Phase 2

Set the change of Slope (ΔS) = 0, and solve for Gamma.

ΔS= Slope2-Slope 1= 0

$$\frac{Y\ upper}{\ln(x_3-g)-\ln(x_2-g)} = \frac{Y\ lower}{\ln(x_2-g)-\ln(x_1-g)}$$

But Y upper=Y lower:

$$\therefore \ln\left(\frac{x_2-g}{x_1-g}\right) = \ln\left(\frac{x_3-g}{x_2-g}\right)$$

Take Anti-Ln of each side.

$$(x_2-g)^2 = (x_1-g)(x_3-g)$$

$$x_2^2-2.x_2g+g^2 = x_1x_3-x_1g-x_3g+g^2$$

$$x_2^2-x_1x_3 = g(2x_2-x_1-x_3)$$

$$g= \frac{x_2^2-x_1x_3}{2x_2-x_1-x_3}$$

where X_2 must be chosen opposite to y_{MID} on the Data Plane

Fig 11

DETERMINATION OF GAMMA (g) FOR ACTUAL CASE OF 10. POINTS. - Phase 1.

1. DIVIDE DATA INTO UPPER AND LOWER DATA, WITH THE DIVIDING LINE TAKEN AS THE AVERAGE (x_D)

$$x_D = \frac{27.5+31.+34.+38.+41.+44.+47.+51.+57.+64.}{10.}$$

$x_D = 43.4$

2. SET x_3 = AVERAGE OF UPPER.

$x_3 = (44.+47.+51.+57.+64.)/5. = 52.6$

3. SET x_1 = AVERAGE OF LOWER.

$x_1 = (27.5+31.+34.+38.+41.)/5. = 34.3$

4. NEXT, FIND $x_2(Y_D)$:

Where Y_D = AVERAGE OF Y_1 AND Y_3.

$Y_L = Y_1$ AND $Y_U = Y_3$ FOR CONVENIENCE

Fig 12

DETERMINATION OF GAMMA (g) FOR ACTUAL CASE OF 10. POINTS - Phase 2.

FIND Y_L FOR $x_L = 34.3$, ($X_L = \ln 34.3 = 3.54$)

1. INTERPOLATE BY DRAWING CURVE IN THE FUNCTIONAL, OR KAO PLANE, THROUGH THE 3 POINTS CLOSEST TO X_L.

THESE ARE $x_1 = 31$, $x_2 = 34$, and $x_3 = 38$.

x	X	y	Y
31	3.43	18	1.63
34	3.52	27	1.3
38	3.64	36	0.82

$Y_L = a + b(X_L) + c(X_L)^2$

WHERE a, b and c follow.

$Y_L = 1.26$

Fig 13

DETERMINATION OF GAMMA (g) FOR ACTUAL CASE OF 10. POINTS - Phase 3.

$$a = \frac{Y_1 X_2 X_3 (X_3 - X_2) + Y_2 X_1 X_3 (X_1 - X_3) + Y_3 X_1 X_2 (X_2 - X_1)}{\left[X_2 X_3 (X_3 - X_2) + X_1 X_3 (X_1 - X_3) + X_1 X_2 (X_2 - X_1)\right] = D}$$

$$b = \frac{Y_2 (X_3^2 - X_1^2) + Y_3 (X_1^2 - X_2^2) + Y_1 (X_2^2 - X_3^2)}{D}$$

$$c = \frac{Y_3 (X_2 - X_1) + Y_1 (X_3 - X_2) + Y_2 (X_1 - X_3)}{D}$$

These are the equations for a parabolic fit through 3 points.

$$Y = a + bX + cX^2$$

Fig 14

DETERMINATION OF GAMMA (g) FOR ACTUAL CASE OF 10. POINTS - Phase 4

SIMILAR TO THE METHOD FOR FINDING Y_L, Y_U IS FOUND FOR $x_U = 52.6$ ($X_U = \ln 52.6 = 3.96$)

THE PARABOLA IS DRAWN THROUGH THE 3 POINTS CLOSEST TO 52.6 WHICH ARE SHOWN BELOW.

x	X	y	Y
47	3.85	64.	-.02
51	3.93	73.	-.4
57	4.04	82.	-.52

$Y_U = -.44$ at $X_U = 3.98$

$$Y_D = \frac{(Y_L + Y_U)}{2} = \frac{1.26 - .44}{2} = +.41$$

These are Kao values. They are the values obtained after going thru $\ln \ln \left(\frac{100}{(100 - \% \text{ Failures})} \right)$

Fig 15

DETERMINATION OF GAMMA (g) FOR ACTUAL CASE OF 10 POINTS - Phase 5

KNOWING $Y_D = +.41$; X_D IS FOUND BY DRAWING A PARABOLA THROUGH THE 3. Y POINTS CLOSEST TO $Y_D = +.41$. NOTE THAT THE ROLES OF X AND Y ARE REVERSED.

$$X_D = a + bY_D + cY_D^2$$

THE CLOSEST POINTS ARE

$y_1 = 36.$, $Y = +.8$

$y_2 = 46.$, $Y = +.45$

$y_3 = 55.$, $Y = +.2$

$X_D = 3.72$ at $Y_D = +.41$, and $x_d = 41.59$

2. FINALLY

$$g = \frac{x_D^2 - x_1 x_3}{2x_D - x_1 - x_3} = \frac{(41.59)^2 - (34.3)(52.6)}{(2)(41.59) - 34.3 - 52.6} = \boxed{20.0 = g}$$

This is the true value of Gamma.

Fig 16

Then use

$$\left[g_e = \frac{X_3^2}{X_1} - X_3\right] + \left[\frac{2X_3}{X_1} - 1 - \frac{X_3}{X_1}\right].$$

Then average $g = (g_e + g_h)/2$.

If double precision is available, the averaging may be eliminated.

It has been found that we get better results, using lowest and average of top half of the data, instead of average of the bottom half and (average of top half). This insures that Gamma, (g) will be less than the lowest data value of "X". A value of Gamma greater than the lowest data value of "X", is erroneous and unacceptable due to the basic definition of Gamma. Gamma is the value, below which no other failure, or X value can occur.

Another method will be discussed next. This second method is the Average Rate of Change of Slope Method, (ARCS).

GAMMA ADJUSTMENT BY AVERAGE RATE OF CHANGE OF SLOPES (ARCS)

1. In each life phase, find the Slope between each 2 points, in the Kao Plane using &n values, shown in Fig. 17.

$$S_i = (Y_{i+1} - Y_i)/(X_{i+1} - X_i)$$

2. Find the change of slope, between each two slopes, (RCS) = $(S_{i+1} - S_i)/(Xmid_{i+1} - Xmid_i)$.

3. Find Average Rate of Change of Slope (ARCS)

$$ARCS = \frac{\sum_{1}^{N} RCS}{N}$$

Since this is the ARCS for the original data, for which there was no adjustment for Gamma, call this value ARCS (Gamma = 0), or ARCS(0).

4. Again use the method above to find ARCS (Gamma = first data point), or ARCS(1). To do this, subtract this first (X-data) point from each successive data point, to establish a new set of data. Then put these data into the Kao Plane by taking the "&n" function shown in the sketch. Then find ARCS(1).

5. Next find ARCS (2/3). Subtract (2/3) of the first (X-data) point from each data, to form a new set of data. Again put the data into the Kao Plane. Then find ARCS (2/3).

6. Write a 3rd order equation through the plot of Gamma versus ARCS(A), Gamma = $aA^2 + bA + C$. Then solve for the value of Gamma (Gamma = C), that makes ARCS = 0. The value of $C(A_1, A_2, A_3)$ is:

$$C = \frac{g_3(A_2-A_1) + g_1(A_3-A_2) + g_2(A_1-A_3)}{A_2A_3(A_3-A_2) + A_1A_3(A_1-A_3) + A_1A_2(A_2-A_1)}.$$

7. As the new Gamma is found, this new value of Gamma is subtracted from each of the X-values of the data points, to get the adjusted data points. Then a new value is found for the Average Rate of Change of Slope (ARCS), to check its approach to zero. If this ARCS is within ±0.1 or if this new ARCS does

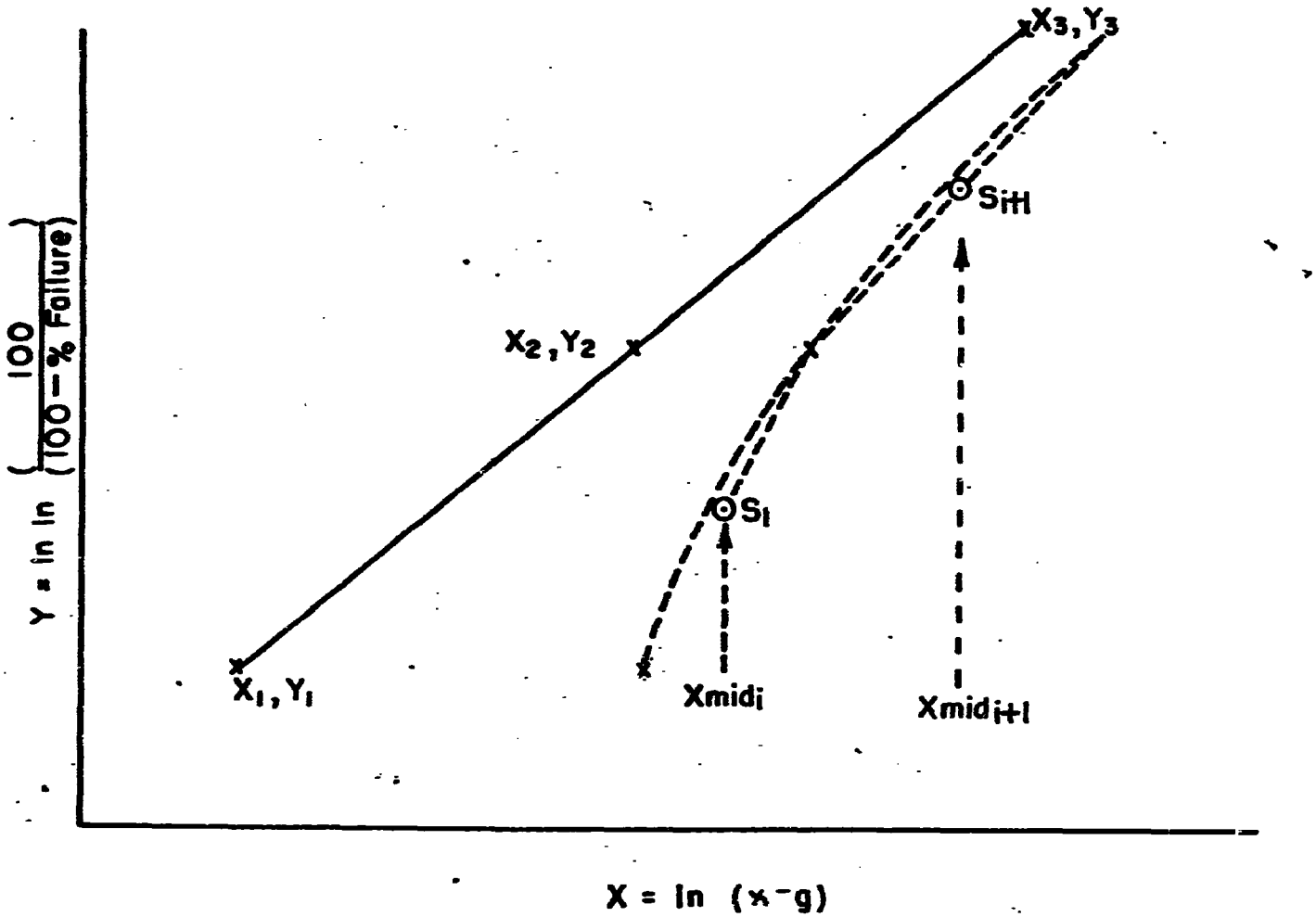

X_3, Y_3
S_{i+1}
X_2, Y_2
S_i
X_1, Y_1
X_{mid_i}
$X_{mid_{i+1}}$
$Y = \ln \ln \left(\frac{100}{100 - \% \text{ Failure}}\right)$
$X = \ln (x - g)$

Fig 17

not improve the previous ARCS by at least 20%, the iteration is complete.

The formula cdns. for stopping iteration are:

1) $|ARCS| < .1$

or

2) $|ARCS_{i+1}/ARCS_i| > .3$

or

3) $(ARCS)_{i+1} > (ARCS)_i$ for $ARCS_i$ positive

or

4) $(ARCS)_{i+1} < (ARCS)_i$ for $ARCS_i$ negative.

8. If these tests pass, the largest ARCS is dropped, and the iteration is continued, using the three lowest values of ARCS with their three corresponding values of Gamma. See the next Figure (18).

FINDING GAMMA (g) BY GEOMETRIC MEAN METHOD This method is an improvement of Dubey's Method, found on Page 293 of Technometrics, May 1967.

The first step is the calculation of Y_{MID} using data values.

The Weibull Plot is a plot of $\ell n\ \ell n\ [1/(1-F(X))] = B\ \ell n\ [(X-g)/N]$.

Taking the inverse ℓn of each side gives:

$$y = \ell n\ [1/(1-F(X))] = \left[\frac{X-g}{N}\right]^B = A(X-g)^B$$

where $A = [1/N]^B$

The constant "g" may be found by writing 3 equations, which are taken from three data points since there are 3 unknowns: g, B, and N.

$$X_1,\ F_1(X);\ X_2,\ F_2(X);\ \text{and}\ X_3,\ F_3(X)$$

$$Y_1 = A(X_1-g)^B$$

$$Y_2 = A(X_2-g)^B$$

$$Y_3 = A(X_3-g)^B$$

Then:

$$Y_1Y_2 - Y_3^2 = A^2\ [(X_1-g)^B\ (X_2-g)^B - (X_3-g)^{2B}]$$

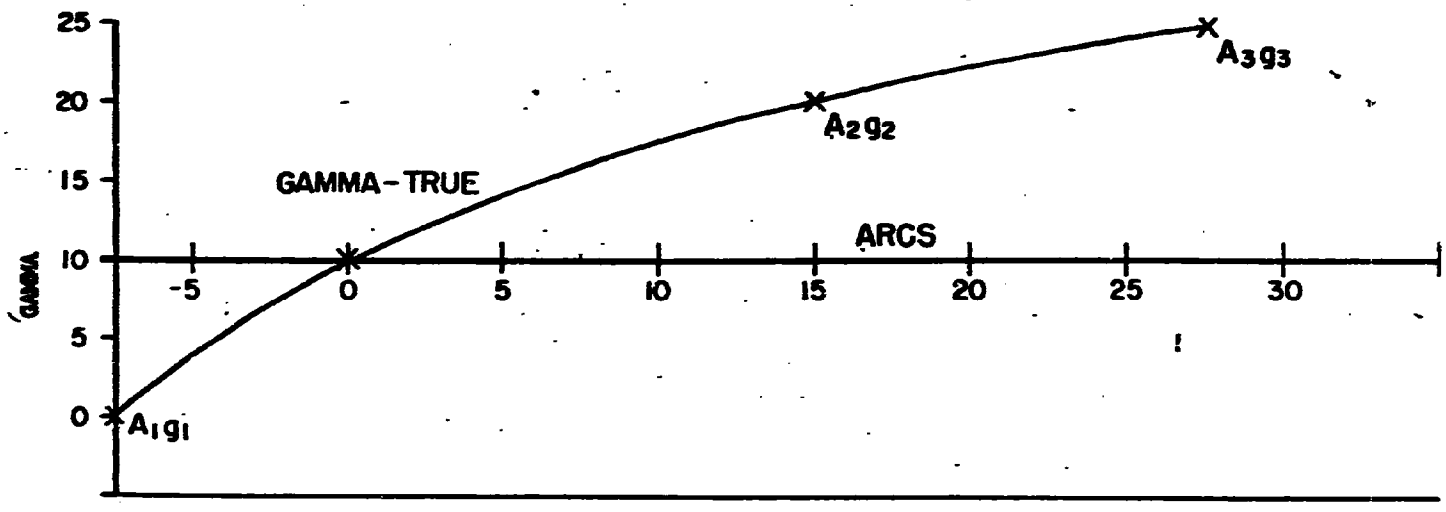
Now if $Y_3 = \sqrt{Y_1Y_2}$, the LHS = 0, and we can solve for "g".

$$(X_1-g)\ (X_2-g) = (X_3-g)^2$$

$$X_1\ X_2 - X_1g - X_2g + g^2 = (X_3)^2 - 2X_3g + g^2$$

$$g[2X_3 - X_1 - X_2] = X_3^2 - X_1\ X_2$$

GAMMA VS RATE-OF-CHANGE OF SLOPE (ARCS) AS 2ND ORDER EQUATION



$$\text{Gamma} = a(\text{ARCS})^2 + b(\text{ARCS}) + c$$

AT ARCS = 0, GAMMA = C, where:

$$c = \frac{g_3(A_2 - A_1) + g_1(A_3 - A_2) + g_2(A_1 - A_3)}{A_2A_3(A_3 - A_2) + A_1A_3(A_1 - A_3) + A_1A_2(A_2 - A_1)}$$

where: A = ARCS

Fig 18

$$\boxed{g = \frac{\bar{x}_3^{\,2} - \bar{x}_1\,\bar{x}_2}{2X_3 - X_1 - X_2}} \qquad \text{Eqn A}$$

Let $X_3 = X_{MID}$, which must correspond to y_{MID}.

$$y_{MID} = \sqrt{y_1 y_2} = \sqrt{[\ell n\ (1-F_1(X))^{-1}][\ell n\ (1-F_2(X))^{-1}]}$$

$$y_{MID} = \sqrt{[-\ell n\ (1-F_{LOW})][-\ell n\ (1-F_{HIGH})]}$$

But we want to convert y_{MID} to a value that a readable in the KAO Plane, which is F_{MID}.

Find F_{MID} in terms of y_{MID}.

$$y_{MID} = \ell n\ [\frac{(1)}{(1-F(X))}] \qquad \text{where } F(X) = F_{MID}(X)\ .$$

Solve for F(X).

$$e^{y_{MID}} = \frac{1}{1-F(X)}$$

$$1-F(X) = e^{-y_{MID}}$$

$$F_{MID}(X) = 1-e^{-y_{MID}} = 1 - \exp - \sqrt{[-\ell n\ (1-F_{LOW})][-\ell n\ (1-F_{HIGH})]}$$

Next find the corresponding X_{MIDK}, by drawing a 3rd order curve through the 3 closest points to y_{MIDK}, using the KAO plane.

$$X_{MIDK} = a + bY_{MIDK} + cY_{MIDK}^2 \ .$$

Both X_{MID} and Y_{MID} are KAO Plane values, as are all the X's, and Y's in the following a, b, and c equations, that are double subscripts.

Where Y_{MID} = KAO Value of F_{MID}; $Y_{MID} = \ell n\ \ell n\ [1/(100-F_{MID})]$

$$a = \frac{X_{11}Y_{22}Y_{33}(Y_{33}-Y_{22}) + X_{22}Y_{11}Y_{33}(Y_{11}-Y_{33}) + X_{33}Y_{11}Y_{22}(Y_{22}-Y_{11})}{Y_{22}Y_{33}(Y_{33}-Y_{22}) + Y_{11}Y_{33}(Y_{11}-Y_{33}) + Y_{11}Y_{22}(Y_{22}-Y_{11})} = \frac{N}{D}$$

$$b = \frac{X_{22}(Y_{33}^{\,2}-Y_{11}^{\,2}) + X_{33}(Y_{11}^{\,2}-Y_{22}^{\,2}) + X_{11}(Y_{22}^{\,2}-Y_{33}^{\,2})}{D}$$

$$c = \frac{X_{33}(Y_{22}-Y_{11}) + X_{11}(Y_{33}-Y_{22}) + X_{22}(Y_{11}-Y_{33})}{D}$$

Then the data value of X_{MID} (Kao Value) is found by taking the anti-ℓn of X_{MID}(Kao Values).

$$X_{MID} = e^{X_{MID_{Kao}}}$$

Then the value of Gamma is given by the same formula Eqn A, used in a previous method.

$$g = \frac{(X_{MID})^2 - X_1X_3}{2X_{MID} - X_1 - X_3}$$

In the above formula, it has been found that the use of ℓn to the base "e" values for the X's, gives a more exact value for g , but only if the equation is used repeatedly until no significant change occurs.

This same concept of repeated usage must be applied to the equation for "g", regardless of the method used, in order to find the true value for "g".

After determination of Gamma, all of the 3 Weibull parameters are complete. Next will be discussed the information obtained from the Probability Distribution. Reliability is first.

Since there will be 3 values of each of the parameters, Gamma, Beta, and Eta, the composite Reliability is given by the sum of 3 terms.

$$R(t) = \underbrace{J \exp - \left[\frac{t-g_1}{N_1}\right]^{B_1}}_{\text{Term 1}} + \underbrace{P \exp - \left[\frac{t-g_2}{N_2}\right]^{B_2}}_{\text{Term 2}} + \underbrace{Q \exp - \left[\frac{t-g_3}{N_3}\right]^{B_3}}_{\text{Term 3}},$$

where $g_3 > g_2 > g_1$, and $J + P + Q = 1$,

where term 3 is set equal to Q for $t < g_3$,

term 2 is set equal to P for $t < g_2$,

J = Percentage of Data Points in Burn-In Phase

P = Percentage of Data Points in Catastrophic or Random Phase

Q = Percentage of Data Points in the Wearout Phase.

$R(t) = 1$ for $t < g_1$

A Plot is shown, Fig. 19.

Next, Hazard Rate and Reliable Life.

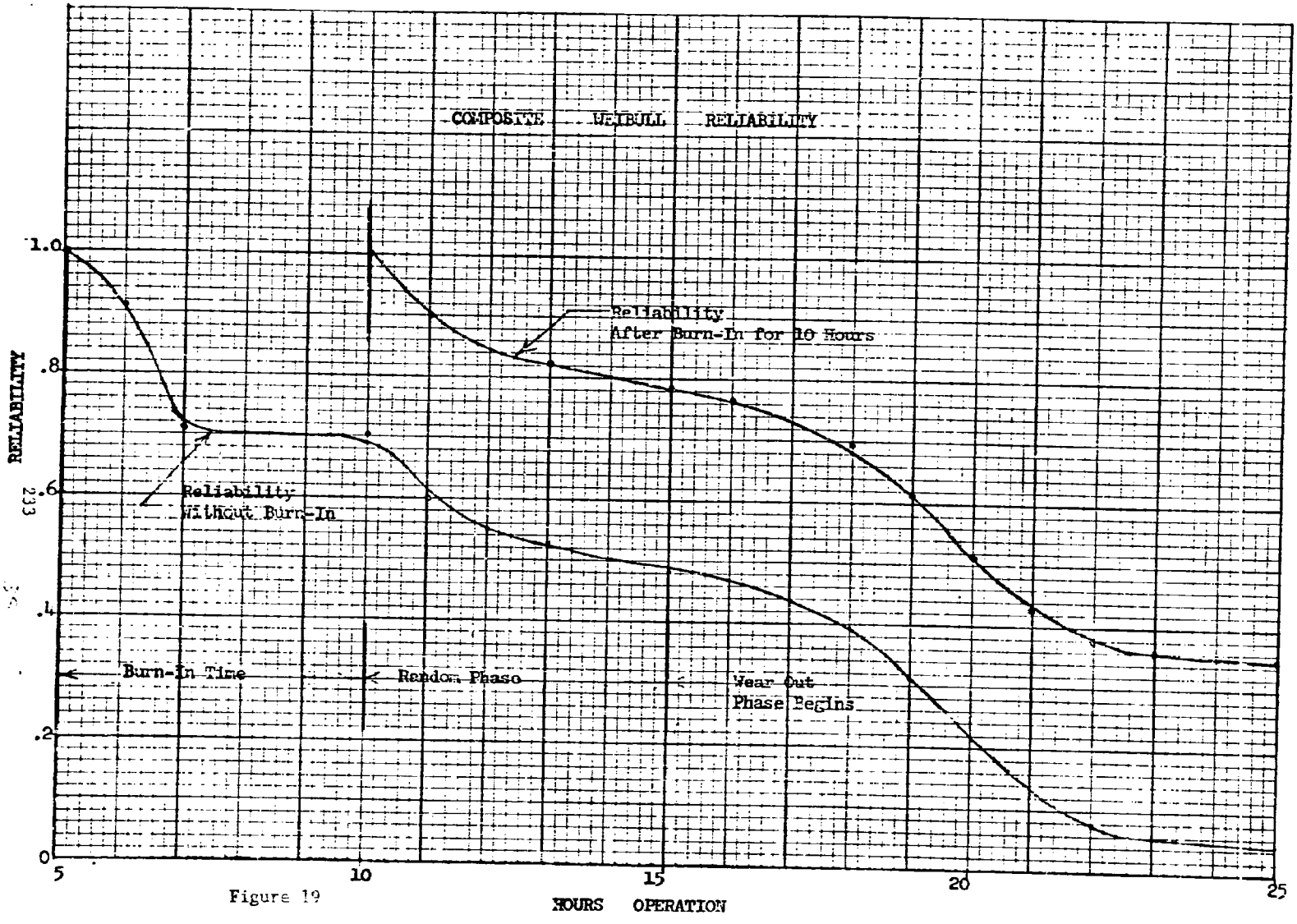


Figure 19

COMPOSITE HAZARD RATE. The Hazard Rate H(t) is the Conditional Probability Density Function of Time to Failure, given that the item has not failed prior to time (t). In other words:

$$H(t)\,dt = P[(t < T < t+dt) \mid (T > t)]$$

It can also be stated as a ratio of Probabilities

$$H(t)\,dt = \frac{P[(t < T < t+dt) \cap (T > t)]}{P(T > t)}$$

where $\cap$ means "Intersect".

$$H(t) = \frac{f_i(t)}{R_i(t)} = \frac{\text{probability density function}}{\text{reliability function}} = \frac{A_i B_i (t-g_i)^{B_i-1}}{(N_i)^{B_i}}$$

A plot of Hazard rate for G.E. task A, Lot 3 is shown (Fig. 7) and a composite Hazard Rate is shown in Fig. 20.

Other important parameters are the Expected Value of Time to Failure and the Variance of the Time to Failure.

The Expected Value (E) and Variance (V) are given for each life phase.

$$E = g_i + N_i\,[\text{Gamma}\ (1/B_i + 1)];\ i = 1,2,3,$$

$$V = N_i^2\,\{\text{Gamma}\ (\tfrac{2}{B_i} + 1) - \text{Gamma}\ (\tfrac{1}{B_i} + 1)\}^2\ ,\ i = 1,2,3.$$

RELIABLE LIFE. The last parameters found is the Reliable Life, RL(C), for a specified confidence level (C). The formula is given next.

$$C = \int_{RL}^{\infty} f(t)\,dt$$

$$RL = (\text{Gamma}) + (\text{Eta})(-\ell nC)^{1/\text{Beta}}$$

RL is found for each component for confidence levels of .85, .90, and .95.

Composite Hazard Rate Evaluation - Synthetic Data

$$H = 0,\quad t < g_1\ ,$$

$$H = J\,B_1\,(N_1)^{-B_1} \cdot (t-g_1)^{B_1-1}\ ,\quad g < t < g_2,$$

$$H = PB_2(N_2)^{-B_2} \cdot (t-g_2)^{B_2-1}\ ,\quad g_2 < t < g_3,$$

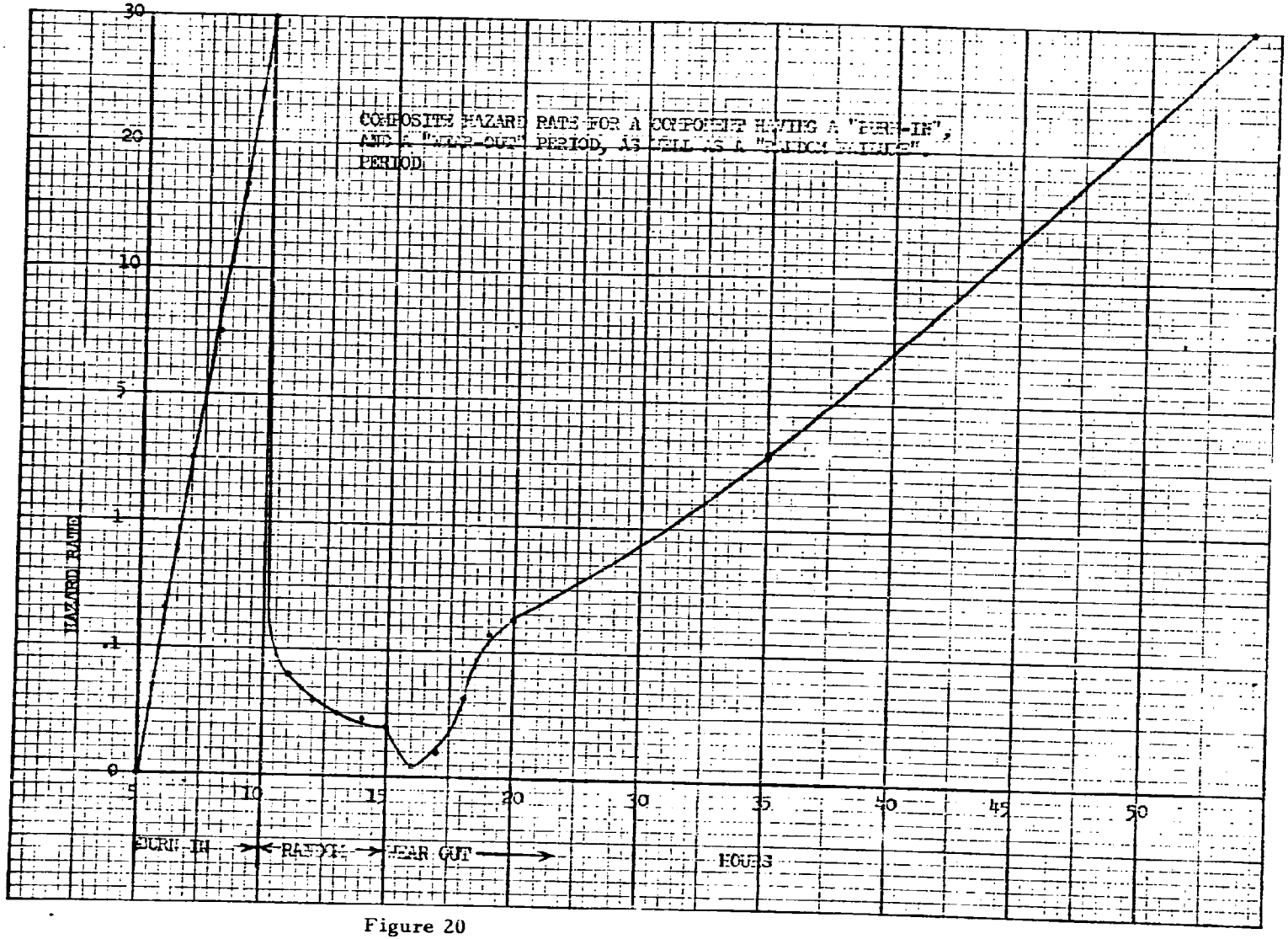


Figure 20

$$H = Q\,B_3\,(N_3)^{-B_3}\,(t-g_3)^{B_3-1} \quad , \quad g_3 < t.$$

DATA

J = .294	P = .353	Q = .353
B_1 = 3.75	B_2 = .621	B_3 = 3.55
N_1 = 1.34	N_2 = 5.1	N_3 = 5.1
g_1 = 5	g_2 = 10.	g_3 = 15.

t	H	t	H
0-5	0	14	.0470
6	.3675	15	.0433
7	2.46	16	.0038
8	7.53	17	.0223
9	16.5	18	.0627
10	30.9	19	.130
11	.0797	20	.231
12	.0613	30	3.8
13	.0526	50	32.6

6. PREPARATION OF TAERS COMPUTER TAPE. Layout for RD-2410 Work-Tape for Weibull Interchangeable Part Program.

1. Prepare RD-CH-47 Work Tape. Save the program used to do this. Table I, which follows Para 10, shows the location of each Variable on the existing ADP-2410 Tape, (contact Mr. C.P. Marquardt of AVCOM) and the desired new location on the RD-CH-47 Work Tape.

2. All Variables, except as noted, must be taken from the ADP-2410 Tape, or calculated from data on this same tape. Two exceptions are the Interchangeable Part Numbers and their associated Manufacturer's Code Number. These are obtained from Program 25F6BE-41, "MDR Component Taers Activity List." This program is available from Mr. Tom Gruenninger (phone extension 2170/ 2176) at AVCOM.

3. The preparation of these two tapes required only records from the ADP-2410 Tape, that are coded 10, CH-47, Copy 1 and 6.

4. Identical information may be taken from Copy 1 or Copy 6, but NOT

both. The copy number is found in column 37. The code "10" may be found on the ADP-2410 Tape, in clumns 35-36. Any other code nullifies its record.

5. Only CH-47 records are to be used. This CH-47 will be found on ADP-2410 Tape in columns 133-144.

6. The blocking factor is (301. X 1).

7. All data on the RD-CH-47 tape must be justified as indicated in the Table. Fill all spaces to the left of Right Justified significant data with zeroes.

8. Records with the following "Fail Code," must be omitted. (The "Fail Code" is located on the ADP-2410 Tape, in columns 264-266). Delete records with Fail Code 0, 120, 130, 138, 256, 301, 446, 464, 500, 503, 530, 540, 950, 796, 797, 798, 799, 800, 801, 802, 803, 804.

9. Delete records with Inspection Action Code (IAC), column 273, equal to "A". This letter "A" indicates that inspection revealed that the item was serviceable without needing repair.

10. The Functional Group Coding must be entered in columns 58-72. It is obtained by reading the Federal Stock Number (FSN), in columns 186-196 of the ADP-2410 Tape. With this FSN, search the Publication Tape, (available from John Witmer, ext. 3221, AVCOM), to find the Functional Group Coding. [Table I starts on the following page.]

Preparation of Recent CH-47 Tape (RD1-CH-47-CGR) and Preparation of Older CH-47 Tape (RD1-CH-47-CGO), From RD-2410 Work Tape.

11. After completion of the RD-2410 Work Tape, separate it into the two tapes named above, based on the Julian records dates, as follows. From this working tape find the most recent date (MDR) of 4 dates, whether FID (Columns 73-76); RID (Columns 77-80); ODNR (81-84); or ODYR (85-88). Prepare 2 tapes: RD1-CH-47-CGR will contain all records whose highest of the 4 dates is between (MRD-25) and MRD-390. Tape RDL-CH-47-CGO will contain records (MRD-390). The cut-off date (COD) is defined as (MRD-25). The COD must be entered into the RD-2410 Work Tape, columns 44-47.

12. The following operations must be applied to each of these 2 tapes.

13. Sort by Component Part Number, Location 13-33, so that the lowest number will be processed first.

14. Sort by Interchangeable Part Numbers, within the "Part Number Sort."

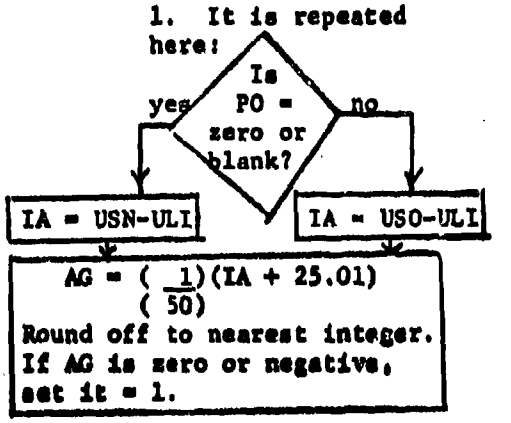
15. Sort by Number of Prior Overhauls such that zeroes, then 1's, etc., will be processed first.

The location of PO is columns 93-94.

TABLE I

Variable	Location On ADP-2410 Tape	New Location On RD-2410 Work Tape	Justify
End Item Model	133-144	1-12 Leave a BLANK between CH and 47	L
Component Part Number Being Processed	145-165	13-33	L
Manufacturing Code for the Component Part Number Being Processed	166-170	34-38	L
Established Time Between Overhaul (TBO) or Established Finite Life (FL)	238-241	39-42	R
Calendar Group - Will be either "R" meaning recent or an "O" meaning older	Not found on this tape. It is determined by the method explained later, in para 11.	43	R
Cut-Off Julian Date (COD)	Not on Tape, see para 11.	44-47	R
Functional Group Code	Not on Tape, see para 10.	48-55	R
Component Serial Number	12-28	56-72	L
First Installation Date (FID)	This date is found in columns 299-302, for copy 6, under the condition that a copy 1 does NOT exist for this control number. If a copy 1 does exist, then fill 73-76 with zeroes.	73-76	R

Variable	Location On ADP-2410 Tape	New Location On RD-2410 Work Tape	Justify
Date of Re-Install (RID)	This date is found in columns 299-302, for copy 6, under the condition that a copy 1 does exist for this control number. If a copy 1 does NOT exist for this control number, fill 77-80 with blanks.	77-80	R
Removal Date, When No Re-Installation Occurs (ODNR)	This date is found in 299-302, for copy 1 under the condition the NO copy 6 exists for this control number. If a copy 6 exists, fill 81-84 with blanks.	81-84	R
Removal Date When Re-Installation Does Occur (ODYR)	This date is found in 299-302, for copy 1 under the condition that copy 6 exists for this control number.	85-88	R
Unfailed Flying Hours	This is not on the ADP-2410 Tape. See para 4.	89-92	R
Number of Prior Overhauls (PO)	136-137	93-94	R
Hours Usage Since New (USN)	123-127	95-99	R
Hours Usage Since Last Installed (ULI)	128-131	100-103	R

Variable	Location On ADP-2410 Tape	New Location On RD-2410 Work Tape	Justify
Hours Usage Since Overhaul (USO)	132-135	104-107	R
Age Group (AG), as a function of: PO located ADP tape 136-137 USN located ADP tape 123-127 ULI located ADP tape 128-131 USO located ADP tape 132-135	Not on ADP tape. It is calculated as in Incl 1. It is repeated here: 	108-109	R
Failure Code of Component (FC)	264-266	110-112	R
Failure Detected During (FDD)	270	113	R
Effect on Mission (EOM)	271	114	R
Inspect & Action Code	273	115	R
Component Noun	61-84	116-139	L
Standard Unit Price	255-263	140-149	R
Organization Ident Code	43-49	150-157	L
End Item Serial Number, Tail Number	212-221	158-167	L
First Interchangeable Part Number 1	Not on ADP-2410 Tape. Use program number 25F6BE-41, "MDR COMPONENT TAERS ACTIVITY LIST." Contact Tom Gruenninger of AVCOM. (See para 3)	168-188	L

Variable	Location On ADP-2410 Tape	New Location On RD-2410 Work Tape	Justify
Manufacturer Code Number 1, Related to Interchangeable Part Number 1	Not on ADP-2410 Tape. See para 3	189-193	L
Second Interchangeable Part Number 2	Not on ADP-2410 Tape. See para 3	194-214	L
Manufacturer Code Number 2	Not on ADP-2410 Tape. See para 3	215-219	L
Third Interchangeable Part Number 3	Not on ADP-2410 Tape. See para 3	220-240	L
Manufacturer Code Number 3	Not on ADP-2410 Tape. See para 3	241-245	L
Fourth Interchangeable Part Number 4	Not on ADP-2410 Tape. See para 3	246-266	L
Manufacturer Code Number 4	Not on ADP-2410 Tape. See para 3	267-271	L
Fifth Interchangeable Part Number 5	Not on ADP-2410 Tape. See para 3	272-292	L
Manufacturer Code Number 5	Not on ADP-2410 Tape. See para 3	293-297	L
Number of Repairs on each Serial Number (RN)		298-300	R
End of Tape Indicator	Not on ADP-2410 Tape. See para 22	301	R

16. Within each overhaul group (PO), sort by Age Group (AG). Request sort to allows the lowest Age Group (1) to be set up to be processed first.

Location of (AG) is columns 108-109.

17. Within (AG), sort by Serial Number, so that lowest Serial Number is first.

Location of Serial Number is columns 57-72. This is Component Serial Number.

18. Within each (AG), count the number of records (RN), having the same serial number. Insert this number into columns 58-60 on each (RN) record. Then, within each Age Group(AG), arrange the records so that those records with the lowest (RN) will be processed first.

19. Find the Unfailed Items and the Flying Hours on each one, as follows.

Within each Serial Number Group, find the highest (RID), which is in columns 77-80, and the highest (ODNR), which is in 81-84.

a. If this RID is greater than ODNR, then this record represents an unfailed item. This is an item that has been installed, but has not failed. In general, ODNR will be blank or zeroes, when RID > ODNR.

b. Now that we have found an unfailed item, we must find the number of hours logged on this unfailed item, by the end of the calendar period.

c. This procedure is as follows:

(1) For the serial number of this unfailed item, pick off the last installation date, (RID), from columns 77-80. Also pick off the Tail Number from columns 58-67.

(2) With these 2 inputs, search the 2408-3 tape file to find the hours logged (OFH), on the aircraft, at the time of installation (RID) of the component. On the 2408-3 tape, the tail number is at Block 4, Columns 14-23, Card "A", and the Julian date (RID) is found at Block 11N, Card "C", Columns 31-34; the OFH is at Block 11K, Card C, Columns 16-20.

(3) Then, to find the number of hours (FFH) on the component at the end of the calendar period, (DEP), it is necessary to search the 2408-3 tape again for this tail number and DEP. The DEP equals the Cut-off Date (COD), in columns 44-47 of the RD tape for the RD1-CH-47-CGR tape, and is equal to COD less one year for the RD1-CH-47-CGO tape.

(4) The procedure is similar to the previous search, but (DEP) is used instead of (RID), and FFH replaces OFH.

(5) Then the usage of hours for the unfailed item, (UFH) is

given below:

$$UFH = FFH - OFH$$

(6) Knowing (UFH), a new record (an artificial 2410 record) is generated for the Unfailed Item. It is a reproduction of the record in which the maximum (RID) was found, with the following changes, "a" and "b".

(a) The newly found value of UFH is put into columns 89-92 of the RD tape.

(b) Replace the original ULI of columns 100-103 by 9999. This is necessary to insure that the Unfailed Items will be processed at the end of each Age Group.

20. Within each Age Group, sort by Usage since last installation, (ULI) of columns 100-103. The lowest value of ULI must be processed first.

21. Within each Age Group, sort by UFH. The highest UFH must be processed last.

22. On the last record enter "9" in column 301, following completion of all sortings.

23. Print a listing of each tape.

24. Store each tape for future use, and notify this office of its identification tag, and procedure for recall.

CONCLUSIONS This report shows that a test has been designed which will analyze the effect of overhauls, repairs, modifications of design (MWO), and engineering change proposals (ECP), for interchangeable items, on Time Between Overhaul (TBO), Reliable Life, Reliability, Burn-In Time, and Hazard Rate. These parameters will adjudge contractor compliance. An assumption of Weibull's family of distributions is made.

A second assumption is that items having the same number of overhauls and number of repairs, and age at the time of installations, will have no more than three drastic changes of failure rates. These sudden changes characterize the three life phase: 1) Burn-In Time, 2) Random Failure Phase, and 3) Wear Out Phase.

The program further tests the adequacy of the Army Reporting System (TAERS).

The major mathematical contribution is the development of formulas for the Weibull Probability Distribution parameters. As a result of this development, it will no longer be necessary to laboriously plot data, repeatedly in a trial and error program, to achieve graphical results. The entire Fortran program is being computerized by the Research and Development Division of RD&E Directorate, and the Special Studies Office of AVCOM. The Automatic Data Processing Office of AVCOM is preparing the data tape of the TAERS information.

A TECHNIQUE FOR INTERPRETING HIGH ORDER INTERACTIONS

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INTRODUCTION. The detection and interpretation of high order interactions has been quite difficult in the recent past. This has been primarily due to the large number of calculations required to evaluate all of the single-degree-of-freedom contrasts in a typical experiment. Hence, short-cut formulae were used which often permitted significant high order interactions to slip by undetected.

The recent advent of very high-speed, large-core, third-generation computers, together with the availability of good statistical packages has made adequate evaluation of interactions feasible. The only remaining aspect of the problem and the topic of this paper is the development of a logical and systematic procedure for ferreting out the essential pieces of information which will lead to a valid interpretation of interaction.

CONCEPT. The proposed procedure for isolating and interpreting high-order interaction is based upon a sequential elimination of the factor levels which are not primarily involved in the interaction. A least squares program is used to fit coefficients to a complete set of orthogonal contrasts among the treatment levels of the factorial. In addition, similar analyses are developed on subsets of the data. These subsets are those data within a given level of each factor of the entire experiment.

The computer output from a typical least-squares regression program is normally displayed in ANOVA table form. Each single-degree-of-freedom contrast is listed with an F test of its significance. These F tests are studied to determine which factor levels are involved in a particular interaction. First, the complete analysis is scanned for the highest significant interaction and also for low order interactions which are not components of higher order interactions. Subsequently, the subset analyses are studied to find which factor levels contribute to the high order interaction. Once the contributing factors are determined, the interaction can be resolved by graphical means.

FIELD AND TREATMENT DESIGNS OF THE EXPERIMENT. The procedures and techniques discussed herein can be readily adapted to a wide variety of field and treatment designs. For purposes of example, however, a factorial experiment in a randomized complete block field design is used. To put the example into context, the analysis of variance table is given in Table 1. The 64 treatments of this design are those of a $2^2 \times 4^2$ factorial treatment design. The 2-level factors are temperature (T), 25°C, 28°C and relative humidity (H), 20%, 80%. The 4-level factors are age (A), 28, 50, 70, 93 hours and populations (P), V, F, C, W of

Drosophila melanica. Population V is from Norfolk, Virginia, F from Forest Park, Missouri, C from Cliff, New Mexico and W from Walnut Creek, Arizona. The yield variable throughout this example is the respiratory rate of samples of ten Drosophila melanica pupae.

Table 1. Factorial Arrangement of Treatments for a Four-Factor Design (T,H,P,A).

Source	Degrees of Freedom	Degrees of Freedom for the Example
Blocks (B)	b-1	3
Temperature (T)	t-1	1
Humidity (H)	h-1	1
TH	(t-1)(h-1)	1
Population (P)	(p-1)	3
TP	(t-1)(p-1)	3
HP	(h-1)(p-1)	3
THP	(t-1)(h-1)(p-1)	3
Age (A)	a-1	3
TA	(t-1)(a-1)	3
HA	(h-1)(a-1)	3
THA	(t-1)(h-1)(a-1)	3
PA	(p-1)(a-1)	9
TPA	(t-1)(p-1)(a-1)	9
HPA	(h-1)(p-1)(a-1)	9
THPA	(t-1)(h-1)(p-1)(a-1)	9
Error	(b-1)(thpa-1)	189

Single degree of freedom contrasts must be developed for the main effects and interactions of the factorial model. First, contrasts are defined among the four main effects and the blocks. For the 2-level factors the contrast is simply +1 for the high-level and -1 for the low-level. For the 4-level factors, however, three contrasts need to be defined for each factor. For instance, our example has four populations, two of which are from the arid Southwest and two from the forested eastern half of the continent. Since three orthogonal contrasts are needed and even though any set will suffice for determination of sums of squares, a logical set of contrasts might be:

(1, 1, -1, -1)
(1, -1, 0, 0)
(0, 0, 1, -1)

where the first vector contrasts the non-arid populations with the arid ones, the second contrasts the non-arid populations and the third contrasts the arid populations. These contrasts are represented respectively as P1, P2 and P3.

When meaningful logical contrasts are not obvious, orthogonal polynomial coefficients can be used with adequate results since any set will produce the correct sum-of-squares. For example, the following set of vectors were used for the blocks contrasts as well as for age contrasts:

(3, 1, -1, -3) linear
(1, -1,-1, 1) quadratic
(1, -3, 3, -1) cubic,

where the vectors are represented by B1, B2 and B3 for blocks and by A1, A2 and A3 for ages.

Orthogonal single-degree-of-freedom interaction contrasts can be readily developed by taking all possible products of the already defined main-effect contrasts. For example, the 3 x 3 = 9 PA interaction contrasts are found by multiplying the elements of each of the three contrast vectors for P with each of the three contrast vectors for A. This procedure can be extended directly to the higher order interactions as well; e.g., the nine TPA contrasts may be developed by multiplying the T contrasts with each of the newly found PA contrasts. Of course, interaction contrasts with blocks do not have to be found because in the linear interaction model they all have expectation of zero. Hence they are valid error components and they can be evaluated by subtracting the blocks and treatments sum of squares from the total sum of squares.

Once the orthogonal contrasts have been defined, they can be used as independent variables in a multiple regression analysis. Most statistical packages include a least-squares program which will accomplish the necessary calculations.

The abbreviated ANOVA table, Table 2, is the result of a regression analysis of respiration rates upon the 66 orthogonal contrasts (3 for blocks and 63 for the treatments of the $2^2 \times 4^2$ factorial). Only those contrasts with F values greater than 4.00 are tabulated. The large array of significant interaction is particularly alarming, especially when a 4-factor interaction is highly significant. The first reaction is, "Who missed a decimal point in a couple of data cards?" Since this is not the case, an interpretation is required. The various interactions can be broken into three general categories for discussion. The first group is composed of the interactions which are not components of the highest order interaction, the second is the highest order interaction itself and the third group is composed of the lower order interactions which are components of the highest order interaction.

Table 2. Significant* Contrasts for the Complete $2^2 \times 4^2$ Experiment.

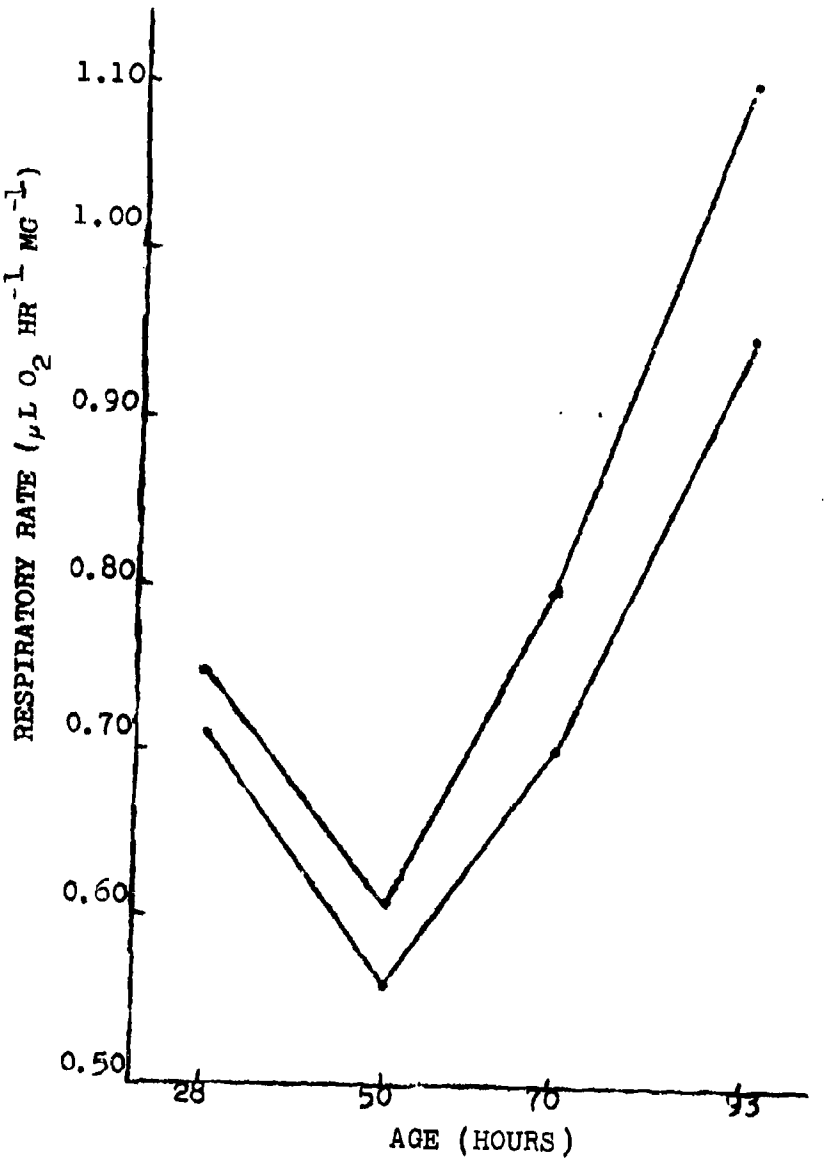
	Contrasts	F Statistic
Main Effects	T	257.71
	H	6.16
	P1	6.45
	P3	30.02
	A1	706.45
	A2	452.66
	A3	8.81
Interactions	TA1	44.64
	TA2	11.44
	TA3	6.23
	HP2	8.98
	P1A1	25.93
	P3A1	7.70
	THP1	11.05
	THP2	8.15
	THA3	12.77
	HP2A1	8.68
	THP2A1	11.73

*Contrasts with F less than 4.00 are not tabulated.

Interactions in the first group such as P3A1, which is not a component of THP2A1, can be easily resolved by graphical techniques. Considering that the contrast P3 is the comparison of Cliff vs. Walnut and that P3A1 does not involve T or H, the mean respiration rate averaged over all temperature and humidity levels was determined for the eight combinations of Cliff and Walnut with age. These are plotted in Figure 1 with respiration rate on the ordinate and age on the abscissa. While the lines for Cliff and Walnut are essentially parallel at the younger ages, they do diverge considerably at age 93. This divergence, of course, is what we detect by the significant F for P3A1. Thus we have resolved P3A1.

A very basic part of the interpretation of high order interactions such as THP2A1 is identification of the particular levels of the effects which are primary contributors to the interaction. To aid in detecting these critical levels, sub analyses were performed on the 3-factor factorials within each level of each main effect. For instance, an analysis was performed on the

FIGURE 1. GRAPHICAL DISPLAY OF THE 2-FACTOR INTERACTION P3A1.



2×4^2 factorial subset where temperature is 28°C. Taking each factor in turn we see the following:

1. Within temperature levels the 25°C data exhibit only two 2-factor interactions while the 28°C data exhibit eight 2 and 3-factor interactions. Table 3.

Table 3. Significant Contrasts for the 3-Factor Experiment within Temperature Levels.

	Contrast	F Statistic	
		25°C	28°C
Main Effects	P1		4.04
	P3	7.79	27.63
	A1	191.31	634.23
	A2	293.79	183.57
	A3		17.11
Interactions	HP1		15.64
	HP2		19.64
	HA1		5.84
	HA3	12.59	
	P1A1	8.89	19.93
	P1A2		4.58
	P3A1		5.98
	HP1A1		4.44
	HP2A1		23.27

2. Within humidity levels we see that both levels exhibit a considerable number of interactions. Table 4.

Table 4. Significant Contrasts for the 3-Factor Experiments within Humidity Levels.

	Contrasts	F Statistic	
		20% R.H.	80% R.H.
Main Effects	T	184.11	103.20
	P1	14.17	
	P2	17.59	
	P3	17.73	14.28
	A1	560.85	253.30
	A2	314.44	186.33

Table 4. (Continued)

	Contrasts	F Statistic	
		20% R.H.	80% R H.
	A3	14.56	
Interactions	TP1	8.51	4.10
	TP2		5.02
	TA1	52.38	9.28
	TA2	11.57	
	TA3		14.65
	P1A1	25.40	7.16
	P2A1	6.11	
	P3A1		4.85
	TP1A1	5.39	
	TP2A1	5.74	6.44

3. Within the populations we see that the Cliff and Walnut strains exhibit only 2-factor interactions while the Virginia and Forest Park strains are both involved in 3-factor interactions. Table 5.

Table 5. Significant Contrasts for the 3-Factor Experiments within Populations.

Contrast	F Statistic			
	Virginia	Forest Park	Cliff	Walnut
T	259.26	29.89	81.89	74.67
H		9.63		
A1	886.49	129.71	163.06	119.71
A2	422.26	77.75	108.71	125.47
A3	7.93		4.29	7.22
TH		6.92		4.86
TA1	61.21	5.71	10.74	10.63
TA2	4.26		6.44	4.86
HA1		5.24		
HA2	5.69			
THA1		9.65		
THA3	7.53			

4. Within the four age levels we find that ages 28, 50 and 70 are not involved in 3-factor interactions while the 93 hour data have significant 3-factor interactions. Table 6.

Table 6. Significant Contrasts for the 3-Factor Experiment within Age Levels.

Contrast	F Statistic			
	28 hrs.	50 hrs.	70 hrs.	93 hrs.
T	7.13	57.67	151.93	79.95
H			6.94	
P1				26.64
P3			7.30	19.80
TH	6.96	6.92		
HP2				11.64
THP1				8.40
THP2				15.80

Thus, it appears that attention should be focused upon the 28°C, 93 hour data from the Forest Park and Virginia strains. The interacting 93 hour data and the non-interacting 50 hour data are demonstrated graphically in Figure 2. It is apparent from these two graphs that the Forest Park and Virginia strains respond differently at the two humidity levels when the temperature is at 28°C. Conversely, when the temperature is at 25°C the response curves are parallel. Thus, we have resolved the 4-factor interaction.

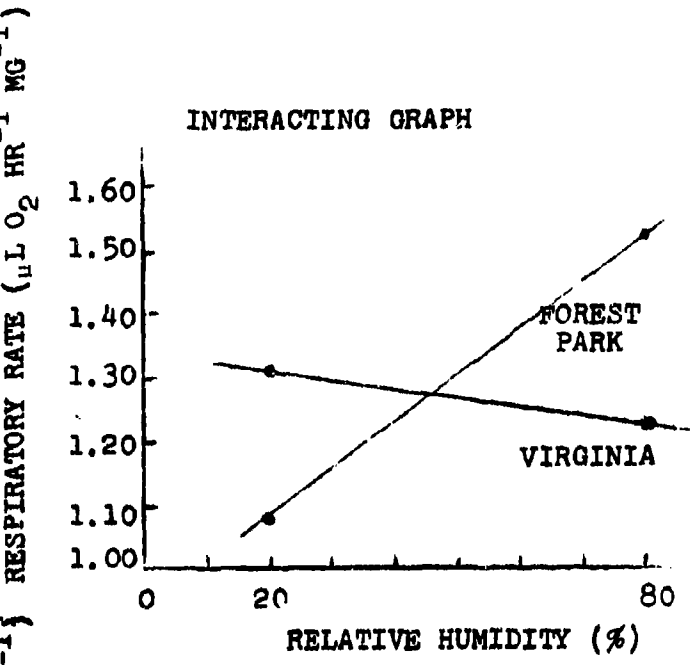
The analyses within age levels indicated that the three youngest ages were involved in only a very few interactions. Thus, we decided to reanalyse the data as a $2^2 \times 3 \times 4$ factorial by eliminating age 93 from the analysis. The results of this analysis, given in Table 7, are quite enlightening. Only two interactions, THA2 and TA1, are really significant. The third interaction, THP1, has an F of only 4.13 which is right at the critical value of F and will be ignored. We also note that the lower order interaction, TA1, is a component of THA2. The most striking result of this $2^2 \times 3 \times 4$ analysis is the complete disappearance of the 4-factor interaction which verifies that its significance is in fact due to a failure of the 93 hour data to conform with the data from all three other ages.

Table 7. Significant Contrasts for the 4-Factor Experiment after Omitting Data from Age Level 93.

	Contrasts	F Statistic
Main Effects	T	178.40
	H	4.20
	P3	12.80

FIGURE 2. GRAPHICAL DISPLAY OF THE 4-FACTOR INTERACTION THP2A1.

TREATMENT COMBINATION	MEAN
28°C, 20%, V, 93hr	1.31
28°C, 80%, V, 93hr	1.23
28°C, 20%, F, 93hr	1.07
28°C, 80%, F, 93hr	1.51



TREATMENT COMBINATION	MEAN
25°C, 20%, V, 50hr	0.50
25°C, 80%, V, 50hr	0.45
25°C, 20%, F, 50hr	0.56
25°C, 80%, F, 50hr	0.50

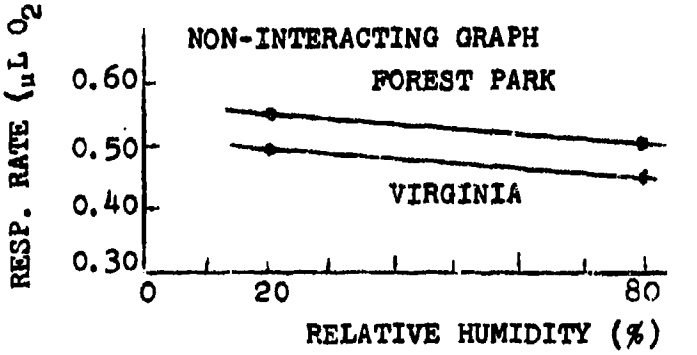


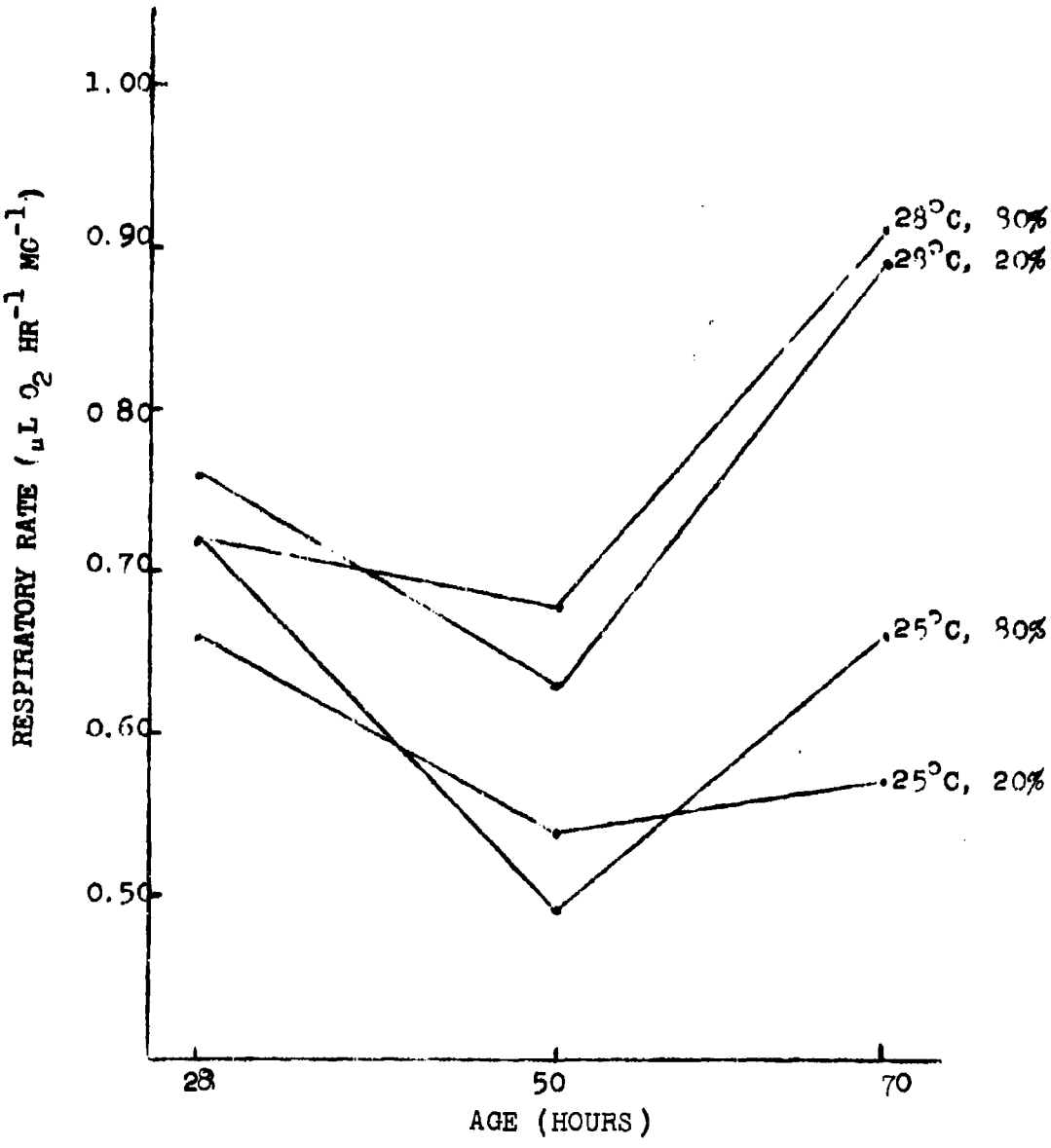
Table 7. (Continued)

	Contrasts	F Statistic
	A1	8.20
	A2	145.60
Interactions	TA1	61.51
	THP1	4.13
	THA2	13.88

The 2^2 x 3 x 4 analysis also points out an interaction of the third type; namely a lower order interaction, THA2, which is a component of a higher order interaction, THP2A1. Because the 4-factor interaction involved age 93 and because the interaction THA2 is significant throughout the remainder of the experiment, we should determine the implications of THA2. Populations are not involved in THA2 so we can plot (Figure 3) the respiration rate, averaged over populations, against age for the four combinations of temperature and humidity. The two 28°C curves are similar whereas the two 25°C curves are quite divergent from each other and also from the 28°C curves. This figure quite adequately demonstrates the response function for the three youngest ages. Because the interactions with population were nonsignificant in the 2^2 x 3 x 4 analysis, we can infer that the response curve of each population is similar in shape to the response curves in Figure 3.

In conclusion, a procedure is outlined for isolating high-order interactions and developing their logical interpretation. Procedures are also outlined for identifying and interpreting two types of lower order interaction, those which are components of the higher order interaction and those which are not. The analysis is based on a least squares fit to single-degree-of freedom contrasts and a subsequent graphical display of the significant contrasts. The key to the method, however, is the ability to isolate the critical factors by taking advantage of the computers ability to easily and inexpensively reanalyze various subsets of the data.

FIGURE 3 GRAPHICAL DISPLAY OF THE 3-FACTOR INTERACTION THA2



A SIMPLIFIED METHOD FOR FINDING OPTIMUM EXPERIMENTAL DESIGNS*

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ABSTRACT. This paper presents a simplified method for determining an optimum experimental configuration that most nearly satisfies the experimenter's requirements.

Although the LaGrangian multiplier method can be used to find a specific experimental design with nearly minimum variance subject to cost restrictions, the experimenter's flexibility is limited and the calculations are laborious. By use of the simple computer program given in this paper, the objective and cost functions can be readily evaluated for numerous feasible combinations. The distinct advantage of the latter technique is that the experimenter is able to choose that design which most nearly fits his experimental needs.

INTRODUCTION. The success or failure of an experiment is normally determined during the planning phase of the research. Success of a particular experimental design is essentially dependent on the design's ability to test adequately certain hypotheses or to estimate certain effects accurately. This paper considers efficient experimental designs from the standpoint of optimum choice of factor levels once the basic design type has been determined. An exemplary problem is solved with the aid of a very simple computer program.

The basic design type, such as a completely random design, a randomized complete block design, or a split-plot design, is determined to a large extent by design restrictions. For instance, you can't change cameras in a reconnaissance aircraft during flight nor can pilots be switched. Similarly, it is not usually possible to completely randomize aircraft speeds or altitudes due to obvious restrictions, both legal and technical. Because the basic design type is usually prescribed in one way or another, we will restrict our attention to selection of the number of levels of each of the component factors for a split-plot design. Although a large number of combinations are essentially equivalent, a very poor design may often be developed if the experiment is not adequately planned. Since many experimental designs cannot be easily changed during the conduct of the experiment, great care needs to be exercised during the design. If a change is made late in the experimental process, a considerable

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waste of effort and loss of efficiency is usually experienced.

The mere fact that we are concerning ourselves with an optimum design suggests that some trade-offs must be made. Usually these trade-offs are precision versus the cost of performing the experiment. The objective functions with which we are normally concerned are not expressed in common units. This considerably complicates matters when we get to the point where we wish to solve for an optimum solution. The objective functions of design efficiency are normally expressed in terms of variance components and the design parameters. The cost function, on the other hand, is typically a function of dollar or hour cost and of the design parameters. An ideal design, of course,would be one which minimized the cost function. Naturally some compromise must be made. Hence, a combination of design parameters must be found that gives near minimum variance (or at least a tolerable variance) for the smallest cost consistent with design needs.

METHOD. To develop a desirable design, several essential steps must be followed. For purpose of this presentation, we will first outline a systematic procedure in seven major steps. Subsequently, we will follow this procedure through to completion with an example from reconnaissance research. The essential steps are:

1. State the hypotheses to be tested and identify the effects to be estimated.
2. Develop a linear mathematical model of the yield variable in terms of the factors of the design. Of course, this model must be such that it will provide test statistics capable of testing the hypotheses stated in Step 1. Furthermore, it must also provide estimators for any effects that must be estimated.
3. Develop an ANOVA table based on the model. Work out the expectations of the mean squares.
4. Develop an objective function for each hypothesis that is to be tested and one for each effect that is to be estimated. These functions will typically be functions of the design parameters and of the variance components. Hence, a priori estimates of the variance components must be developed. Often these estimates can be derived from similar previously performed research projects.
5. Develop a cost function based on the project's design parameters and their respective unit costs.
6. Solve the set of objective functions for an optimum solution. Since the functions are antagonistic and require integer solutions, a computerized evaluation of the objective functions for feasible combinations of design parameters is recommended.
7. Select the combination of design parameters that most nearly minimizes the objectives functions within the budgetary restrictions of the project.

An example from aerial photographic reconnaissance will be used to demonstrate this procedure. Only the identities of the aircraft and its cameras have been changed for security purposes. Oh yes, the velocities and data are also fictitious for the same reason. Several "sophisticated" aircraft will be available to fly photographic missions during this research period. Due to

commitments we have s sorties assigned to this project, where s is somewhat negotiable. One serious restrictions, however, is that a particular aircraft cannot be guaranteed for a fixed number of sorties. Thus, sorties will be considered as blocks. All aircraft will be fitted with an "Advanced, Model A1-Mod 3" camera for this test. Each sortie can reliably produce 12 images of the target complex. We are interested in evaluating image quality at four different velocities, namely slow, fast, very fast and full throttle (all after burners on). The latter one for obvious reasons. Hence, the four levels of speed are our treatments of primary interest. Since 12 images can be secured during a sortie, the 4 velocities will be replicated 3 times within each sortie. The resulting photographic images are to be evaluated by p photographic interpreters. Each of the p interpreters will be required to evaluate the images d times - on different days, of course, to avoid an among successive - evaluations variance of zero.

In this example, a number of tests of hypotheses are of interest. However, to simplify the presentation, we will only consider efficiency for the hypothesis, $\tau_j - \tau_{j'} = 0$. The basic design structure that we have just outlined will provide a test statistic for this hypothesis. Each of the individual measurements upon the imagery can be described by the model:

$$Y_{ijk\ell m} = \mu + \beta_i + \tau_j + \beta\tau_{ij} + \delta_{ijk} + \gamma_\ell + \gamma\delta_{ijk\ell} + \varepsilon_{ijk\ell m}$$

$$\begin{aligned} i &= 1, \ldots, s \\ j &= 1, 2, 3, 4 \\ k &= 1, 2, 3 \\ \ell &= 1, \ldots, p \\ m &= 1, \ldots, d, \end{aligned}$$

where μ is a constant. β_i is the effect of the ith sortie where the $\beta_i \sim NID(0, \sigma_\beta^2)$, and τ is the effect of the jth velocity when $\Sigma\tau_j = 0$. $\beta\tau_{ij}$ is an additional effect due to the specific combination of the jth speed during the ith sortie where the $\beta\tau_{ij} \sim NID(0, \sigma_{\beta\tau}^2)$. δ_{ijk} is a sampling error within the same sortie-velocity combination where the $\delta_{ijk} \sim NID(0, \sigma_\delta^2)$. γ_ℓ is the effect due to the ℓth interpreter where the $\gamma_\ell \sim NID(0, \sigma_\gamma^2)$. $\gamma\delta_{ijk\ell}$ is an effect due to the way the ℓth interpreter evaluates the ijkth image where $\gamma\delta_{ijk\ell} \sim NID(0, \sigma_{\gamma\delta}^2)$. $\varepsilon_{ijk\ell m}$ is a sampling error among consecutive readings of the ℓth interpreter on the ijkth image where $\varepsilon_{ijk\ell m} \sim NID(0, \sigma_\varepsilon^2)$, and the ANOVA would appear (in abbreviated form) as in Table 1.

The first term of the model gives us a line for the overall mean with one degree-of-freedom. Upon closer scrutiny, it is quite obvious that the next four terms of the model define what is commonly referred to as a whole-plots analysis in a split-plot experiment with the s sorties as blocks and v = 4 velocities as treatments. The model indicates that interaction between sorties

and velocities is possible and that each sorties treatment combination is replicated r times. In this case, however, r = 3 since vxr = 12. Normally, of course, the block by treatment interaction is assumed to be zero. In this case, however, sorties cannot be considered to be true blocks since they are really another treatment; moreover in the real world, we wish to estimate the component $\sigma_{\beta\tau}^2$. These first five lines are thus the whole-plot part of the analysis. The photo interpreters are the split-plot treatments and this leads, in turn, to the three line split-plot analysis of photo interpreters, photo interpreters by whole-plot treatments interactions, and sampling error among the split-plot units.

To simplify the discussion somewhat, we restricted our attention to design optimization for the test of the hypothesis, $\tau_j - \tau_j' = 0$. This test can be made using a t-test; therefore, it is obvious that an optimum design for this test can be achieved by minimizing the variance of the difference between the τ treatment means. Now the estimator for the contrast $\tau_j - \tau_j'$ is $\widehat{\tau_j - \tau_j'} = \frac{1}{srpd}(Y_{.j}... - Y_{.j'}...)$ where the dot indicates summation over that subscript. The estimator $\tau_j - \tau_j'$, written in terms of the model is

$$\widehat{\tau_j - \tau_j'} = \frac{1}{srpd}\left[srpd\,(\tau_j - \tau_j') + rpd \sum_{i=1}^{s}(\beta\tau_{ij} - \beta\tau_{ij'}) + pd \sum_{i=1}^{s}\sum_{k=1}^{r}(\delta_{ijk} - \delta_{ij'k})\right.$$

$$\left. + d \sum_{i=1}^{s}\sum_{k=1}^{r}\sum_{\ell=1}^{p}(\gamma\delta_{ijk\ell} - \gamma\delta_{ij'k\ell}) + \sum_{i=1}^{s}\sum_{k=1}^{r}\sum_{\ell=1}^{p}\sum_{m=1}^{d}(\varepsilon_{ijk\ell m} - \varepsilon_{ij'k\ell m})\right].$$

Now the variance of $\widehat{\tau_j - \tau_j'}$ is:

$$V(\widehat{\tau_j - \tau_j'}) = \frac{1}{s^2r^2p^2d^2}\left[2r^2p^2d^2s\,\frac{v}{v-1}\sigma_{\beta\tau}^2 + 2p^2d^2sr\,\sigma_\delta^2 + \right.$$

$\left. 2d^2srp\,\sigma_{\gamma\delta}^2 + 2\,srpd\,\sigma_\varepsilon^2\right]$. Or rewriting into the form of the EMS of the ANOVA table $V(\widehat{\tau_j - \tau_j'}) = \frac{2}{srpd}\left[\sigma_\varepsilon^2 + d\sigma_{\gamma\delta}^2 + pd\,\sigma_\delta^2 + rpd\,\frac{v}{v-1}\sigma_{\beta\tau}^2\right]$. If we knew the actual values of the variance components σ_ε^2, $\sigma_{\gamma\delta}^2$, σ_δ^2, and $\sigma_{\beta\tau}^2$ we could write out one of the objective functions that we wish to minimize. A priori estimates must be found, by argument if necessary, to evaluate the objective function. For many problems estimates can be derived from previously conducted experiments.

The other objective function that we now have to develop is the cost function. This can also be developed from the mathematical model in terms of the design parameters s, r, p, d, and v plus the actual unit cost of each additional level of the factors. Therefore, let:

C_1 = Cost for each sortie

C_2 = Cost for each photo interpreter, and

C_3 = Cost of replicate photo interpretations.

Furthermore, since r and v are fixed, the cost for the experiment will be $sC_1 + pC_2 + svrpd\ C_3$. The two functions which we wish to minimize are thus:

$$f_1 = \frac{2}{3spd}\ (\sigma_\varepsilon^2 + d\ \sigma_{\gamma\delta}^2 + pd\ \sigma_\delta^2 + 4\ pd\ \sigma_{\beta\tau}^2),$$

and

$$f_2 = s\ C_1 + p\ C_2 + 12\ spd\ C_3.$$

The objective functions f_1 and f_2 are antagonistic because f_1 is a decreasing function of the design parameters; whereas, f_2 is an increasing function. A method of evaluating these functions is obviously needed. An often used, however quite unsatisfying, method is the LaGrangian Multiplier method by which the variance function, f_1, is minimized subject to the cost function, f_2, being equal to some fixed cost. Some criticisms of this method are:

(1) It yields non-integer solutions.
(2) It usually requires the solution of very difficult equations.
(3) It does not reveal nearly optimal solutions.
(4) It does not reveal the solutions with considerably smaller variance at only a moderate increase in cost.

A very simple computer program provides a means for finding an optimum integer solution. In fact, all of the previously mentioned criticisms of the LaGrangian multiplier method are avoided. The only apparent difficulties with this computerized method appear to be:

(1) It requires some programming.
(2) The computer output must be scanned visually to find the design parameter combinations that most nearly satisfy the objective functions.

The latter difficulty, of course, could be a very problematical task if several antagonistic objective functions are to be evaluated simultaneously. Even this task is not too difficult if isobars are drawn in various colors on the computer output sheets.

An example of a simple complete Fortran (WATFOR)[1] program which will evaluate our example is:

[1] WATFOR is a Fortran IV compiler written for IBM 360 computers by the University of Waterloo, Waterloo, Ontario.

```
      DIMENSION OUT (8), OUT2 (8)
    3 READ, CS, CP, CD, VE, VCD, VD, VET
      DO 1  1 = 1, 10
    7 FORMAT ('1')
      WRITE (3, 7)
      DO 1, L = 1,  10
      DO 2, M = 1,  8
      OUT (M) = 1*CS+L*CP+12*1*L*M*CD
    2 OUT2 (M) = 2*(VE+M*VCD+L*M*VD+4*L*M*VET)/(3*1*L*M)
      WRITE (3,4) OUT
    1 WRITE (3,5) OUT2
    4 FORMAT ('0', 8F 9.0)
    5 FORMAT (' ', 8F 9.3)
      GO TO 3
      END
```

Just as a matter of interest, this program took less than three seconds to compile and run on a WATFOR compiler with a 360/75 computer.

Table II is an example of a typical computer sheet. Lines, which represent constant cost and constant variance, have been drawn through the tabulations to aid in locating the optimum combination of design parameters. The upper element of a pair is the value of the cost function, whereas the lower element is the value of the objective variance function. A wide variety of designs with similar costs yield essentially the same precision for the desired test. For instance the design with

s = 2, v = 4, r = 3, p = 10, and d = 1 is comparable to the design

s = 2, v = 4, r = 3, p = 4, and d = 5.

Either of these designs will meet the basic criteria for optimization. We might ask, however, whether a better design exists that has essentially the same variance, lower cost and more flexibility. The design with

s = 2, v = 4, r = 3, p = 5, and d = 2

has a similar variance and it costs only 75% of either of the previously

mentioned designs. The flexibility of each of these designs must be considered in making a final selection. Nonetheless, care should be exercised to ensure that the design is not extremely sensitive to inadequate a priori estimates of the variance components in the objective function. This can be easily accomplished by rerunning the program with a number of alternative sets of a priori estimates.

NOTE. Tables II, III, and IV use the following a priori estimates:

$\sigma_{\varepsilon}^2 = 2$

$\sigma_{\gamma\delta}^2 = 2$

$\sigma_{\beta\tau}^2 = 2$

$\sigma_{\delta}^2 = 0.5$

Cost per sortie = \$500

Cost per photo reading = \$2

Fixed cost per photo interpreter = \$200

Tables III and IV illustrate the flexibility that the experimental planner can acquire by using this programming method. Table III has the design parameters $s = 7$, $v = 4$, and $r = 3$; whereas Table IV has $s = 8$, $v = 4$, and $r = 3$. Of all the combinations in Table III with variance of 0.9, the design with $p = 4$ and $d = 1$ has the smallest cost at \$5980. From Table IV, however, the variance can be maintained at 0.9 with a cost of only \$5360 by using $p = 2$ and $s = 8$. Not only can we realize cost savings, but we can also attain a slight reduction in the variance. The design's flexibility as well as its insensitivity to inadequate a priori estimates must necessarily affect the final choice from the candidate designs.

TABLE I - ANOVA for a Split Plot Deisgn

Mean	1	
Sortie	s-1	
Velocity	v-1	$\sigma_\varepsilon^2 + d\sigma_{\gamma\delta}^2 + rsvd\ \sigma_\gamma^2 + dp\sigma_\delta^2 + \frac{vdpr}{v-1}\sigma_{\beta\tau}^2 + \frac{dprs}{v-1}\Sigma\tau_j^2$
Sortie x velocity	(s-1)(v-1)	$\sigma_\varepsilon^2 + d\sigma_{\gamma\delta}^2 + rsvd\ \sigma_\gamma^2 + dp\sigma_\delta^2 + \frac{vdpr}{v-1}\sigma_{\beta\tau}^2$
Rep/SxV	sv(r-1)	$\sigma_\varepsilon^2 + d\sigma_{\gamma\delta}^2 + rsvd\ \sigma_\gamma^2 + dp\sigma_\delta^2$
Photo Interpreters (PI)	p-1	$\sigma_\varepsilon^2 + d\sigma_{\gamma\delta}^2 + rsvd\ \sigma_\gamma^2$
PI x Image	(p-1)(rsv-1)	$\sigma_\varepsilon^2 + d\sigma_{\gamma\delta}^2$
Error	(d-1) rsvp	σ_ε^2

TABLE II - COST AND VARIANCE OF DESIGNS WITH s = 2, v = 4, and r = 3

Number of Readings by Each Interpreter (d)

Number of Interpreters (p)	1	2	3	4	5	6	7	8
1	1320. 4.167	1440. 3.833	1560. 3.722	1680. 3.667	1800. 3.633	1920. 3.611	2040. 3.595	2160. 3.583
2	1640. 3.500	1880. 3.333	2120. 3.278	2360. 3.250	2600. 3.233	2840. 3.222	3080. 3.214	3320. 3.208
3	1960. 3.278	2320. 3.167	2680. 3.130	3040. 3.111	3400. 3.100	3760. 3.093	4120. 3.087	4480. 3.083
4	2280. 3.167	2760. 3.083	3240. 3.056	3720. 3.042	4200. 3.033	4680. 3.028	5160. 3.024	5640. 3.021
5	2600. 3.100	3200. 3.033	3800. 3.011	4400. 3.000	5000. 2.993	5600. 2.989	6200. 2.986	6800. 2.983
6	2920. 3.056	3640. 3.000	4360. 2.981	5080. 2.972	5800. 2.967	6520. 2.963	7240. 2.960	7960. 2.958
7	3240. 3.024	4080. 2.976	4920. 2.960	5760. 2.952	6600. 2.948	7440. 2.944	8280. 2.942	9120. 2.940
8	3560. 3.000	4520. 2.958	5480. 2.944	6440. 2.938	7400. 2.933	8360. 2.931	9320. 2.929	10280. 2.927
9	3880. 2.981	4960. 2.944	6040. 2.932	7120. 2.926	8200. 2.922	9280. 2.920	10360. 2.918	11440. 2.917
10	4200. 2.967	5400. 2.933	6600. 2.922	7800. 2.917	9000. 2.913	10200. 2.911	11400. 2.910	12600. 2.908

$4200

$\sigma^2 = 2.967$

TABLE IV - COST AND VARIANCE OF DESIGNS WITH s = 8, v = 4, and r = 3

Number of Interpreters (p)	Number of Readings by Each Interpreter (d)							
	1	2	3	4	5	6	7	8
1	4680. 1.042	5160. 0.958	5640. 0.931	6120. 0.917	6600. 0.908	7080. $7200 0.903	7560. 0.899	8040. 0.896 $\sigma^2 = 0.896$
2	5360. 0.875	6320. 0.833	7280. 0.819	8240. 0.813	9200. 0.808	10160. 0.806	11120. 0.804	12080. 0.802
3	6040. 0.819	7480. 0.792	8920. 0.782	10360. 0.778	11800. 0.775	13240. 0.773	14680. 0.772	16120. 0.771
4	6720. 0.792	8640. 0.771	10560. 0.764	12480. 0.760	14400. 0.758	16320. 0.757	18240. 0.756	20160. 0.755
5	7400. 0.775	9800. 0.758	12200. 0.753	14600. 0.750	17000. 0.748	19400. 0.747	21800. 0.746	24200. 0.746
6	8080. 0.764	10960. 0.750	13840. 0.745	16720. 0.743	19600. 0.742	22480. 0.741	25260. 0.740	28240. 0.740
7	8760 0.756	12120. 0.744	15480. 0.740	18840. 0.738	22200. 0.737	25560. 0.736	28920. 0.736	32280. 0.735
8	9440. 0.750	13280. 0.740	17120. 0.736	209.60 0.734	24800. 0.733	28640. 0.733	32480. 0.732	36320. 0.732
9	10120. 0.745	14440. 0.736	18760. 0.733	23080. 0.731	27400. 0.731	31720. 0.730	36040. 0.729	40360. 0.729
10	10800. 0.742	15600. 0.733	20400. 0.731	25200. 0.729	30000. 0.728	34800. 0.728	39600. 0.727	44400. 0.727

TABLE III - COST AND VARIANCE OF DESIGNS WITH s = 7, v = 4, and r = 3

Number of Readings by Each Interpreter (d)

Number of Interpreters (p)	1	2	3	4	5	6	7	8
1	4120. 1.190	4540. 1.095	4960. 1.063	5380. 1.048	5800. 1.038	6220. 1.032	6640. 1.027	7060. 1.024
2	4740. 1.000	5580. 0.952	6420. 0.937	7260. 0.929	8100. 0.924	8940. 0.921	9780. 0.918	10620. 0.917
3	5360. 0.937	6620. 0.905	7880. 0.894	9140. 0.889	10400. 0.886	11660. 0.884	12920. 0.882	14180. 0.881
4	5980. 0.905	7660. 0.881	9340. 0.873	11020. 0.869	12700. 0.867	14380. 0.865	16060. 0.864	17740. 0.863
5	6600. 0.886	8700. 0.867	10800. 0.860	12900. 0.857	15000. 0.855	17100. 0.854	19200. 0.853	21300. 0.852
6	7220. 0.873	9740. 0.857	12260. 0.852	14780. 0.849	17300. 0.848	19820. 0.847	22340. 0.846	24860. 0.845
7	7840 0.864	10780. 0.850	13720. 0.846	16660. 0.844	19600. 0.842	22540. 0.841	25480. 0.841	28420. 0.840
8	8460. 0.857	11820. 0.845	15180. 0.841	18540. 0.839	21900. 0.838	25260. 0.837	28620. 0.837	31980. 0.836
9	9080. 0.852	12860. 0.841	16640. 0.838	20420. 0.836	24200. 0.835	27980. 0.834	31760. 0.834	35540. 0.833
10	9700. 0.848	13900. 0.838	18100. 0.835	22300. 0.833	26500. 0.832	30700. 0.832	34900. 0.831	39100. 0.831

A DEFINITIVE CALIBRATION OF AN AERIAL CAMERA IN ITS OPERATING ENVIRONMENT

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INTRODUCTION. It should be appreciated that the calibration of an aerial camera in its operating environment is more meaningful and effective than a laboratory calibration. However, even though this principal has been acknowledged by many scientists in this area of endeavor, the calibration of aerial mapping cameras has almost universally been relegated to the laboratory. In recent years, the ballistic camera has been used for recording flashes from active earth bound satellites or recording reflecting type satellites, such as the Echo Satellite, on photographic glass plates. The ballistic cameras are successfully calibrated in their operating environment using the process of stellar calibration. This has led to suggestions that the technique be applied to aerial mapping cameras. A small amount of work has been expended in calibrating aerial cameras using the stellar calibration technique. However, as with the laboratory methods, this technique still suffers from its failure to simulate the typical operational utilization and environment of an aerial mapping camera' namely, photographing the ground thru a camera window located on the underside of a fast moving aircraft.

The experimental design necessary to calibrate an aerial camera in its operating environment requires extensive knowledge of the scientific disciplines of analytical, aerial photogrammetry, optics, and first order regression processes. It is not the purpose of this paper to explain in detail each of these scientific areas, but we will briefly discuss each of the mathematical models necessary to carry out the equipment.

The photogrammetric model we will adopt has been used successfully in recent years for analytical, aerial triangulation. Also, extensive effort has been expended to develop a mathematical model which describes the displacement of photographic images due to imperfect lenses. One such model is called the Thin Prism Model, and it is used to describe the radial and tangential components of distortion. Alternative models have been derived, such as Conrady's Model, in the year 1919. However, Conrady's model does not agree exactly with the Thin Prism Model. In any case, there have been many investigations through the years concerned with this aspect of optics and, notably, a very recent investigation was carried out by D. Brown [1] whereby he developed a model through extensive analytical, three dimensional ray tracing through a thin prism. Brown derived an analytical expression defining the relationship between the radial and tangential distortion induced by a thin prism at any specified azimuth. This can be considered as an extension to Conrady's Model.

A third model which we must adopt has been well defined for many years and it describes the displacement of an image symmetrically about the optical axis. It has been found that the distortion of a perfectly centered

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lens composed of flawless elements is symmetric about the optical axis. This distortion is commonly referred to as symmetric radial distortion.

With this all too brief narrative summary of photogrammetry and optics, we may consider our final model to consist of three major components; namely, symmetric radial distortion, decentering distortion, and the fundamental projective equation, or colinearity equation. The colinearity equation describes the fact that, with no distortion, the perspective center of the lens, an image on the film and its corresponding point on the ground (object space) all lie on the same straight line.

We will develop the first order regression process which makes practical the solution of a would be very large system of normal equations. The first order regression process will encompass two sets of parameters which will be referred to as stationary and nonstationary parameters. The regression process which simultaneously recovers these sets of parameters is referred to as Aerial SMAC, an acronym for Simultaneous Multistation Analytical Calibration. We will develop the SMAC process to provide for the introduction of external or a priori information associated with any of the stationary and nonstationary parameters.

We shall also discuss in brief the necessary requirements of a photogrammetric test range so that the calibration experiment can yield the best possible recovery of the meaningful parameters resulting from a rigorous data reduction process.

CALIBRATION RANGE

In order to carry out a definitive calibration of an aerial camera in its operating environment, we must conduct the experiment by flying over special target ranges where the horizontal and vertical position of the targets are precisely known relative to each other. A small make-shift 3 by 5 mile range is available in the McClure, Ohio, area. This range was used recently to conduct a SMAC experiment. As a matter of fact, the range was turned into a night photogrammetric test range whereby 56, 500 watt, iodine quartz lamps were placed over the survey markers. Unfortunately, the final results of this experiment are not yet available at the writing of this paper.

From our model, we will see that the X, Y, Z, position of each of these precisely surveyed marks are taken as known quantities. Any small error in their position will be smaller than the noise level of the film measurements at the scale of the photography. However, SMAC suffers the disadvantage of being inherently incapable of yielding a calibration of elements of interior orientation (focal length and principal point) of the camera. It is well known that the variations in the elements of interior

orientation are projectively equivalent to changes in the X^c, Y^c, Z^c, coordinates of the aircraft. On the other hand, when external information is available, a SMAC reduction is possible. As stated in the introduction, the regression process will be developed whereby external information can be introduced into SMAC. The necessary external information will come from either electronic tracking devices, which will track the aircraft as it flies over the test range, or from ballistic cameras observing a flashing light on board the aircraft if the range is a night photogrammetric test range. In either case, the electronic tracking devices, or ballistic cameras, situated around the test range, will provide the X^c, Y^c, Z^c, position of the aircraft from an independent data reduction process. Let it suffice to say that with rigorous data reduction processes, it is possible to recover the position of the aircraft to within 2 feet, especially since we are considering excellent geometry.

Figure 1 illustrates the type of permanent photogrammetric test range to be used in the future and Figure 2 illustrates the flight patterns over this range.

In order to provide the reader with some idea of the accuracies which we hope to achieve, we will say that the film measuring accuracy should be close to 5 microns and then the estimated elements of interior orientation are expected to have standard deviations of approximately 2 microns. The standard deviations of the calibrated functions of radial and tangential distortion are also expected to be approximately 2 microns. It should be appreciated that these accuracies are achievable with only the most rigorous data reduction process, precision measuring devices, and an accurately surveyed test range.

SYMMETRIC RADIAL DISTORTION

As stated previously, the distortion of a perfectly centered lens is symmetric about the optical axis; that is, the distortion is symmetrical about the principal point and therefore is a function of radial distance only.

Figure 3 will give the reader an idea of the photographic coordinate system with which we are dealing. From this figure, we obtain the concept of what is meant by interior orientation. The vector from the perspective center to the image point is defined as follows:

$$\begin{bmatrix} x - X^c \\ y - Y^c \\ z - Z^c \end{bmatrix} = \begin{bmatrix} x - x_p \\ y - y_p \\ 0 - f \end{bmatrix} \qquad (1)$$

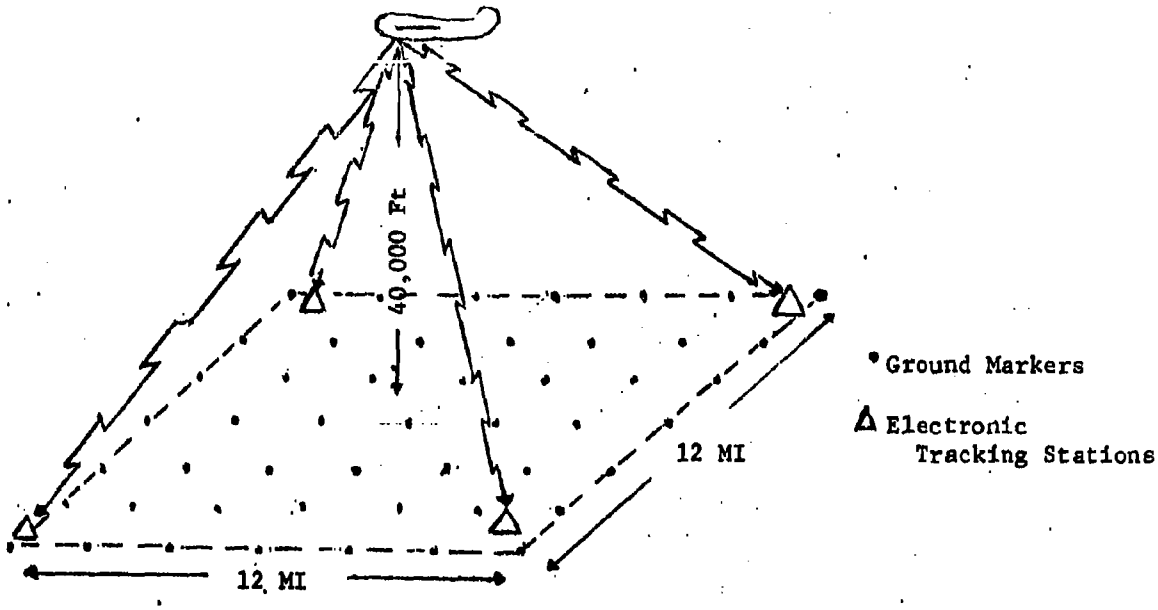

Figure 1. Geometry of Photography and Simultaneous Tracking (SHIRAN). Illustrates a Day Photogrammetric Test Range.

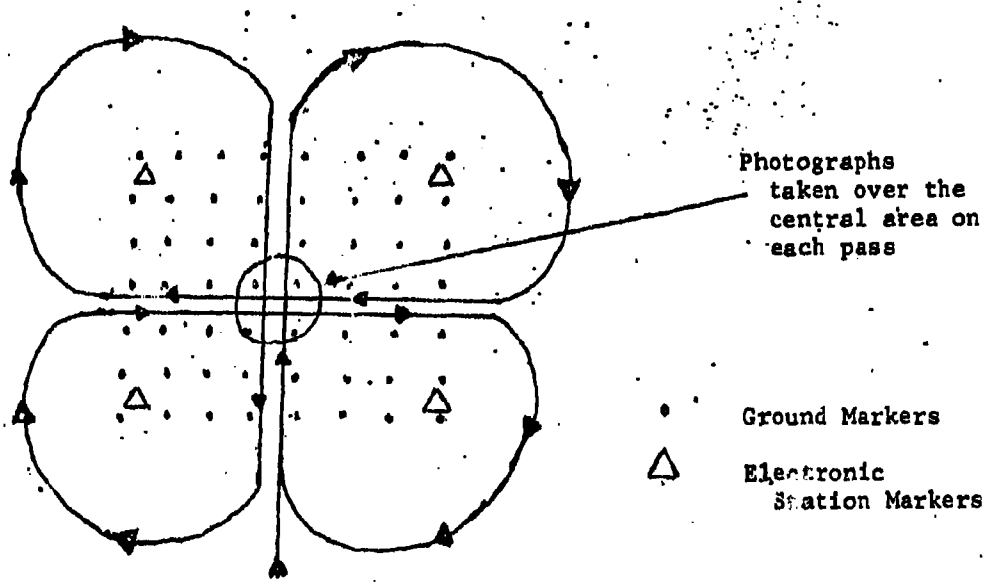

Figure 2. Cloverleaf Flight Path Over the Calibration Range.

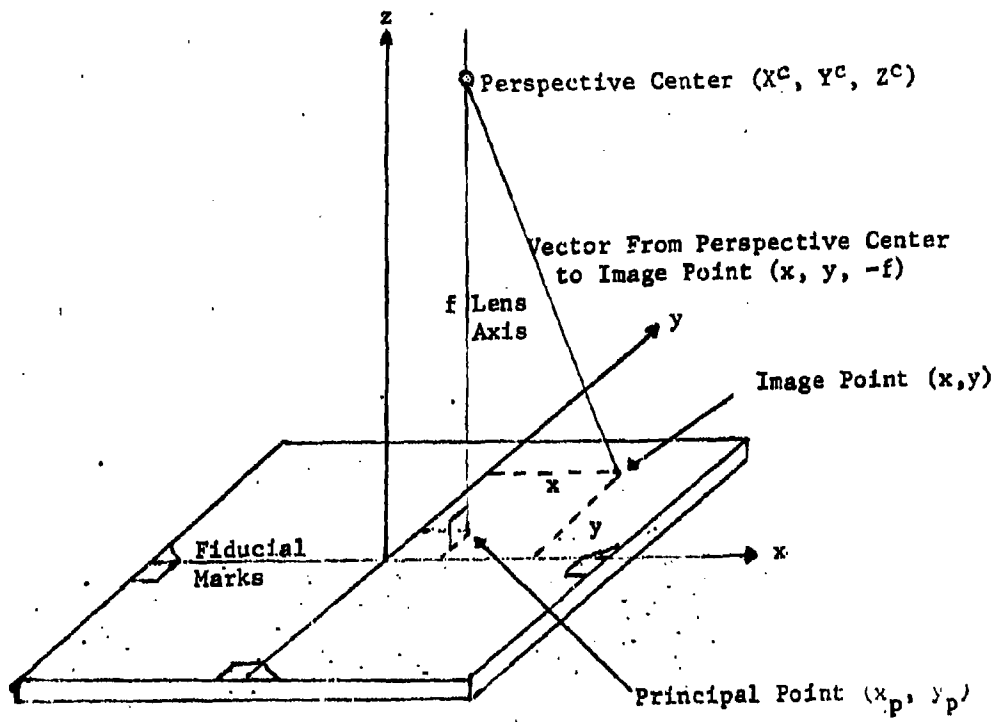

Figure 3. Photographic Coordinate System and Interior Orientation for an Untilted Photograph.

The quantity -f will be called the principal distance and it will be denoted by the letter "c". Brown [3] has shown that the symmetric radial distortion function must be one of two forms depending upon whether or not the principal distance c is carred as an unknown in the calibration process. For our purposes in the SMAC reduction, we wish to carry this parameter as an unknown quantity. Therefore, the distortion model we will adopt is as follows:

$$\delta = K_1 r^3 + K_2 r^5 + K_3 r^7 + \ldots, \qquad (2)$$

where r is the radial distance from the principal point and the K's are the coefficients of distortion. We will carry only three of these coefficients in the SMAC reduction.

DECENTERING DISTORTION

The distortion due to errors in lens centering introduces tangential distortion and asymmetric radial distortion. It should be appreciated that it takes appreciable skill and patience on the part of an optical technician in aligning the lens to suppress this distortion to within the five micron level. A perfectly centered lens means that the centers of curvature of all optical surfaces are collinear, but this goal is never achieved in practice. However, we will use a mathematical model which is successfully being used in the stellar calibration of numerous ballistic cameras and some aerial cameras. As stated previously, the model we will adopt is that one developed by D. Brown [1] as an extended version of Conrady's model. Brown scanned the literature for topics concerning decentered optical systems but found only a few reference books which touched upon this subject. Most of these books and scientific papers published by various authors adopt the aforementioned thin prism model.

The thin prism model describes the phenomenon that there exists on the photographic glass plate an axis passing through the principal point along which the tangential distortion is maximum. At right angles to the axis of maximum tangential distortion is an axis of zero tangential distortion. The tangential distortion along any other axis passing through the principal point is proportional to that along the axis of maximum tangential distortion, the constant of proportionality being the cosine of the angle between the axis in question and the axis of maximum tangential distortion (Brown, Ref. [1]). Analytically, the model is restricted to tangential distortion while ignoring the radial component of decentering distortion. Brown shows that the behavior of radial distortion is precisely the same as that for tangential distortion except for a 90° phase shift. Thus, the axis of maximum radial

distortion corresponds to the axis of zero tangential distortion and vice versa. At phase angles of $\theta - \theta_o = n\frac{\pi}{4}$, the radial and tangential components are of equal magnitude for a specified radial distance (Brown, Ref. [1]). The angle θ is the angle between the positive x-axis and radius vector from the origin to the point whose coordinates are x, y, and the angle θ_o is the angle between the positive x-axis and axis of maximum tangential distortion.

In order to circumvent the problem of finding a suitable approximation for the angle θ_o and the other parameters in the model for decentering distortion, Brown recasts the extended expressions for Conrady's model into the form

$$\Delta_x = [P_1 (r^2 + 2x^2) + 2 P_2 xy] [1 + P_3 r^2 + P_4 r^4 + \ldots] \quad (3)$$

$$\Delta_y = [2 P_1 xy + P_2 (r^2 + 2 y^2)] [1 + P_3 r^2 + P_4 r^4 + \ldots], \quad (4)$$

where

$$P_1 = - J_1 \sin \theta_o$$

$$P_2 = J_1 \cos \theta_o$$

$$P_3 = J_2/J_1$$

$$P_4 = J_3/J_1$$

and

$$r = (x^2 + y^2)^{1/2} .$$

In our experiment we will carry only three parameters of decentering distortion; namely, J_1, J_2 and θ_o. This model should hold for any number of decentered elements for short focal length aerial cameras.

PROJECTIVE EQUATIONS

We come finally to the projective equations which relate corresponding vectors in image space (aerial photograph) with those in object space (terrain). These equations are equivalent to another set of equation known as the colinearity condition equations since they describe the fact that the object point, the image point, and the perspective center in the lens lie on the same straight line. These equations are fundamental to many photogrammetric problems. Figure 4 will enable the reader to gain some insight into the role played by the various parameters in the colinearity equations for tilted photographs. The role of the 3 x 3 matrix [M] shown in Figure 4 is that of an orthogonal transformation from the photographic reference system (image space) to the terrain system (object space) and vice versa. It represents three sequential rotations in 3-space which when multiplied together in the proper order, yields the 3 x 3 orthogonal matrix [M]. The matrix [M] involves three more parameters which must be determined from our experiment. These three angular parameters will be denoted as α, ω, κ and are inherent in the matrix [M] as follows:

$$[M] = \begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix} = \begin{bmatrix} -\cos\kappa\sin\kappa & 0 \\ \sin\kappa\cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin\omega\cos\omega \\ 0 & \cos\omega\sin\omega \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-\cos\alpha\cos\kappa-\sin\kappa\sin\omega\sin\alpha) & (\cos\kappa\sin\alpha-\sin\kappa\sin\omega\cos\alpha) & (\sin\kappa\cos\omega) \\ (\sin\kappa\cos\alpha-\cos\kappa\sin\omega\sin\alpha) & (-\sin\kappa\sin\alpha-\cos\kappa\sin\omega\cos\alpha) & (\cos\kappa\cos\omega) \\ (\cos\omega\sin\alpha) & (\cos\omega\cos\alpha) & (\sin\omega) \end{bmatrix} \quad (5)$$

If we now put together equations (1) and (5) we will have our colinearity equations which relate the coordinates of image points with those of object space. Since the original projective equations include a scale factor, the colinearity equations eliminate this parameter through division of the first two matrix equations by the third thereby yielding the colinearity equations

$$x = x_p + c\,\frac{A(X-X^C) + B(Y-Y^C) + C(Z-Z^C)}{D(X-X^C) + E(Y-Y^C) + F(Z-Z^C)}, \quad (6)$$

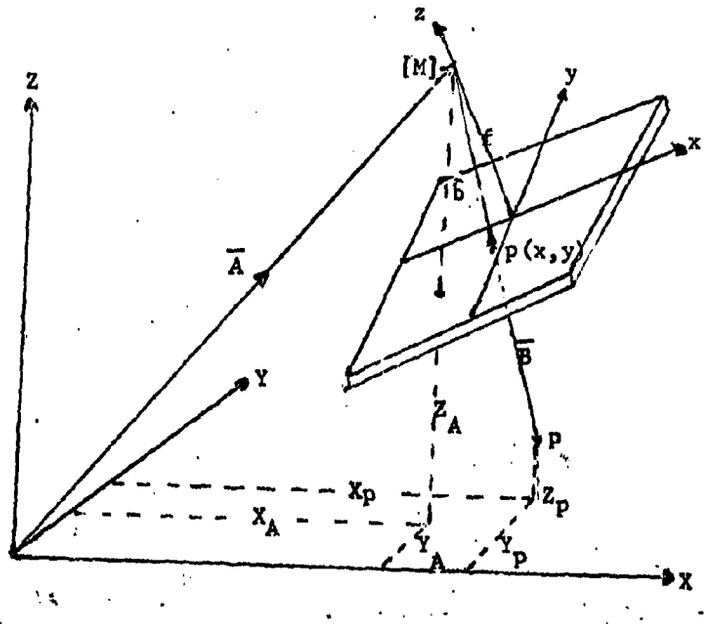

Figure 4. Tilted Photograph and Colinearity Equations

$$y = y_p + c\,\frac{A'(X-X^c) + B'(Y-Y^c) + C'(Z-Z^c)}{D(X-X^c) + E(Y-Y^c) + F(Z-Z^c)} \qquad (7)$$

As stated previously, we have

c = principal distance ($c = -f$)

x,y = coordinates of image points (undistorted)

x_p,y_p = coordinates of principal point

X,Y,Z = coordinates of points in object space

X^c,Y^c,Z^c = coordinates of the perspective center of the lens (aircraft position)

$$[M] = \begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix}$$ = orthogonal orientation matrix defining the rotational relationship between the x, y, z axes of image space and the X, Y, Z axes of object space.

At this point, we have all the necessary models to conduct the experiment for the calibration of an aerial camera in its operating environment.

OBSERVATIONAL EQUATIONS

If we now collect the various models we have developed to describe the undistorted values of the observed quantities, x, y, then the distorted $\bar{x}$, $\bar{y}$ coordinates corrected for symmetric radial distortion and decentering distortion are as follows:

$$(x - x_p) = (1 + \frac{\delta}{r})\,(\bar{x} - x_p) + \Delta_x \qquad (8)$$

$$(y - y_p) = (1 + \frac{\delta}{r})\,(\bar{y} - y_p) + \Delta_y \qquad (9)$$

The terms for radial distortion in equation (8) and (9) are arrived at from the fact that if only radial distortion exists in the lens, then the correction to the measured points with respect to the principal point in the photo coordinate system is

$$\delta x' = \frac{x'}{r}\,\delta \ , \ \delta y' = \frac{y'}{r}\,\delta$$

where

$$x' = (\bar{x} - x_p) \text{ and } y' = (\bar{y} - y_p).$$

Therefore, the undistorted photo images with respect to the principal point are

$$(x-x_p) = x'+\delta x' = x'+\frac{x'}{r}\delta = x'(1+\frac{\delta}{r}) = (\bar{x}-x_p)\ (1+\frac{\delta}{r})$$

and

$$(y-y_p) = y'+\delta y' = y'+\frac{y'}{r}\delta = y'(1+\frac{\delta}{r}) = (\bar{y}-y_p)\ (1+\frac{\delta}{r}),$$

which are the required radial distortion terms in (8) and (9).

The following unknowns are implicit in the terms δ, Δ_x and Δ_y:

K_1, K_2, K_3 (coefficients of radial distortion)

J_1, J_2, θ_o (coefficients and phase angle of decentering distortion)

x_p, y_p (implicit in δ and r where

$$r = [(\bar{x} - x_p)^2 + (\bar{y} - y_p)^2]^{1/2}$$

The substitution of equations (8) and (9) into the colinearity equations (6) and (7) introduces the coefficients of radial and decentering distortion into the observational equation. Therefore, we may express

the colinearity equations (6) and (7) for the i^{th} photograph and the j^{th} measured point as

$$\left(1 + \frac{\delta_{ij}}{r_{ij}}\right)(\bar{x}_{ij} - x_p) + \Delta_{x_{ij}} = c\,\frac{D_{ij}}{F_{ij}} \tag{10}$$

and

$$\left(1 + \frac{\delta_{ij}}{r_{ij}}\right)(\bar{y}_{ij} - y_p) + \Delta_{y_{ij}} = c\,\frac{E_{ij}}{F_{ij}} \tag{11}$$

where

$$\begin{bmatrix} D_{ij} \\ E_{ij} \\ F_{ij} \end{bmatrix} = \begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix} \begin{bmatrix} X_{ij} - X_i^c \\ Y_{ij} - Y_i^c \\ Z_{ij} - Z_i^c \end{bmatrix} \tag{12}$$

In these equations

$\bar{x}_{ij}$, $\bar{y}_{ij}$ are obtained by direct measurement,

X_{ij}, Y_{ij}, Z_{ij} are taken from precisely surveyed values, j^{th} point on the i^{th} photo,

X_i^c, Y_i^c, Z_i^c are the camera station coordinates for the i^{th} photograph

A,B,C,A',B',C',D,E,F are the 3 x 3 matrix, $[M]_i$, elements of the i^{th} photo

α_i, ω_i, K_i are the rotational elements of exterior orientation and they are explicit in the matrix $[M]_i$ for the i^{th} photo,

x_p, y_p, c are the elements of interior orientation (x_p, y_p are also implicit in δ_{ij} and r_{ij}).

Finally, we see that each pair of film measurements gives us to two independent equations involving fifteen parameters, nine of which will be common to all n photos and six of which are considered to be changing from photo to photo. Thus, observational equations for all n photos and all m measured images of precisely surveyed control points constitute a 2mn system of equations in 6n + 9 unknowns.

THE LINEARIZED OBSERVATIONAL EQUATIONS

The pair of equations, (10) and (11), may be considered to be of the functional form:

$$f_1(\bar{x}_{ij},\bar{y}_{ij},X_j,Y_j,Z_j,X_i^c,Y_i^c,Z_i^c,\alpha_i,\omega_i,\kappa_i,K_1,K_2,K_3,J_1,J_2,\theta_o,x_p,y_p,c) = 0 \tag{13}$$

$$f_2(\bar{x}_{ij},\bar{y}_{ij},X_j,Y_j,Z_j,X_i^c,Y_i^c,Z_i^c,\alpha_i,\omega_i,\kappa_i,K_1,K_2,K_3,J_1,J_2,\theta_o,x_p,y_p,c) = 0 \tag{14}$$

In equations (13) and (14) the measured film coordinates are subject to random errors. At present, we will treat the parameters as completely unknown quantities. Later, we will develop observation equations generated by considering these parameters as observed quantities. This means that we will develop a weight constraining procedure based on how well these parameters are known. Because equations (13) and (14) are nonlinear, we will linearize them using a Taylor's series expansion keeping only the zero and first order terms. Therefore, we write

$$\begin{aligned} \bar{x}_{ij} &= \bar{x}^o_{ij} + v_{1ij} \\ \bar{y}_{ij} &= \bar{y}^o_{ij} + v_{2ij} \end{aligned} \tag{15}$$

where $\bar{x}^o_{ij}$, $\bar{y}^o_{ij}$ denote the measured film coordinates for the j^{th} measured image on the i^{th} photograph, and v_{1ij}, v_{2ij} are the corresponding measuring residuals. Also, we set

$$\alpha_i = \alpha_i^{oo} + \delta\alpha_i, \quad \omega_i = \omega_i^{oo} + \delta\omega_i, \quad \kappa_i = \kappa_i^{oo} + \delta\kappa_i, \tag{16}$$

and

$$X_i^c = (X_i^c)^{oo} + \delta X_i^c, \quad Y_i^c = (Y_i^c)^{oo} + \delta Y_i^c, \quad Z_i^c = (Z_i^c)^{oo} + \delta Z_i^c, \tag{17}$$

in which the superscript "oo" are arbitrary approximations and the δ's are the unknown corrections to the approximations. Further, we write

$$\begin{aligned} &K_1 = K_1^{oo} + \delta K_1, \quad K_2 = K_2^{oo} + \delta K_2, \quad K_3 = K_3^{oo} + \delta K_3, \\ &J_1 = J_1^{oo} + \delta J_1, \quad J_2 = J_2^{oo} + \delta J_2, \quad \theta_o = \theta_o^{bo} + \delta\theta_o, \end{aligned} \tag{18}$$

and

$$x_p = x_p^{oo} + \delta x_p, \quad y_p = y_p^{oo} + \delta y_p, \quad c = c^{oo} + \delta c,$$

where the superscript "oo" and δ's have the same meaning as before. The substitution of (15), (16), (17), (18) into (13) and (14) yields the following form:

$$\begin{aligned} f_{1ij} &= f_1(\bar{x}^o_{ij}+v_{1ij}, \bar{y}^o_{ij}+v_{1ij}, \alpha_1^{oo}+\delta\alpha_i, \ldots, c^{oo}+\delta c) = 0, \\ f_{2ij} &= f_2(\bar{x}^o_{ij}+v_{2ij}, \bar{y}^o_{ij}+v_{2ij}, \alpha_1^{oo}+\delta\alpha_i, \ldots, c^{oo}+\delta c) = 0 . \end{aligned} \tag{19}$$

As stated previously, we will linearize these equations via Taylor's series keeping only the zero and first order terms. Therefore, we get

$$v_{1ij}+\dot{b}^1_{1ij}\delta K_1+ \ldots +\dot{b}^9_{1ij}\delta c+\ddot{b}^1_{1ij}\delta\alpha_i+ \ldots +\ddot{b}^6_{1ij}\delta Z^c_i = \varepsilon_{1ij}$$

$$v_{2ij}+\dot{b}^1_{2ij}\delta K_1+ \ldots +\dot{b}^9_{2ij}\delta c+\ddot{b}^1_{2ij}\delta\alpha_i+ \ldots +\ddot{b}^6_{2ij}\delta Z^c_i = \varepsilon_{2ij} \qquad (20)$$

where

$$\varepsilon_{1ij} = -f_1(\overline{x}^o_{ij}, \overline{y}^o_{ij}, K_1^{oo}, \ldots, c^{oo}, \alpha_i^{oo}, \ldots, (Z^c_i)^{oo})$$

$$\varepsilon_{2ij} = -f_2(\overline{x}^o_{ij}, \overline{y}^o_{ij}, K_1^{oo}, \ldots, c^{oo}, \alpha_i^{oo}, \ldots, (Z^c_i)^{oo}), \qquad (21)$$

and

$$\dot{b}^1_{1ij} = \frac{\partial f_{1ij}}{\partial K_1}, \ \dot{b}^2_{1ij} = \frac{\partial f_{1ij}}{\partial K_2}, \ \ldots, \ \dot{b}^9_{1ij} = \frac{\partial f_{1ij}}{\partial c}$$

$$\ddot{b}^1_{1ij} = \frac{\partial f_{1ij}}{\partial \alpha_i}, \ \ddot{b}^2_{1ij} = \frac{\partial f_{1ij}}{\partial \omega_i}, \ \ldots, \ \ddot{b}^6_{1ij} = \frac{\partial f_{1ij}}{\partial Z^c_i} \qquad (22)$$

$$\dot{b}^1_{2ij} = \frac{\partial f_{2ij}}{\partial K_1}, \ \dot{b}^2_{2ij} = \frac{\partial f_{2ij}}{\partial K_2}, \ \ldots, \ \dot{b}^9_{2ij} = \frac{\partial f_{2ij}}{\partial c}$$

$$\ddot{b}^1_{2ij} = \frac{\partial f_{2ij}}{\partial \alpha_i}, \ \ddot{b}^2_{2ij} = \frac{\partial f_{2ij}}{\partial \omega_i}, \ \ldots, \ \ddot{b}^6_{2ij} = \frac{\partial f_{2ij}}{\partial Z^c_i}. \qquad (23)$$

The partial derivatives in (22) and (23) are evaluated at the approximation K_1^{oo}, K_2^{oo}, etc. and ε_{1ij}, ε_{2ij} in (21) result from evaluating

equations (13) and (14) at the measured point using these approximations. We will not derive the formulas for the partial derivatives since they are straightforward and they do not add to the purpose of this report.

The linearized equations for the j^{th} measured image on the i^{th} photo are put in the matrix form

$$v_{ij} + \dot{B}_{ij}\,\dot{\delta} + \ddot{B}_{ij}\,\ddot{\delta}_i = \epsilon_{ij} \tag{24}$$

where

$$\underset{(2,1)}{v_{ij}} = \begin{bmatrix} v_{1ij} \\ v_{2ij} \end{bmatrix}, \quad \underset{(2,9)}{\dot{B}_{ij}} = \begin{bmatrix} \dot{b}^1_{1ij} & \dot{b}^2_{1ij} & \cdots & \dot{b}^9_{1ij} \\ \dot{b}^1_{2ij} & \dot{b}^2_{2ij} & \cdots & \dot{b}^9_{2ij} \end{bmatrix}, \quad \underset{(9,1)}{\dot{\delta}} = \begin{bmatrix} \dot{\delta}K_1 \\ \dot{\delta}K_2 \\ \vdots \\ \dot{\delta}c \end{bmatrix}$$

$$\underset{(2,6)}{\ddot{B}_{ij}} = \begin{bmatrix} \ddot{b}^1_{1ij} & \ddot{b}^2_{1ij} & \cdots & \ddot{b}^6_{1ij} \\ \ddot{b}^1_{2ij} & \ddot{b}^2_{2ij} & \cdots & \ddot{b}^6_{2ij} \end{bmatrix}, \quad \underset{(6,1)}{\ddot{\delta}_i} = \begin{bmatrix} \ddot{\delta}\alpha_i \\ \ddot{\delta}\omega_i \\ \vdots \\ \ddot{\delta}Z^c_i \end{bmatrix}, \quad \underset{(2,1)}{\epsilon_{ij}} = \begin{bmatrix} \epsilon_{1ij} \\ \epsilon_{2ij} \end{bmatrix} \tag{25}$$

The matrix equations (24) and (25) represent the smallest matrix units in the entire development; that is, they involve information from only the j^{th} measured point on the i^{th} photograph. Remembering that the measurements on all n photographs contribute to the solution of the nine stationary parameters and that all m measured images on the i^{th} photo contribute to the solution of the six nonstationary parameters per photo, we can express the linearized equations for all n photographs as

$$v_j + \dot{B}_j\,\dot{\delta} + \ddot{B}_j\,\ddot{\delta} = \epsilon_j$$

where

$$\underset{(2n,1)}{v_j} = \begin{bmatrix} v_{1j} \\ v_{2j} \\ \cdot \\ \cdot \\ \cdot \\ v_{nj} \end{bmatrix}, \quad \underset{(2n,9)}{\dot{B}_j} = \begin{bmatrix} \dot{B}_{1j} \\ \dot{B}_{2j} \\ \cdot \\ \cdot \\ \cdot \\ \dot{B}_{nj} \end{bmatrix}, \quad \underset{(2n,6)}{\ddot{B}_j} = \begin{bmatrix} \ddot{B}_{1j} \\ \ddot{B}_{2j} \\ \cdot \\ \cdot \\ \cdot \\ \ddot{B}_{nj} \end{bmatrix}, \quad \underset{(2n,1)}{\varepsilon_j} = \begin{bmatrix} \varepsilon_{1j} \\ \varepsilon_{2j} \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{nj} \end{bmatrix} \tag{27}$$

If in the next step we collect all equations generated by all measured images, we have the matrix equations

$$v + \dot{B}\dot{\delta} + \ddot{B}\delta = \varepsilon \tag{28}$$

in which

$$\underset{(2mn,1)}{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_m \end{bmatrix}; \quad \underset{(2mn,9)}{\dot{B}} = \begin{bmatrix} \dot{B}_1 \\ \dot{B}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{B}_m \end{bmatrix}, \quad \underset{(2mn,6n)}{\ddot{B}} = \begin{bmatrix} \ddot{B}_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \ddot{B}_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \ddot{B}_m \end{bmatrix} \tag{29}$$

$$\underset{(6n,1)}{\dot{\delta}} = \begin{bmatrix} \delta\alpha_1 \\ \delta\omega_1 \\ \cdot \\ \cdot \\ \cdot \\ \delta Z_n^c \end{bmatrix} \quad \underset{(2mn,1)}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_m \end{bmatrix}$$

It is in the expanded matrix equations (27) and (29) that we see the difference between the analytical aerial triangulation and camera calibration problems.

In order to develop the rigor necessary for the complete calibration effort we must consider the possibility of weighting the observed quantities $\bar{x}^o$, $\bar{y}^o$.

WEIGHT MATRICES

We shall denote the covariance matrix of the film coordinates $\bar{x}^o_{ij}$, $\bar{y}^o_{ij}$ by

$$\underset{(2,2)}{\Lambda_{ij}} = \begin{bmatrix} \sigma^2_{\bar{x}^o_{ij}} & \sigma_{\bar{x}^o_{ij}\bar{y}^o_{ij}} \\ \sigma_{\bar{x}^o_{ij}\bar{y}^o_{ij}} & \sigma^2_{\bar{y}^o_{ij}} \end{bmatrix} \tag{30}$$

and we shall denote the weight matrix of $\bar{x}^o_{ij}$, $\bar{y}^o_{ij}$ to be

$$\underset{(2,2)}{W_{ij}} = \Lambda^{-1}_{ij} . \tag{31}$$

Thus we allow the film coordinates for a given point to be correlated. Let it suffice to say that it is possible to get correlation between $\bar{x}^o_{ij}$, $\bar{y}^o_{ij}$ by considering the calibration of the instrument with which the film measurements are made and it is also possible that correlation may arise from calibrating cameras which do not have flat fields. In any case, by employing the full covariance matrix Λ_{ij}, we properly propagate and preserve the informational content of the original observations throughout the camera calibration effort.

We shall assume independence of film measurements of different images. Therefore, we may express the covariance and weight matrices for the j^{th} point seen on all n photos as

$$\underset{(2n,2n)}{\Lambda_j} = \begin{bmatrix} \Lambda_{1j} & 0 & \cdots & 0 \\ 0 & \Lambda_{2j} & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \Lambda_{nj} \end{bmatrix}, \tag{32}$$

and

$$\underset{(2n,2n)}{W_j} = \begin{bmatrix} W_{1j} & 0 & \cdots & 0 \\ 0 & W_{2j} & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & W_{nj} \end{bmatrix}. \tag{33}$$

If, as before, we collect all equations generated by all measured images, the corresponding covariance and weight matrices are

$$\underset{(2mn,2mn)}{\Lambda} = \begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \Lambda_m \end{bmatrix}, \tag{34}$$

and

$$\underset{(2mn,2mn)}{W} = \begin{bmatrix} W_1 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & W_m \end{bmatrix}. \tag{35}$$

We are now in a position to state that equations (28), (34), and (35) contain all the information provided by our original observation equations (10) and (11). However, we have not made provision for overcoming a basic flaw in the camera calibration experiment; namely, as stated previously, we cannot recover the elements of interior orientation of the camera (x_p, y_p, c) since these elements are projectively equivalent to changes in the coordinates of the exposure station. Thus, this is why we have stated that an external tracking system is necessary for the calibration experiment. Therefore, we must now develop additional observation equations which will allow weight constraints to be applied to X^c, Y^c, Z^c according to how well the tracking system triangulates a camera station. Since we must develop at least three additional observation equations, we will develop complete flexibility and write observation equations for all parameters included in the adjustment. This means that we will be able to incorporate into the adjustment any a priori information concerning any, or all, of the parameters involved in the calibration experiment. The a priori information may come, for example, from a previous calibration.

OBSERVATION EQUATIONS GENERATED BY ELEMENTS OF ORIENTATION, RADIAL AND DECENTERING DISTORTION

In order to develop the flexibility of constraining the unknown parameters to within prescribed limits by weighting, we must develop observation equations for all parameters involved in the calibration problem. We shall assume that independent observations are available for all parameters. Thus, using previous notations for observed quantities, we write for the elements of interior orientation, radial and decentering distortion parameters

$$x_p = x_p^o + v_{x_p}, \quad y_p = y_p^o + v_{y_p}, \quad c = c^o + v_c$$

$$K_1 = K_1^o + v_{K_1}, \quad K_2 = K_2^o + v_{K_2}, \quad K_3 = K^3 + v_{K_3} \tag{36}$$

$$J_1 = J_1^o + v_{J_1}, \quad J_2 = J_2^o + v_{J_2}, \quad \theta_o = \theta_o^o + v_{\theta_o},$$

where the v's are observational residuals. If we eliminate the adjusted observations from equations (18) and (36), we arrive at the observation equations

$$
\begin{aligned}
v_{x_p} - \delta x_p &= x_p^{oo} - x_p^{o} = \varepsilon_{x_p} \\
v_{y_p} - \delta y_p &= y_p^{oo} - y_p^{o} = \varepsilon_{y_p} \\
v_c - \delta c &= c^{oo} - c^{o} = \varepsilon_c \\
&\vdots \\
v_{\theta_o} - \delta\theta_o &= \theta_o^{oo} - \theta_o^{o} = \varepsilon_{\theta_o} .
\end{aligned}
\tag{37}
$$

The meanings of the superscripts are the same as before, but now we have additional observed quantities and their residuals. As stated previously, we must be able to enter a priori information concerning the position of the camera station. If we proceed as we have done thus far, we will write additional observation equations for the elements of exterior orientation for the i^{th} photo as follows

$$
\begin{aligned}
&\alpha_i = \alpha_i^o + v_{\alpha_i},\ \omega_i = \omega_i^o + v_{\omega_i},\ \kappa_i = \kappa_i^o + v_{\kappa_i} , \\
&X_i^c = (X_i^c)^o + v_{X_i^c},\ Y_i^c = (Y_i^c)^o + v_{Y_i^c},\ Z_i^c = (Z_i^c)^o + v_{Z_i^c} .
\end{aligned}
\tag{38}
$$

Now that we have written observation equations for all parameters involved in the experiment and have eliminated the adjusted parameters from (16) and (17), we can express equations (37) and (38) in matrix form as

$$
\dot{v} - \dot{\delta} = \dot{\varepsilon} \tag{39}
$$

and

$$
\ddot{v}_i - \ddot{\delta}_i = \ddot{\varepsilon}_i , \tag{40}
$$

respectively, where (40) is developed further for all n photographs as follows:

$$\ddot{v} - \ddot{\delta} = \ddot{\varepsilon} \tag{41}$$

where

$$\underset{(6n,1)}{\ddot{v}} = \begin{bmatrix} \ddot{v}_{\alpha_1} \\ \ddot{v}_{\omega_1} \\ \vdots \\ \ddot{v}_{Z^c_n} \end{bmatrix}, \quad \underset{(6n,1)}{\ddot{\delta}} = \begin{bmatrix} \ddot{\delta}\alpha_1 \\ \ddot{\delta}\omega_1 \\ \vdots \\ \ddot{\delta}_{Z^c_n} \end{bmatrix}, \quad \underset{(6n,1)}{\ddot{\varepsilon}} = \begin{bmatrix} \ddot{\varepsilon}_1 \\ \ddot{\varepsilon}_2 \\ \vdots \\ \ddot{\varepsilon}_n \end{bmatrix}. \tag{42}$$

Matrix equation (39), which involves observations on the stationary parameters, is expanded as follows:

$$\underset{(9,1)}{\dot{v}} = \begin{bmatrix} \dot{v}_{x_p} \\ \dot{v}_{y_p} \\ \vdots \\ \dot{v}_{\theta_o} \end{bmatrix}, \quad \underset{(9,1)}{\dot{\delta}} = \begin{bmatrix} \dot{\delta}_{x_p} \\ \dot{\delta}_{y_p} \\ \vdots \\ \dot{\delta}_{\theta_o} \end{bmatrix}, \quad \underset{(9,1)}{\dot{\varepsilon}} = \begin{bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \vdots \\ \dot{\varepsilon}_9 \end{bmatrix}. \tag{43}$$

If we assume that observations on the stationary and nonstationary parameters are independent of each other, the covariance and weight matrices associated with the observational vectors (39) and (41) are

$$\underset{(9,9)}{\dot{\Lambda}} = \begin{bmatrix} \sigma^2_{x_{o_p}} & 0 & \cdots & 0 \\ 0 & \sigma^2_{y_{o_p}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma^2_{\theta_o} \end{bmatrix}, \quad \underset{(9,9)}{\dot{W}} = \underset{(9,9)}{\dot{\Lambda}^{-1}} \tag{44}$$

for the stationary parameters and

$$\underset{(6n,6n)}{\ddot{\Lambda}} = \begin{bmatrix} \ddot{\Lambda}_1 & & \cdots & 0 \\ 0 & \ddot{\Lambda}_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \ddot{\Lambda}_n \end{bmatrix}, \quad \underset{(6n,6n)}{\ddot{W}} = \begin{bmatrix} \ddot{W}_1 & 0 & \cdots & 0 \\ 0 & \ddot{W}_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \ddot{W}_n \end{bmatrix} \tag{45}$$

for the nonstationary parameters. In (45), we let $\ddot{\Lambda}_i$ denote the covariance matrix of the observations of the elements of exterior orientation for the i^{th} photograph and let $\ddot{W}_i = \ddot{\Lambda}_i^{-1}$. It is not necessary for these covariance and weight matrices to be diagonal. They can be completely filled without creating undue strain on the computations.

At this stage, we have three matrix observation equations arising from

1. Measured film coordinates,
2. Stationary parameters,
3. Nonstationary parameters.

We are now in a position to form normal equations.

NORMAL EQUATIONS

Writing the matrix observation equations as follows:

$$\begin{aligned} v + \dot{B}\dot{\delta} + \ddot{B}\ddot{\delta} &= \varepsilon \\ \dot{v} - \dot{\delta} &= \dot{\varepsilon} \\ \ddot{v} - \ddot{\delta} &= \ddot{\varepsilon} \end{aligned} \tag{46}$$

we can merge these matrix observations equations into the single matrix equation

$$\begin{bmatrix} v \\ \dot{v} \\ \ddot{v} \end{bmatrix} + \begin{bmatrix} \dot{B} & \ddot{B} \\ -I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \varepsilon \\ \dot{\varepsilon} \\ \ddot{\varepsilon} \end{bmatrix} \tag{47}$$

which is reduced to

$$\bar{v} + \bar{B}\,\bar{\delta} = \bar{\varepsilon}. \tag{48}$$

The dimension of $\bar{v}$ is $(2mn + 9 + 6n, 1)$; that of $\bar{B}$ is $(2mn + 9 + 6n, 6n + 9)$, and $\bar{\delta}$ is $(6n + 9, 1)$. The dimension of the discrepancy vector $\bar{\varepsilon}$ is the same as $\bar{v}$. In merging the covariance and weight matrices of the three matrix observation equations, we have

$$\underset{(\bar{n},\bar{n})}{\bar{\Lambda}} = \begin{bmatrix} \underset{(2mn,2mn)}{\Lambda} & 0 & 0 \\ 0 & \underset{(9,9)}{\dot{\Lambda}} & 0 \\ 0 & 0 & \underset{(6n,6n)}{\ddot{\Lambda}} \end{bmatrix}, \quad \underset{(\bar{n},\bar{n})}{\bar{W}} = \underset{(\bar{n},\bar{n})}{\bar{\Lambda}^{-1}} = \begin{bmatrix} \underset{(2mn,2mn)}{W} & 0 & 0 \\ 0 & \underset{(9,9)}{\dot{W}} & 0 \\ 0 & 0 & \underset{(6n,6n)}{\ddot{W}} \end{bmatrix} \tag{49}$$

where

$$\bar{n} = (2mn + 9 + 6n).$$

The coefficient matrix and constant column of the normal equations can be formed by matrix multiplication as follows

$$(\bar{B}^T \bar{W} \bar{B})\,\bar{\delta} = \bar{B}^T \bar{W}\,\bar{\varepsilon}. \tag{50}$$

Brown [4] shows that the solution of the vectors $\bar{v}$ and $\bar{\delta}$ simultaneously leads to the minimization of the quadratic form of the residuals

$$s = \bar{v}^T \bar{W} \bar{v} \ . \tag{51}$$

If we use matrix equations (47) and (50) and realize that the structure of the $\dot{B}$ matrix is a column matrix of matrices, we can write the normal equation system as

$$\begin{bmatrix} \dot{B}^T & -I & 0 \\ \ddot{B}^T & 0 & -I \end{bmatrix} \begin{bmatrix} W & 0 & 0 \\ 0 & \dot{W} & 0 \\ 0 & 0 & \ddot{W} \end{bmatrix} \begin{bmatrix} \dot{B} & \ddot{B} \\ -I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} \dot{B}^T & -I & 0 \\ \ddot{B}^T & 0 & -I \end{bmatrix} \begin{bmatrix} W & 0 & 0 \\ 0 & \dot{W} & 0 \\ 0 & 0 & \ddot{W} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \dot{\varepsilon} \\ \ddot{\varepsilon} \end{bmatrix} \tag{52}$$

which, after multiplication, becomes

$$\left[\begin{array}{c|c} \left(\sum_{i=1}^{n} \dot{N}_i\right) + \dot{W} & \bar{N} \\ \hline \bar{N}^T & \ddot{N} + \ddot{W} \end{array}\right] \begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \left[\begin{array}{c} \left(\sum_{i=1}^{n} \dot{c}\right) - \dot{W}\dot{\varepsilon} \\ \hline \ddot{c} - \ddot{W}\ddot{\varepsilon} \end{array}\right] , \tag{53}$$

The individual matrix components of the normal equations (53) and their dimensions are

$$\underset{(9,9)}{\left(\sum_{i=1}^{n} \dot{N}_i\right)} = \underset{(9,2mn)}{\dot{B}^T} \quad \underset{(2mn,2mn)}{W} \quad \underset{(2mn,9)}{\dot{B}} ,$$

$$\underset{(9,1)}{\left(\sum_{i=1}^{n} \dot{c}_i\right)} = \underset{(9,2mn)}{\dot{B}^T} \quad \underset{(2mn,2mn)}{W} \quad \underset{(2mn,1)}{\varepsilon} ,$$

$$\underset{(6n,6n)}{\ddot{N}} = \underset{(6n,2mn)}{\ddot{B}^T} \quad \underset{(2mn,2mn)}{W} \quad \underset{(2mn,1)}{\ddot{B}} , \tag{54}$$

$$\underset{(6n,1)}{\ddot{c}} = \underset{(6n,2mn)}{\ddot{B}^T} \quad \underset{(2mn,2mn)}{W} \quad \underset{(2mn,1)}{\varepsilon} ,$$

$$\underset{(9,6n)}{\bar{N}} = \underset{(9,2mn)}{\dot{B}^T} \quad \underset{(2mn,2mn)}{W} \quad \underset{(2mn,6n)}{\ddot{B}} ,$$

where, as stated previously, the $\dot{N}$ and $\dot{c}$ portions of the normal equations are quickly and simply formed by virtue of the structure of the $\dot{B}$ matrix. The general normal equations for the simultaneous adjustment of all n photographs are diagrammatically given as follows:

$\longleftarrow 9 + 6n \text{ cols} \longrightarrow$

$\leftarrow 9 \text{ cols} \rightarrow \quad \leftarrow 6n \text{ cols} \rightarrow \qquad\qquad$ Constant cols

$$\begin{array}{c} 9 \text{ rows} \\ 6n \text{ rows} \end{array} \begin{bmatrix} \left(\sum_{i=1}^{n} \dot{N}_i\right) + \dot{W} & \bar{N}_1 & \bar{N}_2 & \cdots & \bar{N}_n \\ \bar{N}_1^T & \ddot{N}_1 + \ddot{W}_1 & 0 & \cdots & 0 \\ \bar{N}_2^T & 0 & \ddot{N}_2 + \ddot{W}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \bar{N}_n^T & 0 & 0 & \cdots & \ddot{N}_n + \ddot{W}_n \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \vdots \\ \ddot{\delta}_n \end{bmatrix} = \begin{bmatrix} \left(\sum_{i=1}^{n} \dot{c}_i\right) - \dot{W}\dot{\varepsilon} \\ \ddot{c}_1 - W_1\ddot{\varepsilon}_1 \\ \ddot{c}_2 - \ddot{W}_2\ddot{\varepsilon}_2 \\ \vdots \\ \ddot{c}_n - \ddot{W}_n\ddot{\varepsilon}_n \end{bmatrix}$$

(55)

This is the first order regression scheme referred to earlier in the report. This type of structured normal equations were successfully solved by Brown [5]. Practically, a solution is possible no matter how large n may be and it is found that the computations increase only linearly with n. In our camera calibration experiment, the largest matrix to be handled is of order 9. It is not our purpose to give full details of the computation algorithm since these details are available in many of Brown's reports, for example, reference [5]. Let it suffice to say that in practice we will handle approximately 20 photographs selected from four passes over a test range, and that we hope to obtain

at least 40 images on each photo. These images will be common to all 20 photographs. Therefore, we will have 800 measured images contributing directly to the calibration of the stationary parameters whereas 40 measured images will contribute to the determination of each set of nonstationary parameters. The resulting normal equation system will be of order 129 generated from 1600 observation equations, if only the equations from the linearization are considered.

CONCLUSIONS

We consider SMAC to be a total system calibration since the calibrated distortion parameters will include effects of the camera window and the shockwave of the airstream upon the window. Also, since the systemmatic errors tend to be independent from one frame to the next, the estimates of the stationary parameters will not be unduly influenced by these type errors. We feel that this represents the actual conditions in practice and therefore, SMAC is a significantly superior process as compared to one which might use single frames each employing 10 times as many measured images as on a SMAC frame.

A by-product of the SMAC reduction is the covariance matrix of the adjusted parameters. This permits evaluation of error bounds of the calibrated functions of radial and decentering distortion along with those of the parameters of exterior orientation.

A quantitative calibration of existing decentering distortion in a lens is especially important since the practical limit of the length of an analytical, aerial control extension project is heavily dependent upon the elimination of the effects of decentering distortion.

In conclusion, then, we feel that a SMAC approach to the problem of calibrating aerial cameras in their operating environment is both feasible and practical, and that its potential value in other modes of operation, such as stellar calibrations of aerial and ballistic cameras, is yet to be realized. We hope to have the results of a SMAC calibration in the near future; that is, both a stellar SMAC and an aerial SMAC of at least two aerial mapping cameras so that we will have a comparison of three types of calibration (laboratory, stellar, aerial) of the same cameras. These comparisons should prove the effectiveness of the aerial SMAC approach.

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DESIGN AND ANALYSIS OF A STATISTICAL EXPERIMENT ON HIGH-VOLTAGE BREAKDOWN IN VACUUM*

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The problem of high-voltage breakdown in vacuum has been studied for more than forty years. From these studies many conflicting theories have evolved that still do not reliably define a breakdown criterion nor explain the mechanisms involved in the process. High-voltage breakdown in vacuum has received renewed interest in recent years because of the demands for superpower radar system components, ion thrusters for space propulsion, and high-energy particle accelerators.

The study of the factors that effect a high-voltage breakdown in vacuum is being performed at this laboratory using statistically designed experiments. Initially, the sixteen factors shown in Table I were defined as probable contributors to the breakdown process:

TABLE I - FACTORS EFFECTING BREAKDOWN

Inflexible Factors	Flexible Factors
1. Cathode Material	12. Residual Gas Pressure
2. Anode Material	13. Energy of Supply
3. Cathode Finish	14. Contaminant
4. Anode Finish	15. Magnetic Field
5. Cathode Geometry	16. Electrode Spacing
6. Anode Geometry	
7. Vehicle Bakeout	
8. Envelope Material	

The objective of this program is to analyze the significance of each of these factors as well as their interactions.

The first designed experiment was carried out using seven of the inflexible factors, each at two levels, in a 2^{7-3} plan (Table II) derived from Table M of Davies' Design and Analysis of Industrial Experiments:

*Sponsored by Advanced Research Projects Agency under US Army Electronics Command Contract DA28-043 AMC-00394(E) ARPA Order No. 517 PROJECT DEFENDER

TABLE II - FACTOR LEVELS FOR INVESTIGATION

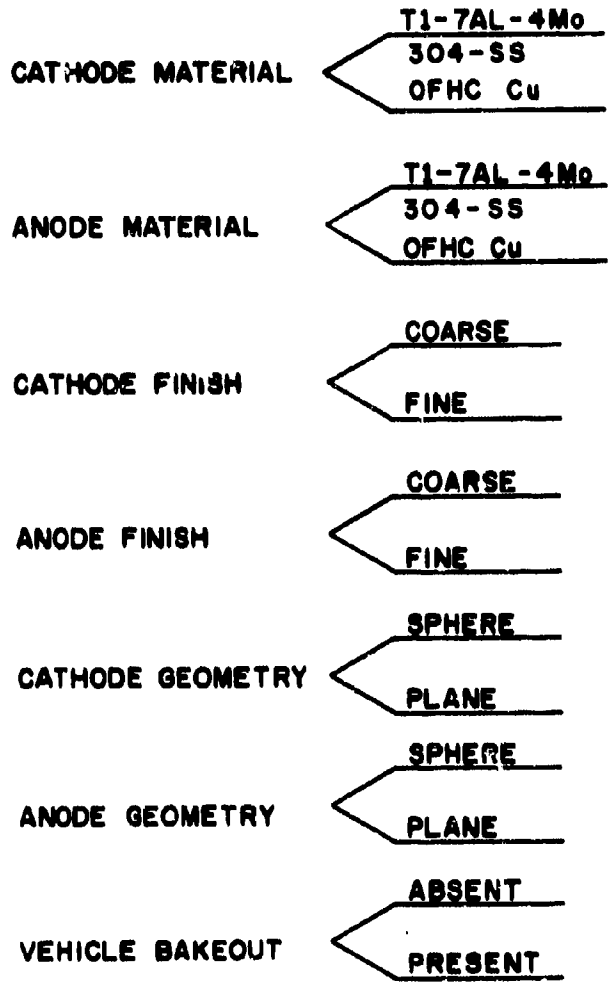

Table III shows the levels of each factor for each of the thirty-two treatments. The minus sign in each treatment means that the factor is either at the low level or absent from the treatment; the plus sign means that the factor is at the high level or present in the treatment. The set is orthogonal; each level of any factor is tested equally against each of the other factor level combinations:

TABLE III - 2^{7-2} PLAN

TREATMENT	A	B	C	D	E	F	G
(1)	-	-	-	-	-	-	-
a c f	+	-	+	-	-	+	-
b c f	-	+	+	-	-	+	-
a b	+	+	-	-	-	-	-
d f	-	-	-	+	-	+	-
a c d	+	-	+	+	-	-	-
b c d	-	+	+	+	-	-	-
a b d f	+	+	-	+	-	+	-
c e	-	-	+	-	+	-	-
a e f	+	-	-	-	+	+	-
b e f	-	+	-	-	+	+	-
a b c e	+	+	+	-	+	-	-
c d e f	-	-	+	+	+	+	-
a d e	+	-	-	+	+	-	-
b d e	-	+	-	+	+	-	-
a b c d e f	+	+	+	+	+	+	-
f g	-	-	-	-	-	+	+
a c g	+	-	+	-	-	-	+
b c g	-	+	+	-	-	-	+
a b f g	+	+	-	-	-	+	+
d g	-	-	-	+	-	-	+
a c d f g	+	-	+	+	-	+	+
b c d f g	-	+	+	+	-	+	+
a b d g	+	+	-	+	-	-	+
c e f g	-	-	+	-	+	+	+
a e g	+	-	-	-	+	-	+
b e g	-	+	-	-	+	-	+
a b c e f g	+	+	+	-	+	+	+
c d e g	-	-	+	+	+	-	+
a d e f g	+	-	-	+	+	+	+
b d e f g	-	+	-	+	+	+	+
a b c d e g	+	+	+	+	+	-	+

The letter assignments, shown in Table IV, were carefully chosen so that in the treatment and analysis of the results the effect of any two-factor interaction involving the bakeout factor, D, would be clear of any other main effect or two-factor interaction of interest:

TABLE IV - LETTER ASSIGNMENT

A - Anode Material

B - Cathode Shape

C - Cathode Material

D - Bakeout

E - Anode Shape

F - Anode Finish

G - Cathode Finish

The treatments were randomized and performed in the test vehicle shown in Fig. 1:

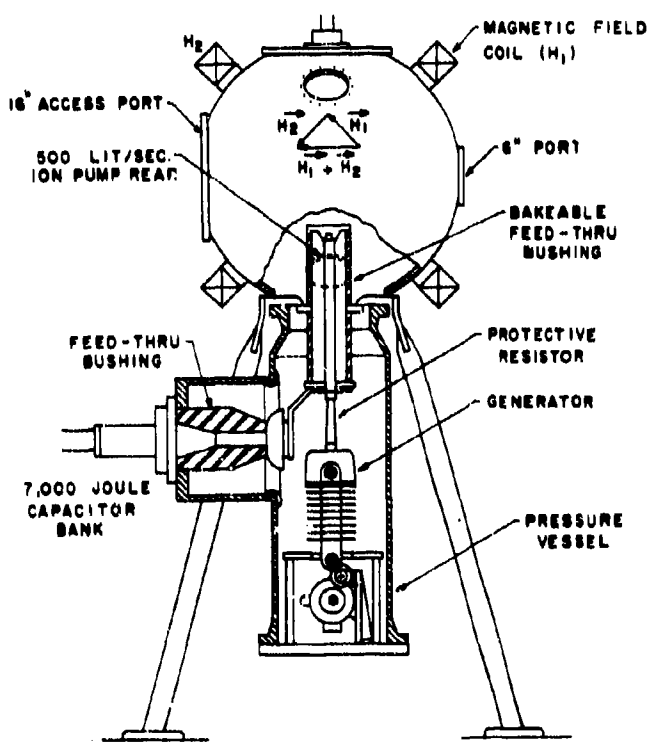

Fig. 1 Test Vehicle

Each treatment was carried out in this manner: The voltage was increased in 10-kV steps, each step held for two minutes. During this procedure, the microdischarges (self-quenching pulses of current), hydrogen evolutions, X-radiation, visible radiation, and prebreakdown current were monitored. The voltage was increased in this stepwise manner until puffs of hydrogen were detected by the mass spectrometer. This voltage was recorded. After the gap was outgassed again, the increase of voltage was continued until sparking occurred. This voltage was recorded as the first breakdown voltage. This procedure was repeated for each treatment at six electrode separations from 0.5 to 3.0 cm. During the application of voltages at each gap setting, the sparking and gas evolution conditions the electrodes so that higher voltages may be held off. These higher voltages were also recorded for the analysis.

Thus, we have three sets of yields of voltages that can be incorporated as the inputs to the design plan for analysis. These numbers inserted in the boxes of the design table and treated with the signs shown will give the deviation from the average of the whole experiment for each factor and factor interaction. The results can be obtained in a more systematic manner by using the Yates Algorithm, which consists of repeatedly adding and subtracting adjacent test results until the results for the mean, main effects and two-factor interactions are obtained, as shown in Table V:

TABLE V - DEFINING RELATION

I = -ABDFG = -CDEFG = ABCE

YIELDS OF YATES ALGORITHM

1	mean	12	ABE + (C)	23	BDG - AF
2	(A)	13	[DE]	24	ABDG - (F)
3	(B)	14	ADE	25	EG
4	AB + CE	15	BDE	26	AEG
5	[(D)]	16	ABDE + [CD]	27	BEG
6	[AD]	17	(G)	28	A BEC + CG
7	[BD]	18	AG	29	DEG - CF
8	ABD - FG	19	BG	30	ADEG
9	(E)	20	ABG - [DF]	31	BDEG
10	AE + BC	21	[DG]	32	ABDEG - EF
11	BE + AC	22	ADG - BF		

This table shows that we can get seven main effects and six two-factor interactions with D (the bakeout) plus the mean. The others may be used for estimating error.

The analysis was carried out using the Yates Algorithm with inputs of the voltages obtained. The results of this analysis indicated a low level of confidence for the effects. Therefore, the voltages were plotted versus distance to the one-half power, since these and many other experimental results have been found to follow this relationship. From these plots a slope was calculated and used as inputs to the Yates program. This slope, using the average of many points, smoothed out the values as well as the error and gave more significant results.

These results are plotted on half-normal graph paper as shown in Fig. 2:

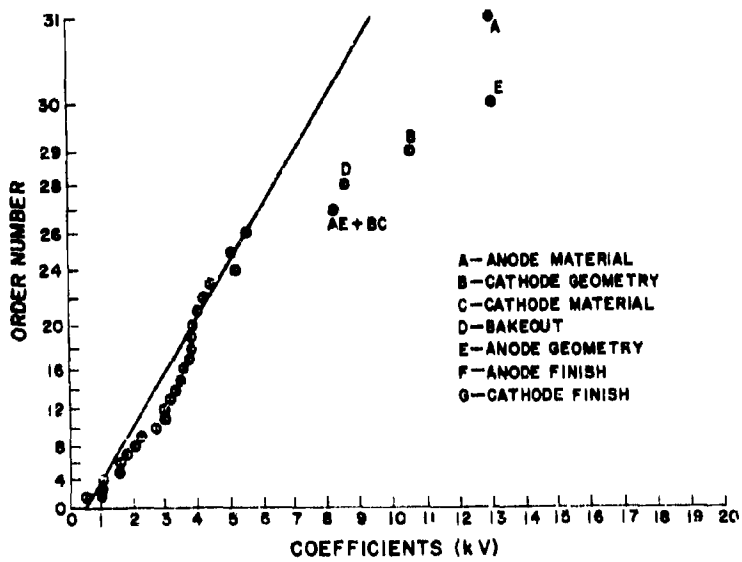

Fig. 2 Half-normal plot of coefficients obtained from the Yates Algorithm.

This graph is designed to give a straight line for any random process. The order number represents the range of values, from smallest to largest, corresponding to the coefficients obtained from the Yates analysis. Deviations from a straight line indicate that the factor has a significant

influence on the distribution of the thirty-two values and that the values are other than random. This plot shows that a straight line can be drawn through most of the points that represent effects of the factors with little or no deviation from the average of the experiment. The points labeled A, E, B, D, and AE+BC are real effects, and the significance is indicated by the distance from the straight line. The AE+BC effect, however, does not donate any information because the AE cannot be distinguished from the BC effect. From this experiment these conclusions can immediately be drawn:

1. A, E (the anode material and geometry) are most important.

2. B (the cathode geometry) is important.

3. D (the bakeout factor) is important, but less than the above.

The level of the anode geometry factor that raised the breakdown voltage is the spherical electrode. This might also be said for the cathode geometry, but with less confidence. When the anode material was titanium alloy, higher breakdown voltages were reached than when it was copper. The bakeout factor, D, was pertinent to this experiment with the test vehicle designed for this study. The two levels of bakeout were complete system and electrode bakeout versus electrode only bakeout. The electrodes were equipped with internal heaters for this purpose. The complete system and electrode bakeout level is superior to electrode only bakeout for attaining higher breakdown voltages.

Along with the statistical analysis, the results of the experiment were analyzed as to the physical processes occurring in the highly stressed electrode system. As previously stated, the hydrogen partial pressures were monitored on the mass spectrometer. Large bursts of gas were coincident with sparking or breakdown. Also, the superiority of spherical electrodes in holding off higher voltages suggested a breakdown mechanism dependent on the amount of gas present in the gap and the pumping conductance of the electrode gap system caused by the shape and size of the electrodes and the gap distance. A theory was proposed whereby the gas conductance of the gap played a major part in the breakdown process.[1] Simply stated, small-area electrodes with a high-conductance gap will hold off higher voltages than large-area electrodes at the same gap spacing. To evaluate this theory, a second statistically designed block-of-eight experiment was derived. The objective of this experiment was to verify the gas pumping conductance theory. The factors chosen were anode processing, cathode processing, and electrode size. The two levels of electrode processing are hydrogen baked versus vacuum baked, and, for size, a 4" versus 4/3" diameter Bruce plane, as shown in a. of Table VI. Because of the simplicity of this full factorial 2^3 experiment, it was decided to incorporate a transverse magnetic field as a factor at the end of each treatment, as shown in b. of Table VI. The treatment was repeated with magnetic field and then again without magnetic field to show up any consistent difference between the first and third breakdown voltages because of the application of the magnetic field:

1. M. J. Mulcahy, A. Watson, and W. R. Bell, "High Voltage Breakdown Study," USAECOM Contract DA28-043 AMC-00394(E) ARPA Order No. 517 (1967).

TABLE VI - FACTORS AND LEVELS FOR BLOCK-OF-EIGHT EXPERIMENT

a. Without Magnetic Field

Factor	Letter	Level	
		High	Low
Anode Processing	A	a - Vacuum Baked	1 - Hydrogen Baked
Cathode Processing	B	b - Vacuum Baked	1 - Hydrogen Baked
Electrode Size	C	c - Large	1 - Small

b. With Magnetic Field

Anode Processing	A	a - Vacuum Baked	1 - Hydrogen Baked
Cathode Processing	B	b - Vacuum Baked	1 - Hydrogen Baked
Electrode Size	C	c - Large	1 - Small
Perpendicular Magnetic Field	D	d - Present	1 - Absent

This is now a complete 2^4 factorial experiment and can be analyzed separately as two 2^3 experiments, as shown in Table VII:

TABLE VII - EXPERIMENTAL ORDER

Order	Description	Main Block	Perpendicular Fields
1	Anode 4-inch Bruce h-baked Cathode 4-inch Bruce h-baked	c	cd
2	Anode 4/3-inch Bruce h-baked Cathode 4/3-inch Bruce h-baked	(1)	d
3	Anode 4-inch Bruce vac-baked Cathode 4-inch Bruce h-baked	ac	acd
4	Anode 4/3-inch Bruce vac-baked Cathode 4/3-inch Bruce vac-baked	ab	abd
5	Anode 4/3-inch Bruce h-baked Cathode 4/3-inch Bruce vac-baked	b	bd
6	Anode 4-inch Bruce h-baked Cathode 4-inch Bruce vac-baked	bc	bcd
7	Anode 4-inch Bruce vac-baked Cathode 4-inch Bruce vac-baked	abc	abcd
8	Anode 4/3-inch Bruce vac-baked Cathode 4/3-inch Bruce h-baked	a	ad

The experiment was performed similarly to the stepwise voltage increase procedure as described before. The resulting voltages are plotted versus distance to the one-half power. In Fig. 3, first the average effect, μ, without magnetic field present, is plotted with the average effect, μ_E, with magnetic field present. It can be seen immediately that the magnetic field lowers the breakdown voltage except at the smallest spacing tested:

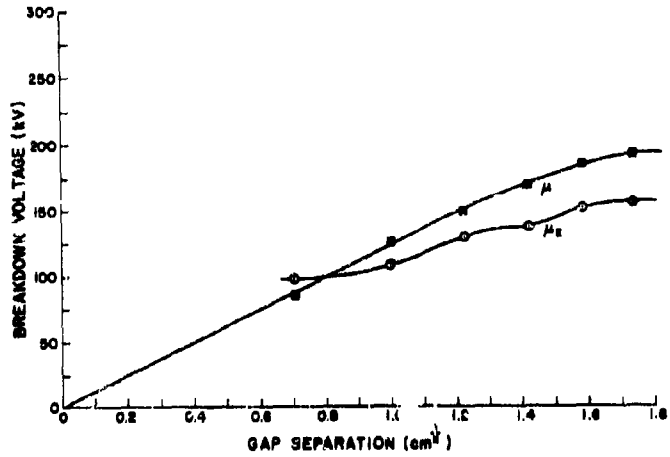

Fig. 3 Breakdown voltage versus gap separation in centimeters to the one-half power for average values with and without magnetic field.

In Fig. 4, the effects of the factors A_1, AE, and A_3 are shown by subtracting the values individually from the corresponding overall average breakdown value, μ. The subscript 1 refers to the conditioned breakdown value prior to applying magnetic field and 3 refers to the breakdown value after application of magnetic field. The differences in these values are indicative of a memory of the conditions imposed by the magnetic field after it was removed. Other main effects and two-factor interactions are plotted similarly:

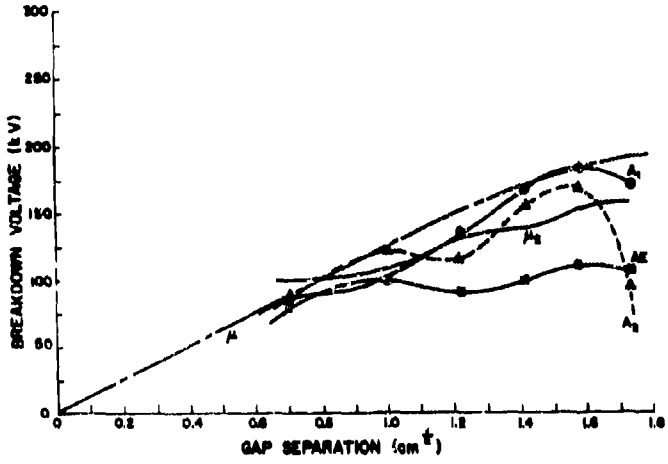

Fig. 4 Breakdown voltage versus gap separation in centimeters to the one-half power for average values with factor A and two-factor interaction AE.

From these curves, the principal conclusions that can be stated with a good measure of confidence are as follows:

1. The hydrogen-baking procedure permitted higher breakdown voltages than did the vacuum-baking. The magnetic field amplified this difference.

2. Large-area electrodes reduced the breakdown voltage, which is consistent with the results obtained in the first experiment. The magnetic field had no effect in this case.

3. The combined effect of hydrogen-baking of the cathode and using small electrodes raises the breakdown voltage. This effect is amplified in the presence of a magnetic field.

The results of these experiments, presented in this manner, show with a good degree of confidence what can be expected when electrodes are designed for high-voltage devices. These data are for copper electrodes. Other materials of interest to vacuum component design engineers will be similarly analyzed.

The next experiment (now being conducted) was designed as a full factorial with six factors at two levels, as a result of some three-factor

interactions showing up in the analysis of the block-of-eight experiment. This is done in order to be complete and assess all the influences in the breakdown process.

Different materials, as well as the other factors initially named, will be introduced into each successive experiment. The results of this program will be compiled in the form of graphs and charts for the high-voltage design engineer.

IMPROVING BINOMIAL RELIABILITY ESTIMATES - A MODERATELY DISTRIBUTION FREE TECHNIQUE FOR SMALL SAMPLE RELIABILITY ESTIMATION

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1. INTRODUCTION. The purpose of this article is to demonstrate how lower confidence bounds for reliability obtained using the distribution free binomial approach can be improved under fairly nonrestrictive assumptions on the random variable involved. The technique to be described, referred to hereafter as the MDF technique, is an extension of the result presented at the 1966 Army Design Conference (see [1]).

2. THE MDF TECHNIQUE. Suppose that the random variable under consideration is continuous (i.e. has an absolutely continuous distribution function) and nonnegative with distribution function $F(x)$ and density function $F'(x)$. Suppose further that the mission for which the reliability is to be estimated can be expressed as a number T in the domain of $F(x)$, and suppose that the reliability is to be estimated on the basis of a sample of n independent systems from the population under investigation. The following Proposition provides the basis for the MDF estimation technique.

Proposition 1. Let Y be the number of mission failures in n trials. For $\gamma \in (0,1)$ let $C(Y)$ be the solution to the equation

$$\sum_{f=0}^{Y} \binom{n}{f} [C(Y)]^f [1 - C(Y)]^{n-f} = 1 - \gamma.$$

Let $M(Y) = \frac{X_{(Y+1)}}{T}$, where $X_{(Y+1)}$ is the $(Y+1)^{th}$ order statistic and T is the mission. Finally, let $k(\gamma,n)$ be determined by the equation

$$\sum_{f=0}^{n^*} \binom{n}{f} [k(\gamma,n)C(0)]^f [1 - k(\gamma,n)C(f)]^{n-f} = 1 - \gamma,$$

where $n^* \leq n$, $1 - k(\gamma,n)C(n^*) > 0$ and $1 - k(\gamma,n)C(n^* + 1) \leq 0$. If $F'(x)$ is monotone nondecreasing on $[0, X_{(Y+1)}]$, then

$$\Pr\{1 - k(\gamma,n)(\frac{C(Y)}{M(Y)}) \leq 1 - F(T)\} \geq \gamma.$$

The proof of Proposition 1 is lengthy and is included in the Appendix. The estimator $1 - k(\gamma,n)[\frac{C(Y)}{M(Y)}]$ will be called the MDF γ-confidence lower bound estimator for the reliability $1 - F(T)$.

It is seen that for each Y the number 1 - C(Y) is simply the binomial γ-confidence lower bound estimator for the reliability based on Y failures in n trials. Values of this estimator are tabulated for selected values of n and γ[2]. Precise determination of the number $k(\gamma,n)$ for each pair (γ,n) is most easily accomplished on a computer. Table 1 presents values of $k(\gamma,n)$ for three confidence levels (.90, .95 and .99) for selected values of n from 5 to 100. Values of $k(\gamma,n)$ intermediate to values of n given can be obtained quite accurately by linear interpolation.

In an application of the MDF technique to a specific problem, a confidence level γ is chosen first; from the observed data one then determines the value of $M(Y) = \frac{X_{(Y+1)}}{T}$, where Y is the observed number of failures of the mission T and $X_{(Y+1)}$ is the $(Y+1)^{th}$ order statistic for the sample. The value C(Y) is obtained from binomial reliability tables (1 - C(Y) is the lower γ-confidence bound estimate for the mission reliability based on Y failures). Finally, the value of $k(\gamma,n)$ is obtained from Table 1. Thus, for example, if Y = f, then according to Proposition 1, $1 - k(\gamma,n) \cdot (\frac{C(f)}{M(f)})$ is the MDF lower γ-confidence estimate of the reliability 1 - F(T) for the mission T.

Example 1. Suppose that for a given reliability estimation problem the mission is T = 3.2 hours, and the times to failure for a sample of 15 systems are given by {11.9, 5.8, 8.1, 13.2, 12.7, 12.6, 25.6, 20.2, 9.2, 20.6, 14.2, 17.8, 19.8, 28.1, 12.2}. Let the confidence level be γ = .90, and suppose it can be assumed (see Example 3) that the probability density function for the time to failure random variable X is monotone nondecreasing on $[0,X_{(Y+1)}]$, where Y is the number of mission failures (i.e. $X_{(Y+1)} = X_{(1)} = 5.8$). In accordance with the above description, the MDF lower .90 confidence bound for the mission reliability is obtained as follows: From the sample data, M(Y) = M(0) = $X_{(1)}/3.2 = 1.81$. From binomial reliability tables C(Y) = C(0) = .142.

By an interpolation in Table 1, k(.90, 15) is determined to be 1.17132. Thus, according to Proposition 1, the MDF .90 confidence lower bound estimate for 1 - F(T), the mission reliability, is

$$1 - 1.17132(\frac{.142}{1.81}) = .9081.$$

(The corresponding binomial estimate is .8577.)

Example 2. Suppose circumstances are the same as for Example 1, except that the data are as follows: {0.9, 4.1, 4.6, 4.7, 7.1, 7.5, 7.9, 11.1, 11.1, 11.5, 15.9, 17.5, 18.1, 21.9, 22.3}. Then

$$Y = 1,\ M(Y) = M(1) = \frac{X_{(2)}}{3.2} = 1.28,\ C(Y) = C(1) = .7356$$

(from binomial reliability table with n = 15, γ = .90) and, as before, k(.90, 15) = 1.17132. Thus, the MDF .90 confidence lower bound estimate for 1 - F(3.2)

is $1 - 1.17132(\frac{.2356}{1.28}) = .7844$. (The corresponding binomial estimate would be .7644.)

3. APPLICABILITY OF THE TECHNIQUE. Whether the MDF technique should be applied to a given problem depends on the extent to which the analyst can justify the necessary assumptions regarding the problem and the distribution function involved. Recall that Proposition 1 requires that the density function $F'(x)$ be monotone nondecreasing on the interval $[0,X_{(Y+1)}]$. This is actually a stronger requirement than necessary for the validity of Proposition 1 - it is noted in the proof of Proposition 1 (Appendix) that the monotonicity requirement is only used to guarantee that $F(X_{(Y+1)}) \geq (\frac{X_{(Y+1)}}{T}) F(T)$; however, this inequality is valid for a much larger class of distributions than that characterized by the monotonicity (nondecreasing) of the density function. Since no simple characterization of the more general class of distributions appears to exist, the purposes of this paper are best served by confining attention to class of distributions with density functions which are monotone nondecreasing on $[0,X_{(Y+1)}]$.

In a particular application then, the analyst must be able to justify the use of the assumption that $f(x)$ is monotone nondecreasing on $[0,X_{(Y+1)}]$. Indications are that the technique is fairly insensitive to other than serious departures from the assumption, and therefore that a relatively loose or insensitive justification technique can be employed. Unfortunately, there appears to be no specific test of the hypothesis that the density function $F'(x)$ is monotone nondecreasing on $[0,X_{(Y+1)}]$ available at present. It is possible that adaptations of certain existing tests, such as the test for a nondecreasing failure rate proposed by Proschan [3], may lead to a suitable test for MDF applications. This possibility is being investigated.

An approach which seems reasonable in view of the apparent insensitivity of the MDF technique to departures from the monotonicity assumption is the following: On the basis of the data, one selects a known distribution function $F_0(x)$ which appears to be a reasonable candidate for the true distribution function of the random variable involved - and is reasonably representative of the data over the interval $[0,X_{(Y+1)}]$. Thus, for example the analyst might decide that a normal distribution with μ and σ equal to the sample mean and sample variance respectively is not an inappropriate selection; again, one might choose a Weibull and estimate the parameters graphically. Having selected $F_0(x)$, one then would apply the Kolmogorov-Smirnov test, using the selected distribution function in the null hypothesis: $H_0: F(x) = F_0(x)$. If the null hypothesis is not rejected, and if the selected function $F_0(x)$ has a monotone nondecreasing first derivative ($F'_0(x)$) on $[0,X_{(Y+1)}]$, then one concludes that it is possible to apply the MDF technique to estimate the reliability. Note that application of the MDF technique in this case is less hazardous

(more conservative) than using the hypothesized distribution function $F_0(x)$ to estimate the reliability. Further, the MDF technique is a much simpler method to apply than classical approaches which involve the hypothesized distribution function $F_0(x)$ in the sense that one does not need to be concerned about estimation of the parameters in $F_0(x)$ and the associated problems encountered in obtaining lower confidence bounds in terms of the estimators.

Example 3. For the data from Example 1, suppose it is hypothesized that the distribution function for this data is normal with $\mu = 15.5$, and $\sigma^2 = 40.4$ (15.5 and 40.4 are the mean and variance, respectively, of the sample.) The Kolmogorov-Smirnov test of the hypothesis $H_0 : F(x) = F_0(x)$, where

$$F_0(x) = \frac{1}{\sqrt{2\pi}\ \sqrt{40.4}} \text{Exp} - \frac{(x-15.5)^2}{80.8},$$

would not reject H_0 at any reasonable level of significance ($\max |F_n(x_{(j)}) - F_0(x_{(j)})| = .179$; the critical value for $\alpha = .20$ is .266). Now $X_{(Y+1)} = 5.8$. Since $F'_0(x) = f_0(x)$ is monotone nondecreasing on [0, 15.5], it is not unreasonable to proceed as if $F'(x)$ is monotone nondecreasing at least on $[0, 5.8] = [0, X_{(Y+1)}]$ and to apply the MDF technique to estimate the reliability for the mission $T = 3.2$ hours, as has been done in Example 1.

To the author's knowledge, there are no readily adaptable goodness of fit type tests available for the situation in which censored data is involved. In this case, justification for use of the MDF technique, or any other technique, must necessarily be based on past experience, on examination of the censored data and, to a large extent, on faith.

The next section discusses seven Monte Carlo studies which were conducted to investigate the behavior of the MDF estimator. Three populations, Weibull, Uniform and Exponential, were considered. The density function for the Weibull population was monotone increasing on the interval $[0, \xi_{.35}]$; the density function for the Uniform distribution is, of course, monotone nondecreasing on its whole domain. However, the density function for the Exponential distribution is monotone decreasing on its whole domain, so that the MDF technique is only an approximate technique for this case. It will be seen that, inspite of the departure of the exponential case from the MDF requirement (monotone nondecreasing density function), the MDF technique generally provided acceptable results in the two Exponential studies conducted.

4. MONTE CARLO STUDIES. In order to obtain an indication of the behavior of the MDF estimator and of the sensitivity of the MDF technique to departure from the monotonicity assumption, seven Monte Carlo studies were conducted as follows: Two studies were based on sampling from a Weibull population

with distribution function $F(x) = 1 - e^{-.005x^2}$ (i.e. $F'(x) = \lambda\theta x^{\theta-1}e^{-\lambda x^{\theta}}$, $\lambda = .005, \theta = 2$). The mission considered was T = 3.2; since F(3.2) = .05, the mission reliability was 1 - F(T) = .95. For Weibull 1, 100 sets of 15 observations each were obtained; for Weibull 2, 100 sets of 30 observations each were obtained.

Three studies were based on sampling from a population with the Uniform distribution on [0,1]. For Uniform 1 and Uniform 2, the mission was T = .05 so that the true reliability was .95. Uniform 1 consisted of 100 sets of n = 10 observations; Uniform 2 consisted of 100 sets of n = 20 observations. For Uniform 3, the mission was T = .15 so that the true reliability was .85; 100 sets of n = 10 observations were drawn for Uniform 3.

Finally, two studies were based on sampling from an Exponential popula-
with distribution function $F(x) = 1-e^{-.01x}$ (i.e. $F'(x) = \lambda e^{-\lambda x}$, $\lambda = .01$). The mission considered was T = 5.129; as with the Weibull studies, since F(5.129) = .05, the reliability was .95. For Exponential 1, 100 sets of n = 20 observations were obtained; for Exponential 2, 100 sets of n = 45 observations were obtained.

For each set of observations in each of the seven studies, the MDF .90 lower confidence estimate was obtained as described in Section 3. Furthermore, for each set of observations in each study, the binomial .90 lower confidence bound was determined. The results obtained by these two methods of estimation are compared in summary form in Tables 2 and 3. Also, for each set of observations in each study the MDF estimate was compared with the binomial estimate for proximity (at the third decimal place) to the true reliability. The results of this proximity evaluation are presented in Table 4.

From Tables 2, 3 and 4 it is seen that use of the MDF technique resulted in substantially better estimates of the true reliability than did the binomial method in the five cases with the smallest sample sizes: Weibull 1 (n=15), Uniform 1 (n=10), Uniform 2 (n=20), Uniform 3 (n=10) and Exponential 1 (n=20). Further, in none of these cases did the observed proportion of errors (estimates in excess of the true reliability) made using the MDF technique exceed the allowable .10, despite the fact that the MDF technique is only an approximate technique for the exponential case. Also, the magnitude of the errors was relatively small in general, as indicated by the proximity of the error median to the true reliability in each case.

Consider now the results of Weibull 2 and Exponential 2: Tables 2, 3 and 4 show that although the superiority of the MDF technique is not as pronounced in these cases as with the three smaller sample cases, it is nevertheless evident; further, Table 3 and 4 indicates that the MDF provides better estimates in these cases (Weibull 2, Exponential 2) often enough to justify at least calculating the MDF estimate to determine whether it gives a larger value than the corresponding binomial estimate.

The MDF technique led to 12 erroneous estimates (>.95) in Exponential 2. Although this proportion exceeds $1 - \gamma$ = .10, it is seen from Table 2 that

the degree of departure from the true reliability is not excessive in three cases (.952, .953, .953). Further, only 5 of the erroneous estimates exceeded the true reliability by more than .01. Again, it is pointed out that the MDF technique is only approximate for the Exponential case. However, it is clear that the more nearly the Exponential distribution function involved is approximated by a Uniform distribution function over the range of interest (i.e. over the interval $[0, X_{(Y+1)}]$), the smaller will be the chance of obtaining erroneous estimates.

To provide an indication of how the MDF technique compares with two other commonly used estimation techniques, the following studies were conducted: 1) For each set of observations in Weibull 1 and Exponential 1, a .90 confidence lower bound (for the reliability) was obtained using the method described by Epstein in [4] (which assumes Exponentiality) for the non-replacement situation with data censored at the third order statistic. 2) The same technique, with data censored at the fifth order statistic was used to obtain a .90 confidence lower bound (for the reliability) for each set of observations in Exponential 2. The results of studies 1) and 2) are summarized in Table 5. 3) For 33 randomly selected trials from Weibull 1 and 32 randomly selected trials from Weibull 2 the technique described by Johns and Lieberman in [5], with data censored at the seventh order statistic, was used to obtain .90 confidence lower bounds for the reliability. The results of these studies are summarized in Table 6.

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TABLE 1. VALUES OF k(γ,n) FOR SELECTED CONFIDENCE LEVELS γ AND SAMPLE SIZES n

n \ γ	.90	.95	.99
5	1.13222	1.10195	1.05967
6	1.14160	1.11117	1.06796
7	1.15098	1.12040	1.07787
8	1.15595	1.12583	1.08310
9	1.16093	1.13125	1.08958
10	1.16399	1.13369	1.09343
11	1.16706	1.13613	1.09729
12	1.16836	1.13857	1.09907
13	1.16956	1.14101	1.10086
14	1.17049	1.14198	1.10194
16	1.17215	1.14393	1.10441
18	1.17382	1.14588	1.10691
20	1.17548	1.14782	1.10928
22	1.17644	1.14887	1.11053
24	1.17739	1.14992	1.11179
26	1.17834	1.15097	1.11287
28	1.17910	1.15178	1.11337
30	1.17986	1.15259	1.11431
35	1.18177	1.15462	1.11691
40	1.18236	1.15529	1.11770
45	1.18296	1.15596	1.11850
50	1.18320	1.15626	1.11874
60	1.18367	1.15687	1.11974
70	1.18415	1.15740	1.12033
80	1.18477	1.15807	1.12089
90	1.18501	1.15835	1.12144
100	1.18525	1.15861	1.12176
200	1.18653	1.16004	1.12344

TABLE 2. COMPARISON OF DISTRIBUTIONS OF BINOMIAL AND MDF .90 CONFIDENCE BOUNDS FOR THE RELIABILITY FOR EACH MONTE CARLO STUDY

SUMMARY OF RESULTS

STUDY	SAMPLE SIZE	INTERVAL	BINOMIAL	MDF	MDF ERRONEOUS ESTIMATES
Weibull 1	15	$\leq .536$	1	1	
		(.536, .607]	5	2	
		(.607, .683]	11	4	
		(.683, .764]	35	12	
		(.764, .858]	48	42	
		(.858, .900]	0	29	
		(.900, .950]	0	10	
Weibull 2	30	$\leq .713$	4	1	.951
		(.713, .751]	2	1	
		(.751, .791]	7	6	
		(.791, .832]	22	14	
		(.832, .876]	32	24	
		(.876, .926]	33	33	
		(.926, .950]	0	20	
		(.950, 1.00]	0	1	
Uniform 1	10	(.448, .550]	8	1	.953
		(.550, .663]	35	5	.955
		(.663, .794]	57	23	.962
		(.794, .850]	0	20	
		(.850, .900]	0	23	
		(.900, .950]	0	25	
		(.950, 1.00]	0	3	

TABLE 2. (Continued)

STUDY	SAMPLE SIZE	INTERVAL	BINOMIAL	MDF	MDF ERRONEOUS ESTIMATES	
Uniform 1	10	(.448, .550]	8	1	.953	
		(.550, .663]	35	5	.955	
		(.663, .794]	57	23	.962	
		(.794, .850]	0	20		
		(.850, .900]	0	23		
		(.900, .950]	0	25		
		(.950, 1.00]	0	3		
Uniform 2	20	≤ .639	2	1	.951	
		(.639, .696]	2	0	.952	
		(.696, .755]	12	1	.952	
		(.755, .819]	51	19	.956	
		(.819, .891]	33	33	.958	
		(.891, .950]	0	40	.976	
		(.950, 1.00]	0	6		
Exponential 1	20	≤ .639	0	0	.951	.972
		(.639, .696]	3	3	.952	
		(.696, .755]	21	4	.954	
		(.755, .819]	36	12	.956	
		(.819, .891]	40	28	.960	
		(.891, .950]	0	45	.962	
		(.950, 1.00]	0	8	.969	

TABLE 2. (Continued)

STUDY	SAMPLE SIZE	INTERVAL	BINOMIAL	MDF	MDF ERRONEOUS ESTIMATES	
Exponential 2	45	$\leq .804$	4	4	.952	.963
		(.804, .830]	6	1	.953	.965
		(.830, .858]	18	11	.953	.965
		(.858, .886]	35	18	.956	.967
		(.886, .916]	25	26	.957	.972
		(.916, .950]	12	28	.960	
		(.950, 1.00]	0	12	.960	
Uniform 3*	10	$\leq .354$	4	1	.859	
		(.354, .448]	20	5	.863	
		(.448, .550]	24	13	.865	
		(.550, .663]	35	31	.865	
		(.663, .794]	17	29	.885	
		(.794, .850]	0	16		
		(.850, .900]	0	5		
		(.900, 1.00]	0	0		

*The true reliability in this case was .850.

TABLE 3. SUMMARY COMPARISON OF MDF AND BINOMIAL LOWER .90 CONFIDENCE BOUNDS: MEDIANS, MEANS, ERROR PERCENTAGES AND ERROR MEDIANS FOR 100 SETS OF OBSERVATIONS FOR EACH STUDY

EST. TECHNIQUE / STUDY	SAMPLE SIZE	MDF				BINOMIAL		
		MEDIAN	AVERAGE	PERCENT ERROR	ERROR MEDIAN	MEDIAN	AVERAGE	PERCENT ERROR
Weibull 1	15	.842	.822	0.0	—	.764	.790	0.0
Weibull 2	30	.879	.873	1.0	.951	.876	.868	0.0
Uniform 1	10	.854	.839	3.0	.955	.794	.729	0.0
Uniform 2	20	.888	.876	6.0	.954	.819	.829	0.0
Exponential 1	20	.899	.875	8.0	.958	.819	.831	0.0
Exponential 2	45	.904	.902	12.0	.960	.886	.890	0.0
Uniform 3*	10	.661	.665	5.0	.865	.663	.603	0.0

*Recall that the true reliability in this case was .85. In all other cases, the true reliability was .95.

TABLE 4. PROXIMITY* COMPARISON OF MDF AND BINOMIAL LOWER .90 CONFIDENCE ESTIMATES

EST. TECHNIQUE / STUDY	MDF	BINOMIAL	TIE
Weibull 1	74	25	1
Weibull 2	59	40	1
Uniform 1	86	13	1
Uniform 2	80	19	1
Exponential 1	85	12	3
Exponential 2	55	37	8
Uniform 3	67	31	2

*For example, in Weibull study 1, the MDF technique gave an estimate which was closer to the true reliability than the corresponding binomial estimate in 74 instances, the binomial method gave the closer estimate in 25 cases and there was one tie.

TABLE 5. COMPARISON OF ESTIMATES OBTAINED USING EXPONENTIAL TECHNIQUES WITH CORRESPONDING MDF ESTIMATES (BASED ON ALL 100 TRIALS FROM EACH STUDY)

EST. TECHNIQUE / STUDY	MDF				EPSTEIN [4]			
	MEDIAN	AVERAGE	PERCENT ERROR	ERROR MEDIAN	MEDIAN	AVERAGE	PERCENT ERROR	ERROR MEDIAN
Weibull Study 1	.842	.822	0.0	—	.822	.809	0.0	—
Exponential Study 1	.899	.875	8.0	.958	.908	.890	12.0	.957
Exponential Study 2	.904	.902	12.0	.960	.925	.910	9.0	.955

TABLE 6. COMPARISON OF ESTIMATES OBTAINED BY JOHNS-LIEBERMAN METHOD WITH CORRESPONDING MDF ESTIMATES (33 TRIALS FROM WEIBULL STUDY 1, 32 TRIALS FROM WEIBULL STUDY 2)

EST. TECHNIQUE / STUDY	MDF			JOHNS-LIEBERMAN			
	MEDIAN	AVERAGE	PERCENT ERROR	MEDIAN	AVERAGE	PERCENT ERROR	ERRONEOUS ESTIMATES*
Weibull Study 1	.813	.802	0.0	.863	.847	12.1	.952 .955 .960 .960
Weibull Study 2	.879	.870	0.0	.895	.889	6.3	.959 .966

*All erroneous estimates are based on extrapolation in Table 1 of [5].

APPENDIX

This section presents a proof of Proposition 1 upon which the MDF estimation technique is based. Proposition 2, which is proved in Reference 1, is restated here for reference ease.

Proposition 2. Let a continuous nonnegative random variable X have probability density function $F'(x)$ and distribution function $F(x)$. Suppose that $F'(x)$ is monotone nondecreasing on an interval $[0,\xi_p]$. If $m \geq 1$ and $MT \in [0,\xi_p]$, then $F(MT) \geq MF(T)$.

Proposition 1. Let Y be the number of mission failures in n trials. Let C(Y) be the solution to $\sum_{j=0}^{Y} \binom{n}{j}[C(Y)]^j[1 - C(Y)]^{n-j} = 1 - \gamma$; let $M(Y) = \frac{X_{(Y+1)}}{T}$, where T is the mission; and let $k(\gamma,n)$ be determined by the equation

$$\sum_{f=0}^{n^*} \binom{n}{f}[k(\gamma,n)C(0)]^f[1 - k(\gamma,n)C(f)]^{n-f} = 1 - \gamma,$$

where $n^* \leq n$, $1 - k(\gamma,n)C(n^*) > 0$ and $1 - k(\gamma,n)C(n^* + 1) \leq 0$. If $f(x)$ is monotone nondecreasing on $[0,X_{(Y+1)}]$, then $\Pr\{1 - k(\gamma,n)(\frac{C(Y)}{M(Y)}) \leq 1 - F(T)\} \geq \gamma$.

Proof. We show that $\Pr\{1 - k(\gamma,n)(\frac{C(Y)}{M(Y)}) \geq 1 - F(T)\} \leq 1 - \gamma$.

We have that $1 - k(\gamma,n)\left(\frac{C(Y)}{M(Y)}\right) \geq 1 - F(T) = 1 - \frac{C(Y)}{M(Y)} \geq$ $1 - F(T) + \left[1 - \frac{1}{k(\gamma,n)}\right]F(T)$. Let $1 - \frac{1}{k(\gamma,n)} = k^*$. Thus, the probability that the MDF estimator actually exceeds the reliability is equal to the probability that $1 - \frac{C(Y)}{M(Y)}$ exceeds the reliability by an amount not less than $k^*F(T)$. Thus, we want to show that $\Pr\{1 - \frac{C(Y)}{M(Y)} \geq 1 - F(T) + k^*F(T)\} \leq$ $1 - \gamma$. To do this, we consider two cases: Case 1: $F(T) \leq k(\gamma,n)C(0)$; Case 2: $F(T) \geq k(\gamma,n)C(0)$.

Case 1. $F(T) \leq k(\gamma,n)C(0)$.

$$\Pr\{1 - \frac{C(Y)}{M(Y)} \geq 1 - F(T) + k^*F(T)\}$$

$$= \Pr\{0 \text{ failures and } X_{(1)} \geq [\frac{k(\gamma,n)C(0)}{F(T)}]T\}$$

$$+ \Pr\{1 \text{ failure and } X_{(2)} \geq [\frac{k(\gamma,n)C(1)}{F(T)}]T\}$$

$$+ \Pr\{2 \text{ failures and } X_{(3)} \geq [\frac{k(\gamma,n)C(2)}{F(T)}]T\}$$

$$+$$

$$\vdots$$

$$+ \Pr\{n^* \text{ failures and } X_{(n^*)} \geq [\frac{k(\gamma,n)C(n^*)}{F(T)}]T\}$$

$$\vdots$$

$$+$$

$$\vdots$$

$$+ \Pr\{n - 1 \text{ failures and } X_{(n)} \geq [\frac{k(\gamma,n)C(n)}{F(T)}]T\}.$$

Now, $\Pr\{Y \text{ failures and } X_{(Y+1)} \geq [\frac{k(\gamma,n)C(Y)}{F(T)}]T\}$

$$= \Pr\{Y \text{ failures and } F(X_{(Y+1)}) \geq F([\frac{k(\gamma,n)C(Y)}{F(T)}])\}.$$

Since $F(T) \leq k(\gamma,n)C(0)$, $F(T) \leq k(\gamma,n)C(Y)$ for $Y \in \{0,1,2, \ldots,n\}$. Thus, $\frac{k(\gamma,n)C(Y)}{F(T)} \geq 1$, so that by Proposition 2 $F(\frac{k(\gamma,n)C(Y)}{F(T)}\, T) \geq [\frac{k(\gamma,n)C(Y)}{F(T)}] \cdot F(T) = k(\gamma,n)C(Y)$. Therefore,

$$\Pr\{Y \text{ failures and } F(X_{(Y+1)}) \geq F([\tfrac{k(\gamma,n)C(Y)}{F(T)}]T)\}$$

$$\leq \Pr\{Y \text{ failures and } F(X_{(Y+1)}) \geq k(\gamma,n)C(Y)\}$$

$$= \Pr\{Y \text{ failures and } X_{(Y+1)} \geq g_{k(\gamma,n)C(Y)}\}$$

$$= \binom{n}{Y}[F(T)]^{Y}[1 - k(\gamma,n)C(Y)]^{n-Y},$$

when $k(\gamma,n)C(Y) \leq 1$ (i.e. for $n \leq n^*$). If $k(\gamma,n)C(Y) \geq 1$ (i.e. for $n > n^*$), $\Pr\{Y \text{ failures and } F(X_{(n-Y)}) \geq F([\frac{k(\gamma,n)C(Y)}{F(T)}]T)\} = 0$. Hence,

$$\Pr\{1 - \frac{C(Y)}{M(Y)} \geq 1 - F(T) + k^*F(T)\} \leq$$

$$\sum_{f=0}^{n^*} \binom{n}{f}[F(T)]^{Y}[1 - k(\gamma,n)C(Y)]^{n-Y}.$$

Now, $F(T) \leq k(\gamma,n)C(0)$. Thus,

$$\sum_{f=0}^{n^*} \binom{n}{f}[F(T)]^{Y}[1 - k(\gamma,n)C(Y)]^{n-Y} \leq$$

$$\sum_{f=0}^{n^*} \binom{n}{f}[k(\gamma,n)C(0)]^{Y}[1 - k(\gamma,n)C(Y)]^{n-Y}.$$

By hypothesis, the sum on the right equals $1 - \gamma$; thus when $F(T) \leq k(\gamma,n)C(0)$,

$$\Pr\{1 - \frac{C(Y)}{M(Y)} \geq 1 - F(T) + k^*F(T)\}$$

$$= \Pr\{1 - k(\gamma,n)(\frac{C(Y)}{M(Y)} \geq 1 - F(T)\} \leq 1 - \gamma.$$

Case 2. $F(T) \geq k(\gamma,n)C(0)$. In this case there exists $f = f^*$ such that $F(T) \leq k(\gamma,n)C(f^*)$ and $F(T) > k(\gamma,n)C(f^* - 1)$. We will thus have $\Pr\{1 - \frac{C(Y)}{M(Y)} \geq 1 - F(T) + k^*F(T)\} = \Pr\{f^* \text{ or fewer failures }\} + \Pr\{f^* + 1 \text{ failure and } X_{(f+2)} \geq [\frac{k(\gamma,n)C(f^*+1)}{F(T)}]T\} + \ldots + \Pr\{n^* \text{ failures and } X_{(n^*+1)} \geq [\frac{k(\gamma,n)C(n^*)}{F(T)}]T\}$

$$= \sum_{f=0}^{f^*} \binom{n}{f}[F(T)]^f[1 - F(T)]^{n-f} + \sum_{f=f^*+1}^{n^*} \binom{n}{f}[F(T)]^f \cdot$$

$[1 - k(\gamma,n)C(f)]^{n-f} = H(F(T))$. The maximum value the second sum in $H(F(T))$ can take on occurs for $F(T) = k(\gamma,n) \cdot C(f^*)$, since $F(T) \leq k(\gamma,n)C(f)$ for $f \geq f^*$. Thus, $H(F(T))$ is dominated by the function $G(F(T))$, where

$$G(F(T)) = \sum_{f=0}^{f^*} \binom{n}{f}[F(T)]^f[1 - F(T)]^{n-f} + \sum_{f=f^*+1}^{n^*} \binom{n}{f} \cdot$$

$$[k(\gamma,n)C(f^*)]^f[1 - k(\gamma,n)C(f)]^{n-f}.$$

Furthermore,

$$\frac{dG(F(T))}{dF(T)} = -n[1 - F(T)]^{n-1} + \sum_{f=1}^{f^*} \binom{n}{f}\{fF(T)^{f-1}[1 - F(T)]^{n-f}$$

$$- (n - f)F(T)^f[1 - F(T)]^{n-f-1}\}$$

$$= -n[1 - F(T)]^{n-1} + \sum_{f=1}^{f^*} \binom{n}{f}[F(T)]^{f-1}[1 - F(T)]^{n-f-1}\{f - nF(T)\}.$$

Now recall that for $f \in \{0,1,2,\ldots,f^*\}$ $F(T) \geq k(\gamma,n)C(f^*)$, and $C(f^*) \geq f^*/n$. Thus, since $k(\gamma,n) \geq 1$, $nF(T) \geq \frac{nf^*}{n} = f^*$, so that $f - nF(T) < 0$ for $f \in \{0,1,2,\ldots,f^*\}$. Hence $\frac{dG(F(T))}{dF(T)}$ is negative; i.e. $H(F(T))$ is dominated by the monotone decreasing function $G(F(T))$. The value of $G(F(T))$ when $F(T) = k(\gamma,n)C(0)$ is

$$G(k(\gamma,n)C(0)) = \sum_{f=0}^{f^*} \binom{n}{f}[k(\gamma,n)C(0)]^f[1 - k(\gamma,n)C(0)]^{n-f}$$

$$+ \sum_{f=f^*+1}^{n^*} \binom{n}{f}[k(\gamma,n)C(f^*)]^f[1 - k(\gamma,n)C(f)]^{n-f}.$$

However, it is clear that when $F(T) = k(\gamma,n)C(0)$, $f^* = 0$, so that $G(k(\gamma,n)C(0)) = \sum_{f=0}^{n^*} \binom{n}{f}[k(\gamma,n)C(0)]^f[1 - k(\gamma,n)C(f)]^{n-f}$.

Thus $H(F(T)) \leq G(k(\gamma,n)C(0))$. By hypothesis, $G(k(\gamma,n)C(0)) = 1 - \gamma$, and the theorem is proved.

THE DEVELOPMENT OF A PROBABALISTIC MODEL FOR ACOUSTIC SOUND RANGING

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It was decided last year that the Environmental Sciences Division of the Atmospheric Sciences Laboratory at White Sands Missile Range would study the possibility of improving acoustic sound ranging by applying more elaborate meteorological corrections.

Accordingly, a very elaborate test was set up. Charges consisting of two and one half pounds of TNT were to be exploded at twenty minute intervals and the resulting acoustic waves picked up by eighteen broadband microphones placed so as to give various microphone configurations. The outputs from these microphones were fed to three magnetic tape recorders, eight signals to one, five signals to each of the other two. The locations for the microphones were laid out very carefully and a final first order survey run to precisely determine the microphone coordinates.

It was felt that the microphone coordinates were not in error by more than one foot with respect to each other and to the firepoint. By digitizing the analog tapes at one millisecond intervals, since sound propagates approximately one foot per millisecond, the timing and microphone location errors should be of the same order of magnitude. It was hoped that this would remove timing and microphone placement errors as sources of error but, if necessary the digitizing rate could be increased ten-fold and more sophisticated methods could be used to compensate for microphone placement errors.

Figure 1 shows the geometry of a six microphone acoustic ranging array and the equations to be solved. A derivation of these equations is given in the Appendix to "Probabalistic Model for Acoustic Sound Ranging", ECOM-5159, October 1967 by the author. Briefly, assuming the wave front to be a plane wave at ten degrees centigrade with no wind, the time difference, t_i, in acoustic signal arrival times at two adjacent microphones divided by s, the acoustic travel time between these adjacent microphones, gives the sine of the angle, θ_i, between the normal to the plane wave and the normal to the microphone array. Any two of these rays can then be solved for intersection point. Five such rays will intersect at ten points. The average of these ten points will give a preliminary location from which approximate distances, R_i, to the midpoints between the microphones can be calculated. Next, corrections for wind, temperature, and wave front currature are applied to the t_i and the computations repeated. For

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most of the area in front of the microphones one iteration is sufficient.

As the tapes were digitized, plotted, and first motion read, it became evident that all was not well. Table 1 shows a tabulation of the differences in signal arrival times at the five microphones (not in a straight line) recorded at Van 6015. As mentioned above the analog signals were digitized at 1000 samples per second and first motion was, in general, well defined. It was estimated that the one sigma time reading error should fall between one and two milliseconds. This is almost an order of magnitude less than the seven to ten milliseconds rms variation shown in Table 1. Table 2 summarizes the recordings made at another five microphones during the same period of time at Van 6026. Again the rms variation is too great. In addition, a statistical analysis will show the essential randomness of these residuals.

A few moments reflection will show that nothing we have presented so far will account for these results. The firing point and microphone locations were fixed and errors in these would cause no variation in the time recordings. Temperature changes would apply equally to all parts of the acoustic wave front and variations from microphone to microphone should be highly correlated. The same should be true if the wind across the array changed. During the time this data was assembled, acoustic ray trace calculations were underway to find out if the observed time differences could be checked by this method and to study the effect of the ray path on the meteorological corrections to be applied. Very typical results are shown in Table 3. Although complete horizontal stratification was assumed in the winds and temperatures, a common assumption in acoustic ray tracing, due to the large azimuth changes from one ray to another and varying distances,each ray travels a path characterized by different parameters. The effective temperature for each path was obtained by dividing the distance from the fire point to the microphone by the time as shown by acoustic ray tracing for the sound to propagate between these two points and then determining a mean temperature based on this propagation velocity. Similarly, the wind displacement normal and parallel to the baseline for each increment of ray path was summed and divided by the elapsed time to give the effective wind components.

Reexamining the equations of Figure 1 with the data from Table 3 in mind, it can be seen that implicit in these equations is the assumption that there exists a unique temperature and a unique wind velocity vector valid for the entire area in front of the microphone array and that if these were exactly known proper corrections could be made for wind and temperature. Table 3 indicates that in addition to errors in estimating the effective temperature and effective wind component parallel to the baseline (the wind component normal to the baseline does not appear in the equations) and to random timing errors and microphone placement errors, there exists errors due to the variations in effective temperature and in both components of the effective wind velocity vector from ray path to ray path.

The ratio of the rms acoustic source location error in meters to the rms gaussian timing error in milliseconds was examined by choosing a point in the XY plane, calculating the acoustic propagation time to each microphone, adding to this elapsed time a gaussian random number (with zero mean and specified standard deviation), substituing the perturbed times into the sound ranging equations, and calculating the apparent acoustic source location. After storing the source location errors in X and Y and the timing errors, a new set of timing errors was drawn and the process repeated until 100 samples had been produced. The ratio of the rms output error to the rms input error was then determined for that point in the XY plane. Calculations were made at sufficient points to permit plotting the error contour curves shown in Figure 2.

The effect of microphone placement errors was examined by generating two random numbers for each microphone location, one uniformly distributed between 0 and 360 to represent the angle with the baseline, the other with a gaussian distribution to represent the magnitude of the microphone placement error. Microphone positions corresponding to these errors in placement were calculated as were the acoustic arrival times to these new locations. These arrival times were than substituted into the array equations, where they were used as if the microphone were at the original locations. The resulting acoustic source locaton errors were stored as well as the input microphone placement errors and the process repeated until a sample of 100 had been determined. The ratio of the rms acoustic source location error to the rms microphone placement error was determined for enough points to permit drawing the contours of Figure 3.

The equations for sound ranging assure that there is an effective mean temperature over the entire area in front of the array. Since temperature is a single number applied to the entire array, a bias type of error results. It is only necessary to make a single computation at any point in the XY plane to determine the error in acoustic source location per degree centigrade. This error has both magnitude and direction. The error contours of Figure 4 are based on magnitude only. There will also be a unique direction associated with each point in the plane but no attempt has been made to plot this.

If it is assumed as indicated in Table 3 that each ray from the acoustic source to a microphone encounters slightly different atmospheric conditions, then there will be small variations from ray path to ray path in effective temperature. To obtain the error contours of Figure 5, the mean temperature of 10°C for the entire array was perturbed for each ray be adding a random variation drawn from a gaussian population having a mean zero and standard deviation of .1 °C. Acoustic arrival times based on these perturbed temperatures were than plugged back into the sound ranging equations and the error in acoustic source location determined. Again a sample of 100 such calculations for a given point resulted in a reasonably stable ratio of rms error to rms input error.

As with temperature, the equations for sound ranging assume there is an effective mean wind over the area between the acoustic source and the microphone array. This is simplified by the fact that only the wind component parallel to the baseline enters into the correction equations. That the wind component perpendicular to the baseline cancels out is shown in the ECOM Report 5159 references earlier. Since only a single number is involved the error is gias type error having magnitude and direction at each point in the plane in front of the microphone. If an error of one meter per second in assumed wind is used the error contours (magnitude only) of Figure 6 result. There will be a unique direction for each point in the plane. No attempt has been made to show this.

When calculations are based on perturbed effective wind velocity for each acoustic ray it is necessary consider variations in the wind component normal to the baseline as well as variations in the component parallel to the baseline. The procedure for producing the error calculations from which Figures 7 and 8 are derived resembles those using temperature variations except that now the random numbers drawn represent component wind perturbations.

The importance of including terms representing the perturbations in temperature and wind velocity over the area in front of the microphone array can be seen by comparing Figure 4 with Figure 5 and Figure 6 with Figures 7 and 8. From the magnitude of the contours shown is obvious a small variation from ray path to ray path in mean effective wind or mean effective temperature will have considerably greater effect than a similar error in estimating the mean effective wind or the mean effective temperature. Figures 2 through 8 indicate that the following parameter variations produce approximately equivalent acoustic source location errors:

(1) 5 millisecond rms timing errors
(2) 2.5 meter rms microphone placement errors
(3) 1°C error estimating effective temperature
(4) .1°C rms variation from one path to another in effective temperature
(5) 1 meter per second error in estimating the component of the effective wind parallel to the baseline
(6) .2 meter per second rms variation from one path to another in the component of the effective wind parallel to the baseline
(7) .1 meter per second rms variation from one path to another in the component of the effective wind normal to the baseline.

Since effects (4), (6) and (7) will show up as random variations in acoustic arrival times at each of the microphones, they are sufficient to account for the random variations shown in Tables 1 and 2.

Table 4 deserves special mention. Here three of the minmum time rays rose to a peak height of about 40 meters, the other three to a peak height of approximately 150 meters. With the large differences in effective wind and effective temperature between the two groups, the equations for sound ranging have no valid solution.

TABLE 1

DIFFERENCES IN SIGNAL ARRIVAL TIMES
AT ADJACENT MICROPHONES

Data Recorded 3 Feb. '67
Van 6015

Signal	Mike 2-Mike 1 Milliseconds		Mike 4-Mike 2 Milliseconds		Mike 3-Mike 4 Milliseconds		Mike 5-Mike 3 Milliseconds	
	Difference	Res.	Difference	Res.	Difference	Res.	Difference	Res.
1	553	- 9	1047	- 4	637	- 3	559	+ 1
2	558	- 4	1045	- 6	637	- 3	602	+ 4
3	551	- 11	1073	+ 22	630	- 10	589	- 9
4	552	- 9	1056	+ 5	630	- 10	601	+ 3
5	573	+ 11	1046	- 5	648	+ 8	604	+ 6
6	559	- 3	1044	- 7	645	+ 5	595	- 3
7	570	+ 8	1053	+ 2	652	+ 12	615	+ 17
8	574	+ 12	1047	- 4	647	+ 7	605	+ 7
9	572	+ 10	1059	+ 8	632	- 8	609	+ 11
10	560	- 2	1065	+ 14	641	+ 1	600	+ 2
11	581	+ 19	1041	- 10	658	+ 18	600	+ 2
12	556	- 6	1047	- 4	636	- 4	591	- 7
13	561	- 1	1044	- 7	639	- 1	594	- 4
14	549	- 13	1050	- 1	634	- 6	590	- 8
15	566	+ 4	1039	- 12	644	+ 4	590	- 8
16	543	- 19	1065	+ 14	626	- 14	593	- 5
17	*		*		*		*	
18	555	- 7	1052	+ 1	640	0	590	- 8
19	576	+ 14	1040	- 9	647	+ 7	597	- 1
Mean	562	± 10	1051	± 9	640	± 8	598	± 7

* Signal 17 overlooked in digitizing analog tape.

TABLE 2

DIFFERENCES IN SIGNAL ARRIVAL TIMES AT ADJACENT MICROPHONES

Data Recorded 3 Feb. '67
Van 6015

Signal	Mike 5-Mike 1 Milliseconds		Mike 2-Mike 5 Milliseconds		Mike 4-Mike 2 Milliseconds		Mike 3-Mike 4 Milliseconds	
	Difference	Res.	Difference	Res.	Difference	Res.	Difference	Res.
1	664	+ 13	323	- 9	633	+ 1	836	+ 4
2	662	+ 11	319	- 15	632	0	846	+ 14
3	*		343	+ 9	623	- 9	839	+ 7
4	652	+ 1	338	+ 4	633	+ 1	834	+ 2
5	634	- 17	337	+ 3	633	+ 1	829	- 3
6	650	- 1	340	+ 6	624	- 8	845	+ 13
7	650	- 1	331	- 3	643	+ 9	822	- 10
8	651	0	346	+ 12	630	- 2	828	- 4
9	654	+ 3	333	- 1	631	- 1	831	- 1
10	645	- 6	355	+ 21	620	- 12	826	- 6
11	654	+ 3	329	- 5	628	- 4	838	+ 6
12	649	- 2	321	- 13	646	+ 14	823	- 9
13	650	- 1	357	+ 23	625	- 7	841	+ 9
14	643	- 8	338	+ 4	622	- 10	844	+ 12
15	657	+ 6	318	- 16	645	+ 13	811	- 21
16	651	0	322	- 12	646	+ 14	821	- 11
17	650	- 1	323	- 9	629	- 3	830	+ 2
18	642	- 9	344	+ 10	631	- 1	824	- 8
19	654	+ 3	320	- 14	625	- 7	833	+ 1
Mean	652	± 7	334	± 12	632	± 8	832	± 9

* Unable to find signal 3 in analog record from Microphone 1.

TABLE 3
DATA OBTAINED BY ACOUSTIC RAY TRACING FROM SOUND SOURCE TO EACH OF SIX MICROPHONES

Microphone	TS-803	TS-804	TS-805	TS-808	TS-809	TS-810
Minimum time from explosion to microphone (seconds)	26.340	26.440	26.653	26.990	27.437	28.002
Peak Height reached above ray (meters)	37.	37.	38.	38.	38.	38.
Effective temperature (0^{0}C)	12.8	12.7	12.6	12.5	12.4	12.5
Effective Wind Component Parallel to Baseline (meters per second)	0.5	0.7	0.9	1.1	1.3	1.5
Effective Wind Component Normal to Baseline (meters per second)	2.6	2.8	3.0	3.1	3.3	3.5

TABLE 4
DATA OBTAINED BY ACOUSTIC RAY TRACING FROM SOUND SOURCE TO EACH OF SIX MICROPHONES

Microphone	TS-803	TS-804	TS-805	TS-808	TS-809	TS-810
Minimum time from explosion to microphone (seconds)	26.330	26.426	26.646	26.940	27.384	27.934
Peak height reached by above ray (meters)	37.	37.	38.	142.	150.	151.
Effective temperature (0^0C)	12.9	12.8	12.7	13.3	12.9	12.5
Effective Wind Component Parallel to Baseline (meters per second)	-1.6	-1.6	-1.6	1.1	1.6	1.8
Effective Wind Component Normal to Baseline (meters per second)	.15	.14	.14	3.2	3.6	3.7

$$\theta_i = \arcsin\left(\frac{t_i}{s}\right)$$

$$x = x_i + R_i \sin\theta_i$$

$$y = y_i + R_i \cos\theta_i$$

$$t_w = \frac{w}{v} s \cos\phi$$

$$t_Q = t_i\left(\sqrt{(273 + Q)/283} - 1\right)$$

$$t_C = t_i\left(\sqrt{1 + (s^2 + t_i^2)v^2/R_i^2} - 1\right)$$

$$t_i' = t_i + t_w + t_Q + t_C$$

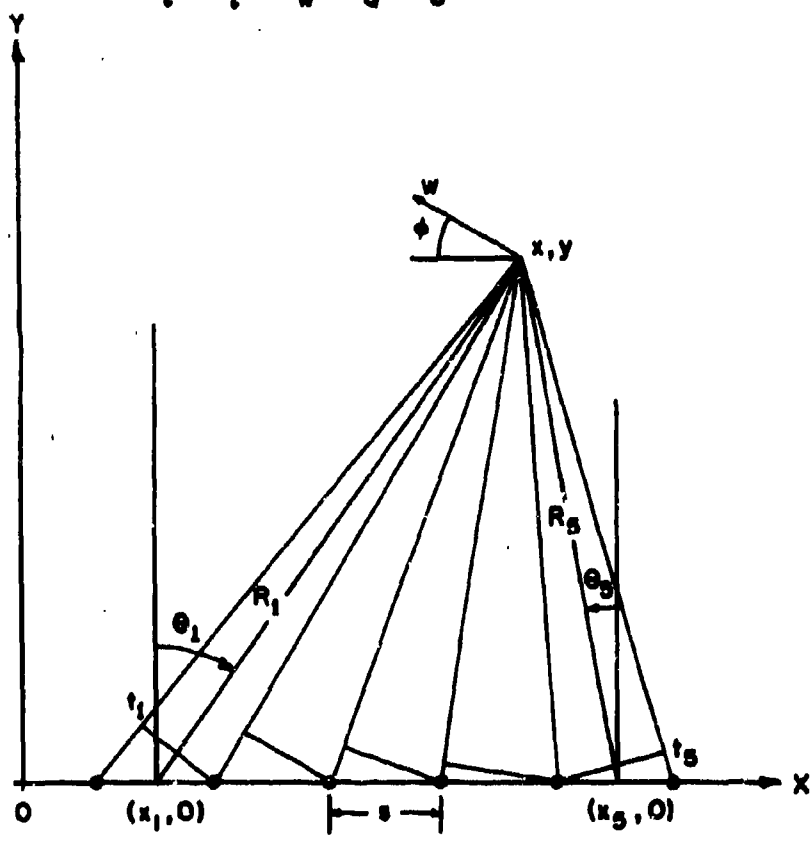

FIGURE 1
GEOMETRY OF A PLANE WAVE SOLUTION

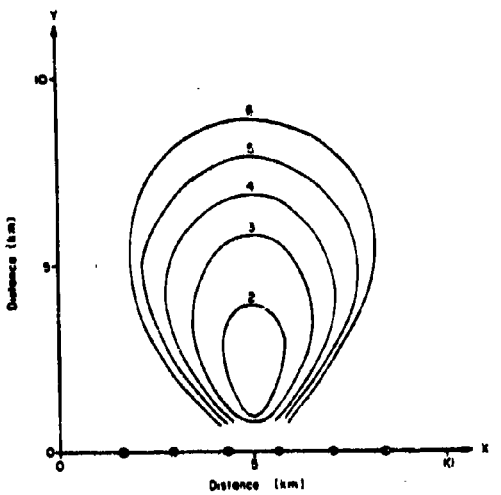

FIG 2 RATIO OF RMS ACOUSTIC SOURCE LOCATION ERROR (METERS) TO THE INPUT RMS TIMING ERROR (MILLISECONDS)

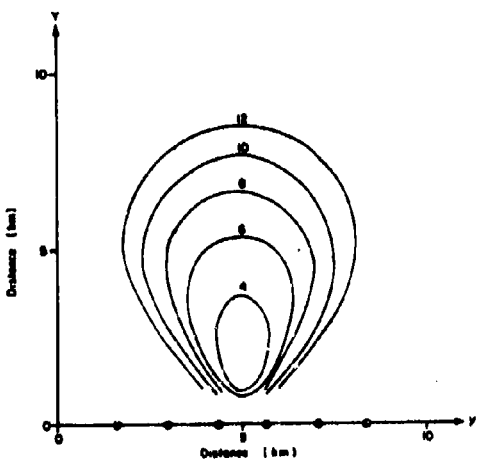

FIG 3 RATIO OF RMS ACOUSTIC SOURCE LOCATION ERROR (METERS) TO RMS MICROPHONE POSITION ERROR (METERS)

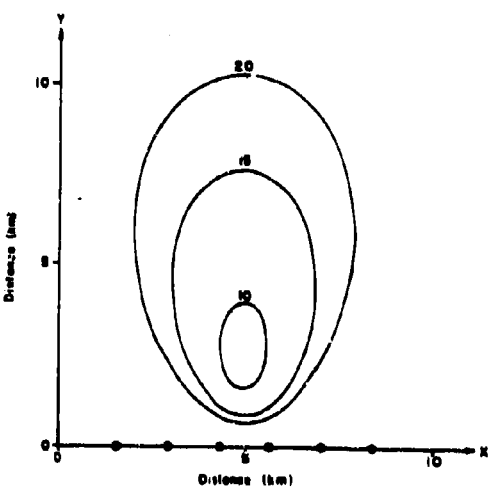

FIG 4 RATIO OF THE MAGNITUDE OF ACOUSTIC SOURCE LOCATION BIAS ERROR (METERS) TO EFFECTIVE TEMPERATURE ERROR (°C)

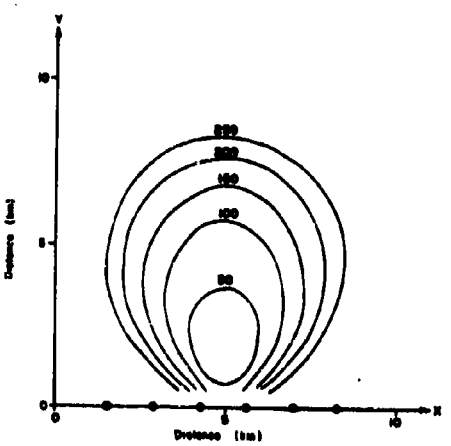

FIG. 5 RATIO OF THE RMS ACOUSTIC SOURCE LOCATION ERROR (METERS) TO RMS VARIATION IN EFFECTIVE TEMPERATURE (°C) FROM RAY PATH TO RAY PATH.

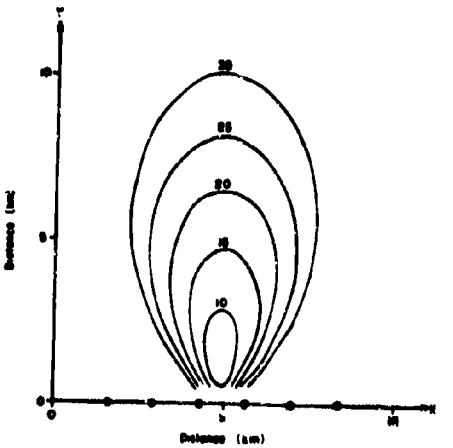

FIG. 6 RATIO OF THE MAGNITUDE OF ACOUSTIC SOURCE LOCATION BIAS ERROR (METERS) TO ERROR IN EFFECTIVE WIND COMPONENT (METERS/SEC) PARALLEL TO THE BASELINE.

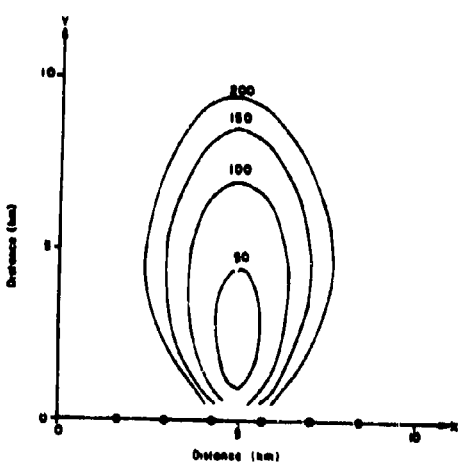

FIG. 7 RATIO OF RMS ACOUSTIC SOURCE LOCATING ERROR (METERS) TO RMS VARIATION IN EFFECTIVE WIND COMPONENT (METERS/SEC) PARALLEL TO THE BASELINE.

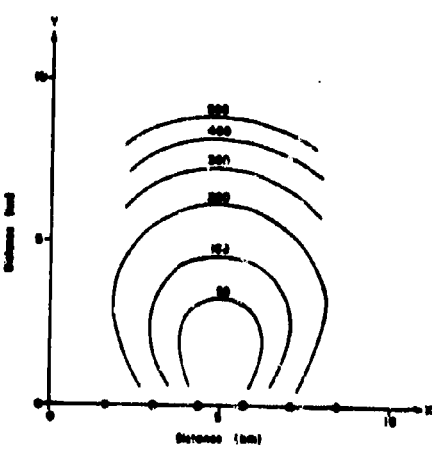

FIG. 8 RATIO OF RMS ACOUSTIC SOURCE LOCATION ERROR (METERS) TO RMS VARIATION IN EFFECTIVE WIND COMPONENT METERS/SEC) NORMAL TO BASELINE.

ON EXPERIMENTS CONCERNED WITH THE SAMPLING DISTRIBUTION OF LANCHESTER'S PARAMETERS

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Research modeling activities of the past five years have led to the construction of a number of computerized war gaming models. These are simulations of combat carried on in more or less detail, and carried out under the guidance of player groups who prescribe computer input both at the start of and during the simulation. Such a simulation might represent the activities of opposing Red and Blue Divisional forces accounted for down to the company and battery level of resolution. The simulation proceeds in accordance with player input and assessment rules internal to the computer which are generally probabilistic in nature.

It is often desired to use such gaming models to compare the relative effectiveness of some aspect of organization, tactics, equipment, supply, or armament of a combat organization. The procedure would be to game two or more alternative possibilities and then somehow to compare the results obtained from the game. In this case, the complex probabilistic nature of these models raises questions of experimental design, since the outcome of a given game would vary over repeated trials.

In a deterministic war game, the problem of variability does not exist. For example, consider the Lanchester "Laws" of combat. These are:

1. Lanchester's first linear law:

$$\frac{dx}{dt} = k_1$$

$$\frac{dy}{dt} = k_2$$

In this case, the attrition of the strength of the force x, dx/dt is proportional to some environmental constant k_1, unrelated to the strengths of the combattants.

2. Lanchester's square law:

$$dx/dt = k_1 y$$

$$dy/dt = k_2 x$$

Here, the proportion is to the strength of the opponent.

3. Lanchester's second linear law:

$$dx/dt = k_1 xy$$

$$dy/dt = k_2xy$$

The first and third law are both called linear since in both cases

$$dx/dy = k_1/k_2 \text{ (a constant)}$$

therefore, the graph of x against y is linear.

In the case of a supposed Lanchester model experimental design would be simple. It is only necessary to play a game through two time periods. This will provide the data necessary to calculate k_1 and k_2.

However, in probabilistic models the problem is far more complex. No explicit mathematical model seems capable of being prescribed.

In a recent paper, (1) David G. Smith has explored the possibility of relating Lanchester Theory to the study of simulation results. The idea is to suppose that the underlying process of warfare (or war simulation) is reasonably depicted by a Lanchester formulation, but that the results actually observed in terms of measurable characteristics of war (in Smith's case, casualties) are taken from sampling probability distribution around the pure Lanchester form.

Giving Smith's results for the linear law; using the sides with initial strengths m and n:

$A(m,n)$ Prob. that red will sustain next casualty.

$P(x,y,m,n)$ Prob. that point x,y, is reached from m,n.

$P(R,m)$ Prob. of red win.

for $dm/dt = -\alpha f_1(m,n)$

$dn/dt = -\beta f_2(m,n)$

(1) $A(m,n) = \rho f_2(m,n)/[f_1(m,n) + \rho f_2(m,n)]$, $\rho = \frac{\beta}{\alpha}$

or in the case of the linear law

(2) $A(m,n) = \rho/(1 + \rho)$

and

$$P(x,y,r,s) = A(x,y+1)\ P(x,y+1,r,s) + 1 - A(x+1,y)\ P(x+1,y,r,s) \quad (3)$$

leading for the linear law:

$$P(x,y,m,n) = \binom{m + n - x - y}{m - x} \cdot \left(\frac{\rho}{1 + \rho}\right)^{n - y} \cdot \left(\frac{1}{1 + \rho}\right)^{m - x} \quad (4)$$

$$\bar{P}(\bar{R},m) = \sum_{s=n}^{m+n-1} \binom{m+n-1}{s} \frac{n^s}{(1+\rho)^{m+n-1}} = I_p(m,n) \qquad (5)$$

where $I_p(n,m)$ = Incomplete Beta Functions and $p = \frac{\rho}{1+\rho}$

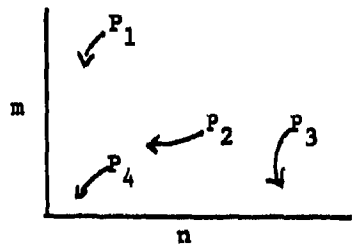
Smith has had some success in applying these formulations to the results of war games.

In Smith's case, these results pertain to human casualties as simulated. Supposing these to be of primary importance, he suggests the use of p as a measure to the effectiveness of m over n. Since a measure of military effectiveness is of great importance in operations research it is of interest to see whether this approach can be extended to cover situations of greater complexity.

The first difficulty is that of estimating p in practical cases. Formula (5) offers an apparent method since there the problem is one of a familiar distribution parameter. H. Weiler in reference 2 provides methods of estimation. However, this requires that the simulation continue until a side is annihilated. Except in small-unit cases, this does not occur.

Equation (4) might seem to offer possibilities since it is independent of the final outcome and might be amendable to the sort of treatment prescribed for Bernoulli Procces in Raffia and Schlaiffer (Reference 3). The details need to be worked out.

Considerable practical difficulty surrounds the construction of an experimental plan. Using an existing computerized war gaming model, such as STAG's LEGION model, a very great variety of initial conditions can be arranged. If these can be standardized on some basis, the problem might be reduced to one as shown on the diagram.

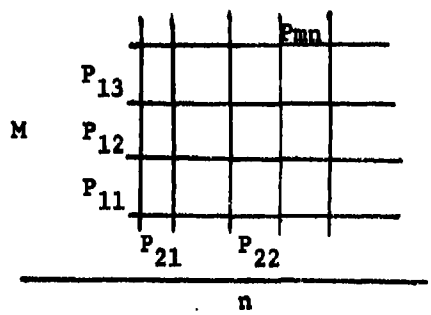


The space spanned by the m and n axes represents the strength of the m and n forces. A point in the space such as P, represents a starting point for a battle. The line leading from P, represents a possible course of battle, subsequent strengths of the two sides.

If either of Lanchester's "linear" laws were a valid representation of the process, the straight line leading from P_4 would be descriptive. Also, the initial point chosen, P, would be immaterial since the laws of all possible battles would be a family of parallel lines. (Battles, that is, between opponents with relative effectiveness). In this case, it would be sufficient to obtain an estimate of ρ, the slope of the line. A least squares estimate

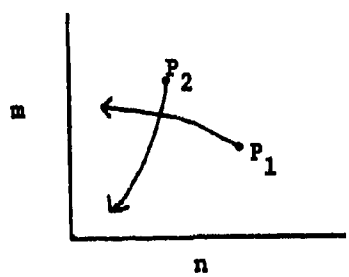
based on all available points would suffice.

However, if the object of the experiment is to study the appropriateness of the linear, as well as other models, there seems to be no prior reason for preferring any particular starting point as an experimental tool, since it might be argued that every P could be associated with a unique locus. Such considerations would lead to the experimental design:



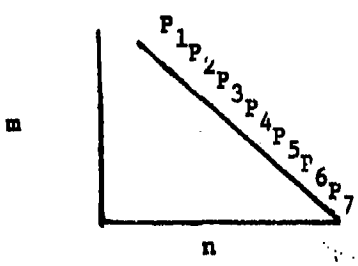
Since the quarter space is unbounded, the design is arbitrarily large, not an encouraging state of affairs.

If, on the other hand, one might suppose that the following condition cannot reasonably exist.

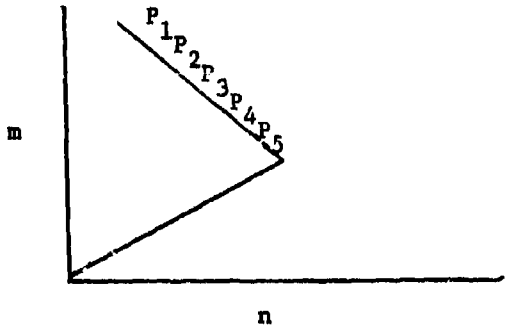


Hypothesis: Every point lies on a unique locus of battle (supposing m and n to be true indices of force).

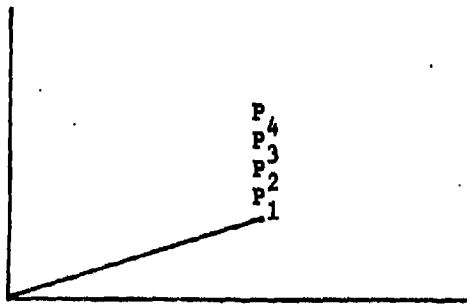
This suggests the following experimental design



But since m and n are interchangeable symmetry would be expected suggesting the following design:



A drawback of this design seems to be that in the case of the linear law or anything similar to it, the design is prejudicial toward starting points close to the diagonal. To correct this, we might choose something like:



The experimental design also requires consideration of the size of time interval and number of internals to be followed from each origin.

The foregoing seems to indicate the need for considerable research using war gaming models to investigate the merits of various models concepts, and develop a methodology for conducting studies using these models.

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1. Smith, David G. "The Probability Distribution of the Number of Survivors in a Two-Sided Combat Situation - Comparison with Experimental Data" Operational Research Quarterly 16.429

2. Weiler, H. "The Use of Incomplete Beta Functions for Prior Distribution in Binomial Sampling" Technometrics Vol 7 No 3 August 1965

3. Raffia H. and Schlaiffer R. "Applied Statistical Decision Theory" Harvard University 1961

ESTIMATES OF P(Y<X) AND THEIR APPLICATION TO RELIABILITY PROBLEMS FOR BOTH CONTINUOUS AND QUANTAL RESPONSE DATA

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1. Introduction. In this manuscript, we provide a brief summary of two papers, which will be published in more complete form elsewhere.

We assume that X and Y are independent random variables with cumulative distribution functions $F_X(x)$ and $G_Y(y)$ respectively. The distribution of Y is assumed to be known and a random sample of n observations distributed as X is obtained, say $X_1, X_2, \ldots, X_n$. The objective is to estimate $P\{Y<X\}$.

Two models for this problem are studied. In both models, $F_X(x)$ is assumed to be an absolutely continuous cumulative distribution function. In the case of the first model, the values of the random variables $X_1, X_2, \ldots, X_n$ are directly observed and we refer to this as the continuous model. In the second model, n real numbers, $y_1, y_2, \ldots, y_n$, are selected by the experimenter, who then only acquires the information $X_i < Y_i$ or $X_i > Y_i$, $i = 1, 2, \ldots, n$, from his experiment. The case $X_i = y_i$ can be ignored, since this is an event of probability zero as a consequence of the assumption that $F_X(x)$ is absolutely continuous. This model is referred to as the quantal response model.

In both models we will specifically assume that both X and Y are normally distributed.

2. Physical Background of the Problem. Suppose that X is the strength of a component which is subjected to a stress Y. Then, the component fails whenever $X \leq Y$ and there is no failure when $Y < X$. In addition, the stresses may be expensive to sample, such as might be the case in missile flights, but the physical characteristics of the missile system, such as the propulsive force, angles of elevation, changes in atmospheric condition, and so on, may all have known distributions, consequently the distribution of the stresses may be calculated. Therefore, we have assumed that $G_Y(y)$ is known. In addition, when the distribution of Y is known, this permits estimating $P\{Y<X\}$ by laboratory experiments measuring the strengths of components without recourse to extensive flight testing.

In view of this physical model, $P\{Y<X\}$ is the probability that a component whose strength is distributed by $F_X(x)$ does not fail when subjected to the random stress Y distributed by $G_Y(y)$. Thus, it is natural to call $P\{Y<X\}$ the reliability of the component and we denote this by R and estimators will be generally denoted by $\hat{R}$.

3. A Summary of Results for the Continuous Model. When X and Y are both normally distributed, we can take $G_Y(y)$ to be the standard normal distribution $\Phi(y)$. Then,

$$R = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right), \tag{1}$$

where μ and σ^2 are the mean and variance of X respectively. Thus a natural choice of estimator for R is

$$\hat{R} = \Phi\left(\frac{\bar{X}}{\sqrt{1+s^2}}\right) \tag{2}$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$. It can be shown that $V = \frac{\bar{X}}{1+s^2}$ is asymptotically normally distributed with mean $\mu_V = \frac{\mu}{\sqrt{1+\sigma^2}}$ and variance

$$\sigma_V^2 = \frac{\sigma^2}{1+\sigma^2}\left(\frac{1}{n} + \frac{\mu^2\sigma^2}{2(n-1)(1+\sigma^2)^2}\right). \tag{3}$$

Thus V is a consistent estimator of $\mu/(1+\sigma^2)$ and by continuity, it follows that $\hat{R}$ is a consistent estimator of R. From these remarks, one can readily write down one and two-sided confidence intervals for R as follows:

$$P\{R > \Phi(V-\Phi^{-1}(1-\gamma)\hat{\sigma}_V\} \sim 1-\gamma, \tag{4}$$

$$P\{\Phi(V-\Phi^{-1}(1-\tfrac{\gamma}{2})\hat{\sigma}_V < R < \Phi(V-\Phi^{-1}(1-\tfrac{\gamma}{2})\hat{\sigma}_V\} \sim 1-\gamma, \tag{5}$$

where $\hat{\sigma}_V^2$ is obtained from (3) by replacing μ, σ^2 on the right hand side of (3) by the estimates $\bar{X}$, s^2. Since $\hat{\sigma}_V - \sigma_V = 0(n^{-1})$, (4) and (5) are satisfactory approximate confidence intervals for R. The asymptotic distribution of $\hat{R}$ is given by

$$P\{\Phi(V) < u\} \sim \Phi\left(\frac{\Phi^{-1}(u)-E(V)}{\sigma_V}\right). \tag{6}$$

4. <u>A Summary of Results for the Quantal Response Model</u>. For fixed $y_1 \le y_2 \le \ldots \le y_n$, not all equal, the likelihood function is given by

$$L(\mu, \sigma^2) = \prod_{i=1}^{n}\left[\Phi\left(\frac{y_i-\mu}{\sigma}\right)\right]^{u_i}\left[1-\Phi\left(\frac{y_i-\mu}{\sigma}\right)\right]^{1-u_i} \tag{7}$$

where $u_i = 1$ if $X_i \le y_i$ and $u_i = 0$ if $X_i > y_i$, $i = 1, 2, \ldots, n$. Analogously with (2), we propose to use

$$\hat{R} = \Phi\left(\frac{\hat{\mu}}{\sqrt{1+\hat{\sigma}^2}}\right)$$

Here $\hat{\mu}$ and $\hat{\sigma}^2$ are the maximum likelihood estimates of μ and σ^2 determined from (7).

Before stating a theorem on the existence of proper maximum likelihood estimates of μ, σ^2, that is $-\infty < \hat{\mu} < \infty$ and $0 < \hat{\sigma}^2 < \infty$, it is worthwhile to exhibit the type of sample sequences which lead to improper estimates of μ and σ^2. The sample sequence u_i, $i = 1, 2, \ldots, n$ is a collection of n ordered ones and zeros, which we denote by $\tilde{u}$. If $\tilde{u} = (0, 0, \ldots, 0)$, this means that no component failed, hence you are lead to conclude that the mean strength is high (relative to $y_1, y_2, \ldots, y_n$) and in fact, the logical estimator $\hat{\mu}$ is $+\infty$. Similarly if $\tilde{u} = (1, 1, \ldots, 1)$, one is inclined to set $\hat{\mu} = -\infty$. Similarly, if $\tilde{u} = (1, 0, 1, 0, \ldots, 1, 0)$, the experiment suggests strongly that the probability of failure does not change as the y's change. In fact, it appears to be about $\frac{1}{2}$, independent of the y_i's. This should suggest that the variance is very large and $+\infty$ is the reasonable choice. Finally, the sequence $(0, 0, \ldots, 0, 1, \ldots, 1)$ suggests a very small variance, which it is reasonable to take to be zero. These degenerate cases do not invalidate the estimation of R. In fact all but the last case can be treated as a binomial sample, in that the probability of failure is essentially constant, that is, independent of $y_1, y_2, \ldots, y_n$, and the usual binomial estimates apply. In the last case, X has a very sharply peaked distribution with almost all its mass located between the last zero and the first one, and thus R can be readily estimated since $G_Y(y)$ has been assumed known. From these intuitive considerations, the following theorem is suggested.

Theorem A necessary and sufficient condition that $\hat{\mu}$ and $\hat{\sigma}^2$ be finite is that the correlation between $\tilde{u}$ and $\tilde{y} = (y_1, y_2, \ldots, y_n)$ is positive.

Intuitively, this says that the probability of a failure should appear to increase as the y's increase.

To obtain the estimates $\hat{\mu}$ and $\hat{\sigma}^2$, we reparametrize by setting $\omega_1 = \frac{\mu}{\sigma}$ and $\omega_2 = \frac{1}{\sigma}$. Then, $L_1(\omega_1, \omega_2)$ given by replacing μ and σ in (7) by ω_1/ω_2 and $1/\omega_2$ respectively is a strictly concave function and has a unique maximum which can be determined by any of a variety of numerical methods'. Substituting in $\hat{R}$ gives the estimate for R.

NUMBERS NEEDED FOR DETECTING IMPORTANT DIFFERENCES FOR COUNT DATA

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Chi-square tests are widely used in testing statistical questions where an actual computed variance can be compared with an expected variance defined on the theory under test. Percentage counts and population counts fall in this class; in the simplest model of each the expected variance is that of the binomial and Poisson respectively. Chi-square is thus useful for enumerations, but cannot be used directly for measurement statistics, with no theoretical expectation for variance. Enumerations arise commonly in measuring biological populations, in comparing alternative forms of treatment or in comparing frequencies of accidents or other chance occurrences.

Where the actual internal variance on the model chosen exceeds the theoretical variance, chi-square tests may show a significance not borne out by repeated work. Caution must thus be used. Snedecor[6] (1956) shows procedure in comparing population counts; of making sure the main experimental treatments do not show significant internal chi-squares between subsamples (thus are "homogeneous"). The main treatments can then be compared by chi-square. A quick chi-square test will often be time-saving; if populations do not show differences by chi-square, they will not show significant differences by any test.

While various forms of comparison of variances are carried out as chi-square tests, the most frequent form is a somewhat approximate test of frequency distributions. An actual distribution of numbers in several classes is compared with counts expected by some theory; or two or more sets of classes are compared to see if they differ, with their average serving as the theoretical distribution. The chi-square is defined as the sum of ratios $(O-C)^2/C$, where C is the calculated and O the observed number in each class. The test is related to Poisson expectation.

Holt[4] et al (1967) have recently published an article on numbers required for chi-square tests in forest insect work. They show formulae for estimating the numbers which would bring a non-significant chi-square to significance <u>if</u> the difference already found held up in further sampling. Examples used include a comparison of actual frequencies to a negative binomial, and a 2 X 2 comparison of sex differences in response to two attractants. Both are on insect data, and both are suited to use of "two-tail" probabilities. "One-tail" odds are adapted where the interest is only in a difference in one direction.

It would seem of more value to attempt to define numbers to detect a real difference of a given size, if present, than to define numbers which would make a certain observed difference significant. With greatly enlarged numbers almost any trifling difference will test as significant. A useful aim is to define numbers which will practically ensure detection of any important difference, but will not waste effort in detecting unimportant differences. The level of difference to be regarded as "unimportant" will depend on the experimental problem.

A set of counts (Fleming & Baker[2], 1936) of Japanese beetle larvae on individual square feet is available for study. Two areas of 375 square feet each were defined, which seemed fairly homogeneous. The total count for each area yielded averages of almost exactly 3 and 5 larvae per square foot respectively.

Random samples of ten individual square feet were taken in each area. Sample means were 2.9 and 4.9, with variances of 4 to 6. Chi-square between the two totals was about 5, definitely significant at 5%. In the Poisson the variance equals the mean. Hence, in the plot with a population mean infestation of 3 larvae per square foot, the variance will also be 3 in the absence of additional factors inflating the variance, and the 5 per square foot plots will have a variance of 5. Use of these theoretical Poisson errors in a *t* test yielded a t of about 2.2, corresponding well to the equivalent chi-square test. In this case 10 units of each were sufficient to show the observed difference to be significant. This observed difference is very close to the true difference in this case.

In further study, 20 samples of 7 units each were taken from each population. Each sample was later expanded to 10, next to 12 and then to 26 units. Chi-squares were calculated between the 20 pairs of samples in each case.

TABLE I

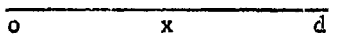
RESULTS OF TESTS

No. of Units per Sample	Number of Chi-squares		
	Significant	Non-significant	Marginal
7	11	8	1
10	14	5	1
12	16	4	0
26	19	1	0

The 7-unit samples are evidently near the level at which half will be significant. With 7 pairs of units, one from the 3 per square foot population, one from the 5, the expected total will be 21 + 35 = 56 larvae. Using the expected Poisson variance, the variance of the difference of the sample totals

will also be approximately 56. On an individual square foot basis this becomes 8; on a basis of means of 7 units, 1.14 (Snedecor, section 3.6). Actually, 8 square foot counts per sample meet the half-significance level in theory.

As the observed results of Table I show, the number of significant chi-squares varies as the number of square foot units per sample does. If we wish to be quite sure to detect a difference as large as 2 larvae per square foot in the rate of infestation it will be necessary to sample a large number of units. The situation is diagramed below.



In a given but very large sample from two populations whose two counts differ by d units, the observed difference, x, will be held close to the true value, d, with high probability. At the same time, a test of significance evaluates the departure of x from zero, also with high probability since the sample is large. Reversing the argument, we may choose the significance test probability level and also the probability level that x will be sufficiently large to show significant departure from zero and then calculate how large the sample must be. There is only one chance process involved--not two, one with a true difference d and one with zero difference. It is postulated that a true difference of d exists, so this is the one real chance process that yields the observed value of x. The role of the test of significance for zero difference is solely to determine by how large an interval the observed chance variable x can fall short of the population mean d and yet yield a decision of significance against the null hypothesis. Thus, in the actual experiment, that value of sample size N is sought which ensures that the observed difference will fall in the interval from d-x to infinity with the chosen probability.

The distribution of the difference of two Poisson variables when a true difference exits is complex.[3,5] But in most cases the normal approximation will be entirely adequate[1] and here the existence of a true difference does not disturb the normality of the distribution. If, for example, a 95% confidence level is chosen that, if a true difference of d units exists, the null hypothesis will be rejected, then the value of N is so chosen that in only 5% of the trials, the observed difference will fall to the left of x in the diagram. The test is hence a one-tailed test. The usual test becomes

$$t_1 = \frac{\bar{x}_1 - \bar{x}_2}{s_d} = \frac{d - x}{s_d} \qquad (1)$$

where $\bar{x}_1$ and $\bar{x}_2$ are the means of the samples from the two areas in the field being compared. $d = 2$ is the population difference we wish to detect. The value of x is set by the null test significance level, as follows. If the usual 5% level is chosen, then

$$t_2 = \frac{\bar{x}_1 - \bar{x}_2}{s_d} = \frac{x}{s_d} \qquad (2)$$

will be the t calculated on the null hypothesis. x is the observed difference in sample means.

Hence $x = t_2 S_d$

From the diagram it is seen that from (1) and (2)

$$d - x + x = t_1 S_d + t_2 S_d$$

$$\text{or} \qquad d = (t_1 + t_2) S_d \tag{3}$$

The null hypothesis test of significance will be referred to standard t tables. At the 5% two-tailed level of significance and infinite degrees of freedom $t_{05} = 1.96$. But the t tables are two-tailed, whereas what is wanted is a one-tailed level of performance (only too small an observed x will be judged nonsignificant). Since the t distribution is symmetric, the value taken from the tables at the 10% level is the 5% level for a one-tailed test. This value is 1.64. Substituting these values in equation (3) gives

$$2 = (1.96 + 1.64) S_d$$

$$\text{or} \qquad S_d = 2 \div 3.6 = 0.56 \tag{4}$$

This leads to a needed variance of $(0.56)^2 = 0.31$ to be fairly sure (95%) of detecting a difference at a 5% significance level.

The discussion above is an elaboration of the argument of Cochran[1] (Appendix to Ladell, 1938). This treatment is reproduced in Cochran and Cox for both one-tail and two-tail tests and with 80, 90, and 95% confidence that the hypothesized true per cent treatment difference will yield a significant test against the null hypothesis.

It remains to apply this result to our data to determine the size of the sample needed to discriminate between two areas, one infested at a rate of 3 larvae per square foot and one at a rate of 5 larvae per square foot. If equal numbers of one square foot units, sampled from each of the areas to be compared for level of infestation, are examined, then the theoretical variance of the difference between the means is:

$$V(d) = \frac{M_1 + M_2}{N}$$

(Snedecor, section 7.10)

$$\text{or} \qquad N = \frac{M_1 + M_2}{V(d)} = \frac{3 + 5}{0.31} = \frac{8}{0.31} = 26 \tag{5}$$

That is, if we wish to discriminate between two fields where one has an infestation rate of 3 larvae per square foot and the other 5 larvae per square

foot, then 26 square feet will have to be examined for each. Of course, the usual requirements for a random selection of the 26 units from the total area of each field will have to be observed.

Practical workers will often have a good idea of the level of infestation economically acceptable, and how much larger the infestation must be to make treatment application worthwhile. Hence, the above approach should often prove useful.

If, however, the infestation rate is taken as derived solely on the basis of a preliminary trial, the formula can be modified to employ the estimated variance from the preliminary trial. This might well be indicated if the investigator felt doubtful about accepting the applicability of the Poisson variance in his work. Suppose a sample of 7 square foot units distributed at random over each of the two areas is taken and that (the Poisson distribution being applicable, as proved true in Fleming and Baker's data) the observed variance of the difference turned out to be 1.14. What is desired is the size of trial needed to discriminate a difference of 2 larvae per square foot in the infestation rate of the two areas. A basic formula is that the variance of a mean is inversely proportional to the number of units in that mean (Snedecor, section 3.6)

$$N_1 V(\bar{x}_1) = N_2 V(\bar{x}_2) = \text{Constant}, \qquad (6)$$

where $\bar{x}_1$ is the mean of N_1 units and $\bar{x}_2$ is the mean of N_2 units from the same basic population.

$$\text{Hence} \qquad N_2 = 7(1.14) \div (0.31) = 26. \qquad (7)$$

The answer in (7) is the same as that in (5) since, the Poisson formula applies in practice. The approach represented by formula (6) is general, however.

It is only because the t-test applies both to equation (1) and equation (2) that it could be used for both. In the general case (for non-normal base distributions) equation (2) would be referred to the non-central form of the applicable distribution.

Further, in the example we used, we only wanted a one-sided test, since we specified which population would be larger, if one were. If instead we had only wished to determine whether the fields could be considered equally infested, and decided that such a conclusion would be acceptable unless the infestation differed by at least two larvae per square foot, the t-probability would be used for 5% significance to accommodate a difference in either direction. Actually of course a reference to the normal table might well be more convenient, where the one-tailed probability for 2.5% would be used to obtain 95% confidence that a true difference of 2 larvae would be detected, where either field could be the more heavily infested.

The theory is verified in Table I by the outcome with 26 units per sample; the 95% expectation is met more exactly than usual. If a smaller difference

is to be detected, a larger number per sample needed can be defined as is done above; a larger difference can be detected by smaller samples. The difference to be detected must be equated to $3.60S_d$ (with the selected probability levels). The general level of population per unit must be approximately known or some preliminary information is needed, perhaps from partial samples because, in the Poisson, the variance is equal to the mean.

The procedure is then to define approximately the level of population and theoretical variance from preliminary samples. The difference to be detected must be specified. This difference is then equated to 3.6 times the needed standard error. With the variance of the mean thus defined, numbers can be specified to yield this variance, and to give high confidence of detection of the specified difference. Numbers are over three times as great as are required for 50% confidence of detection, and still higher for higher confidence levels.

In addition to the comparison of two Poisson counts discussed in detail above, count data arises in the comparison of percentages and in contingency tables. Paulson and Wallace[8] (1947) treat the case of choosing a sample size for the comparison of two percentages. Przyborowski and Wilenski[9] show that when the percentage of successes in each of two series is small compared to the number of observations in each series, the successes alone can be analyzed as if they were observations on a binomial variate, and that the latter provides an exact test for the ratio of parameters from two independent Poisson trials. Hoel[7] (1945), however, showed that an exact test should rarely be needed. The power function for 2 X 2 tables was discussed by Pearson and Merrington.[11] Clark and Downie[12] (1966) provided charts for determining sample sizes for discriminating two proportions at the 50, 80, and 95% probability level. Bennett and Hsu[10] (1960) give the power function for the exact test for the 2 X 2 table. Halperin, Rogot, Gurian and Ederer[13] have prepared tables from which sample sizes required for comparative trials of two forms of repeated treatments can be determined. While intended for medical trials, it is possible that their results could be applied to a situation involving, say, periodic maintenance, or to perennial crop culture or to a regimen for the maintenance of field fertility.

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ON A STATISTICALLY CONSISTENT ESTIMATE OF AN AVERAGE RESPONSE FUNCTION

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ABSTRACT. A rule for selecting runs in a sensitivity test and a method for analyzing the experimental results from those runs are presented in this paper. The analysis techniques provide confidence bounds and limits.

INTRODUCTION. A binomial (or Bernoulli) experiment is one involving the observations of a random variable which represents the outcomes of experimental runs. The outcome of a run has only two attributes which are referred to as go and no-go or as success and failure. A class of binomial experiments, referred to as sensitivity tests, may be described as having the following properties.

1. The experiment involves many specimens which are identical insofar as the experimenter can distinguish.
2. During the experiment, each specimen is subjected to a stimulus which is controlled and measurable. This is a run of the experiment.
3. After the specimen has been subjected to the known stimulus, it is observed to be in one or the other of the two possible outcome states.
4. No specimen is subjected to a stimulus more than once.

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The object of a sensitivity test is to determine the probability, as a function of the applied stimulus, that a specimen will be in one of the two outcome states. We emphasize that, in a sensitivity test, a known amount of stimulus is applied, and then the outcome is observed. There is another class of experiments, called tests to destruction, which may be related to and are often confused with sensitivity tests. In tests to destruction, the applied stimulus is measured that causes the specimen to reach a prescribed state. This is basically a different type of experiment from a sensitivity test even though the prescribed state of the specimen may be labeled success and all other states labeled failure. Thus, the outcome state is known for each run and the random variable is the amount of stimulus required to reach that state. Tests to destruction are not binomial experiments since the observed random variable in such experiments usually possesses a continuum of values. In a test to destruction, one may seek the density function $f(\tau)$ for the random variable τ (the stimulus); and, if there is an equivalent sensitivity test, in which one obtains an estimate for the probability of success $p(\tau)$ as a function of the amount of stimulus, then the following relation exists

$$p(\tau) = \int_{-\infty}^{\tau} f(t)dt .$$

However, given only experimental data for $p(\tau)$ acquired from a sensitivity test, no confidence can be stated for relating a density function to the parameter τ by trying to consider it to be the random variable of a test to destruction. The reason for mentioning tests to destruction is to warn the experimenter that analysis techniques developed for them, those which assume a distribution for the stimulus, are not directly applicable to sensitivity tests.

Often an experiment can only be performed as a sensitivity test; and even when it can be performed either as a sensitivity test or as a test to destruction, sensitivity testing is chosen for reasons of economy and expediency. For this reason, this paper is devoted to developing a rule for selecting runs in a sensitivity test and to an associated method of analysis.

Definitions and Notation. In a sensitivity test, an amount of stimulus τ_k is applied to the k-th specimen and the outcome $\eta(\tau_k)$ is observed where $\eta(\tau_k)$ can assume either the value 1, referred to as go, or the value 0, referred to as no-go. More specifically, we assume that there is a sequence of I values of stimulus τ_i such that

$$\tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{I-1}$$

For each value τ_i, $0 \le i \le I-1$, N_i runs are performed, and

$$N = \sum_{i=0}^{I-1} N_i \qquad (1)$$

is the total number of runs for the sensitivity test. Thus

$$\eta_j(\tau_i) = \begin{cases} 0 & \text{for no-go} \\ 1 & \text{for go} \end{cases} \tag{2}$$

is the outcome of the j-th run, $1 \le j \le N_i$, for $\tau=\tau_i$. The number of go's, $n_{N_i}(\tau_i)$, in the N_i runs for $\tau=\tau_i$ is given by

$$n_{N_i}(\tau_i) = \sum_{j=1}^{N_i} \eta_j(\tau_i). \tag{3}$$

The probability of go for $\tau=\tau_i$ is designated by $p(\tau_i)$, $0 \le p(\tau_i) \le 1$; and the expected number of go's for $\tau=\tau_i$ given N_i runs is

$$E[n_{N_i}(\tau_i)] = N_i p(\tau_i). \tag{4}$$

An estimate of the probability of go, $\hat{p}_{N_i}(\tau_i)$, for $p(\tau_i)$ is given by

$$\hat{p}_{N_i}(\tau_i) = \frac{n_{N_i}(\tau_i)}{N_i} = \frac{1}{N_i} \sum_{j=1}^{N_i} \eta_j(\tau_i). \tag{5}$$

The standard deviation for the number of go's $n_{N_i}(\tau_i)$ when performing N_i runs is

$$\sigma[n_{N_i}(\tau_i)] = \sqrt{N_i p(\tau_i)[1-p(\tau_i)]} \tag{6}$$

and is estimated by

$$\hat{\sigma}[n_{N_i}(\tau_i)] = \sqrt{N_i \hat{p}_{N_i}(\tau_i)[1-\hat{p}_{N_i}(\tau_i)]}. \tag{7}$$

An estimate for the accuracy of $\hat{p}_{N_i}(\tau_i)$ is obtained by knowing that the standardized variable

$$\frac{n_{N_i}(\tau_i) - N_i p(\tau_i)}{\sigma[n_{N_i}(\tau_i)]}$$

is asymptotically normal for large N_i when $n_{N_i}(\tau_i)$ is a binomial random variable. Thus, the cumulative probability P is approximated by

$$P\left\{ -k \le \frac{n_{N_i}(\tau_i) - N_i p(\tau_i)}{\sigma[n_{N_i}(\tau_i)]} \le k \right\} \doteq \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-\tau^2/2} d\tau$$

or

$$P\left\{ \left| p(\tau_i) - \hat{p}_{N_i}(\tau_i) \right| \le \frac{k}{N_i} \sigma[n_{N_i}(\tau_i)] \right\} \doteq \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-\tau^2/2} d\tau; \quad (8)$$

$\frac{k}{N_i} \sigma[n_{N_i}(\tau_i)]$ is the <u>confidence bound</u> for the absolute difference of $p(\tau_i)$ and $\hat{p}_{N_i}(\tau_i)$; and the right side of Eq. (8) is the <u>confidence</u> (limit) with which the bound holds. Since there is no way of determining $\sigma[n_{N_i}(\tau_i)]$ from experimental data, Eq. (8) is approximated by

$$\hat{P}\left\{ \left| p(\tau_i) - \hat{p}_{N_i}(\tau_i) \right| \le \frac{k}{N_i} \hat{\sigma}[n_{N_i}(\tau_i)] \right\} \doteq \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-\tau^2/2} d\tau \quad (9)$$

where the notation $\hat{P}$ implies that an estimated confidence bound has replaced the confidence bound in P. The value of k is determined by setting $\hat{P}$ equal to the desired level of confidence, say 0.95, and solving

$$\frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-\tau^2/2} d\tau = \hat{P}$$

for k. Usually, when $\hat{\sigma}[n_{N_i}(\tau_i)]$ is used in place of $\sigma[n_{N_i}(\tau_i)]$, the Student T distribution is considered more appropriate for the right side of Eq. (9), and the confidence bound is enlarged to accomodate for the uncertainty involved from not knowing σ. However, for purposes of presentation, we use the normal distribution approximation for the asymptotic property of the binomial random variable.

Using the probability of go $p(\tau)$, the average of the probability of go over a 2T interval of τ, centered at $\tau=\tau_i$, is defined as

$$\bar{p}_T(\tau_i) = \int_{\tau_i-T}^{\tau_i+T} p(\tau)d\tau. \tag{10}$$

Let N' experimental runs be performed at each value

$$\tau_{i+m} = \tau_i + m\Delta\tau \tag{11}$$

where $m = -M,\ldots,-1,0,1,\ldots M$ and $\Delta\tau = T/M$, then an estimate for the average probability is given by

$$\hat{\bar{p}}_{M,N'}(\tau_i) = \frac{1}{2M+1}\sum_{m=-M}^{M}\hat{p}_{N'}(\tau_{i+m}) = \frac{1}{2M+1}\sum_{m=-M}^{M}\frac{1}{N'}\sum_{j=1}^{N'}\eta_j(\tau_{i+m}). \tag{12}$$

In the particular case when $N' = 1$,

$$\hat{\bar{p}}_M(\tau_i) = \hat{\bar{p}}_{M,1}(\tau_i) = \frac{1}{2M+1}\sum_{m=-M}^{M}\eta(\tau_{i+m}) \tag{13}$$

where $\eta(\tau_{i+m}) = \eta_1(\tau_{i+m})$. An equation similar to Eq. (9) can be derived for the confidence and confidence bound for the absolute difference of $\bar{p}_T(\tau_i)$ and $\hat{\bar{p}}_M(\tau_i)$. Define

$$\bar{n}_M(\tau_i) = \sum_{m=-M}^{M} \eta(\tau_{i+m}), \tag{14}$$

then

$$\hat{\sigma}[\bar{n}_M(\tau_i)] = \sqrt{(2M+1)\hat{\bar{p}}_M(\tau_i)[1-\hat{\bar{p}}_M(\tau_i)]}; \tag{15}$$

and

$$P\left\{ \left| \bar{p}_T(\tau_i) - \hat{\bar{p}}_M(\tau_i) \right| \le \frac{k}{2M+1} \hat{\sigma}[\bar{n}_M(\tau_i)] \right\} = \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-\tau^2/2} d\tau. \tag{16}$$

Thus, in a sensitivity test one performs runs at various levels of the stimulus τ and observes the outcome $\eta(\tau)$, as defined in Eq. (2), for each run. From the outcomes, one calculates statistics such as $n_{N_i}(\tau_i)$ of Eq. (3) or $\bar{n}_M(\tau_i)$ of Eq. (14) and then makes estimates for the probability of go $\hat{p}_{N_i}(\tau_i)$ using Eq. (5) or the average probability of go $\hat{\bar{p}}_M(\tau_i)$ using Eq. (13). Finally, one makes estimates of the error involved in these probabilities using Eq. (9) or (16). When one follows the procedure that uses Eqs. (3), (5), and (9), he is following an experimental design referred to as the Probit method. In following this method, when evaluating $\hat{p}_{N_i}(\tau_i)$ for

$$\tau_i = \tau_0 + i\Delta\tau, \qquad 0 \le i \le I-1,$$

and when using $N_i = 2M+1$ for all i, the total number of runs is $(2M+1)I$. We present a method that uses Eqs. (13), (14) and (16) which requires $2M+I$ runs at

$$\tau_i = \tau_0 + i\Delta\tau, \qquad -M \le i \le I+M-1.$$

A Selection Rule. In every sensitivity test, the experimenter possesses either explicit or implicit information about the upper and lower values, τ_U and τ_L, of the stimulus he will use in the experiment and about the spacing $\Delta\tau$ between values of τ at which he desires estimates for the probability of go $p(\tau)$. The values of τ_U, τ_L and $\Delta\tau$ are determined by previous knowledge about the specimen, by the total amount of time that can be devoted to the experiment, by the experimental equipment or some combination of these. For example, $\Delta\tau$ may be determined by the ability to measure the stimulus. τ_L is always greater than or equal to zero, and τ_U might be determined through the rationale that if it exceeds a prespecified amount then the specimen is no longer of experimental interest. We assume then that the values of τ_U, τ_L and $\Delta\tau$ are explicitly specified. In addition, the experimenter must specify the confidence $\hat{P}$ and the confiedence bound

$$b = \frac{k}{N_i}\,\hat{\sigma}[n_{N_i}(\tau_i)]$$

for $|p(\tau_i) - \hat{p}_{N_i}(\tau_i)|$ or

$$\bar{b} = \frac{k}{2M+1}\,\hat{\sigma}[\bar{n}_M(\tau_i)] \qquad (17)$$

for $|\bar{p}(\tau_i) - \hat{\bar{p}}_M(\tau_i)|$. Although the experimentor sometimes has additional information that greatly simplifies the selection of experimental runs, he is often faced with only the information that $0 \leq p(\tau) \leq 1$ for τ in an interval of length $D(\tau) \leq \tau_U - \tau_L$ and that this interval lies in $[\tau_L, \tau_U]$. The exact length of $D(\tau)$ and where it is

located in $[\tau_L, \tau_U]$ are not known. The experimenter desires $\hat{p}_{N_i}(\tau_i)$ for those τ_i's where

$$b \leq \hat{p}_{N_i}(\tau_i) \leq 1-b$$

or $\hat{\bar{p}}_M(\tau_i)$ for those τ_i's where

$$\bar{b} \leq \hat{\bar{p}}_M(\tau_i) \leq 1-\bar{b}.$$

The object of the run selection rule is to acquire as much information as is possible about the value of $\tau=\tau^*$ such that $\bar{p}_T(\tau^*) = \frac{1}{2}$ and to remove from further consideration those values of τ for which $\bar{p}_T(\tau) \doteq 1$ or $\bar{p}_T(\tau) \doteq 0$. In the following description we will assume that the experimenter has chosen his measure of the stimulus so that $p(\tau)$ will be a monotonically increasing function of τ. This is an assumption since, in general, there is no guarantee that $p(\tau)$ is monotonic for any choice of τ.

To obtain a selection rule we desire to have a confidence $\hat{P}$ in the selection of the interval of τ which contains $p(\tau^*) = \frac{1}{2}$. For purposes of presentation, we use a specific value of $\hat{P}_S$, namely, $\hat{P}_S = 0.90$. From $\hat{P}_S$, we determine the estimated confidence bound $\hat{b}_S$ as a function of the sample size N_S as follows. Set

$$\hat{P}_S = \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-\tau^2/2} d\tau = 0.90$$

which implies that $k \doteq 1.65$. Since

$$\bar{b}_S = \frac{k}{N_S} \sigma_S$$

and since

$$\sigma_S[\bar{n}_{N_S}(\tau)] = \sqrt{N_S \bar{p}_T(\tau)[1-\bar{p}_T(\tau)]},$$

then the maximum value of $\bar{b}_S$ occurs when σ_S takes on its maximum value which is when $\bar{p}_T(\tau) = \frac{1}{2}$

$$\sigma_{max} = \sqrt{N_S(1/2)(1/2)} = \sqrt{N_S}/2.$$

Therefore

$$\bar{b}_{max} = \frac{k}{2}\frac{1}{\sqrt{N_S}} = \frac{0.825}{\sqrt{N_S}}. \qquad (18)$$

Next, we determine the minimum value of N_S so that the observation from that sample size permit estimates $\hat{\bar{p}}_M(\tau)$ which prohibit $\bar{p}_T(\tau)$ from assuming the value $\frac{1}{2}$ with confidence $\hat{P}_S$. Thus $\bar{b}_{max} < 0.25$ for some value of N_S. Setting $\bar{b}_{max} = 0.25$ and solving Eq. (18) for N_S, we obtain

$$N_S = (0.825)^2/(0.25)^2 = 10.89$$

Choosing $N_S = 11$ gives $\bar{b}_{max} \doteq 0.249$ and $\frac{2}{11} < 0.249 < \frac{3}{11}$.

Therefore for $N_S = 11$, $\bar{p}_T(\tau) < \frac{1}{2}$ for $0 \le \hat{\bar{p}}_S \le \frac{2}{11}$ and $\bar{p}_T(\tau) > \frac{1}{2}$ for $\frac{9}{11} \le \hat{\bar{p}}_S \le 1$.

From the preceding inequalities we see that the following run selection rule is feasible. First set

$$\Delta\tau_0 = \frac{\tau_U - \tau_L}{11}$$

and perform $N_S^{(0)} = 12$ runs for

$$\tau_i^{(0)} = \tau_L + i\Delta\tau_0, \qquad i = 0,1,\ldots, \text{ii} = N_S^{(0)}-1$$

Let the outcomes of these runs be designated by $\eta(\tau_i^{(0)})$. If these outcomes do not include $\eta(\tau_0^{(0)}) = 0$ and $\eta(\tau_{ii}^{(0)}) = 1$, the experimenter should reconsider his experiment. Therefore, assume $\eta(\tau_0^{(0)}) = 0$ and $\eta(\tau_{ii}^{(0)}) = 1$ in the remainder of this discussion. If ten or more sequential outcomes are identical and one of these is at the end value $\tau_0^{(0)}$ or $\tau_{ii}^{(0)}$, then remove that end value from further consideration. Next, choose values of τ that are midway between the consecutive values of $\tau_i^{(0)}$ and perform runs at these values of τ. Now, we possess outcomes for values of τ_i for

$$\tau_i^{(1)} = \tau_0^{(1)} + i\Delta\tau_1, \qquad i = 0,1,2,\ldots, N_S^{(1)}-1$$

where

$$\Delta\tau_1 = \Delta\tau_0/2,$$

$$\tau_0^{(1)} = \begin{Bmatrix} \tau_0^{(0)} \\ \tau_1^{(0)} \end{Bmatrix} \quad \text{if } \tau_0^{(0)} \begin{Bmatrix} \text{was not} \\ \text{was} \end{Bmatrix} \text{discarded,}$$

and

$$N_S^{(1)} = \begin{Bmatrix} 23 \\ 21 \end{Bmatrix} \quad \text{if} \quad \begin{Bmatrix} \text{neither} \\ \text{either} \end{Bmatrix} \text{end value of } \tau_i^{(0)} \text{ was discarded.}$$

Again, check to see if there are ten or more sequential $\eta(\tau_i)$ that are alike where one of these $\eta(\tau_i)$ is either $\eta(\tau_0^{(1)})$ or $\eta(\tau^{(1)}_{N_S^{(1)}-1})$. If there are, remove from further consideration all but the nine identical $\eta(\tau_i^{(1)})$ which are next to an opposite outcome. Next, choose new values of τ that are midway between the consecutive values of the $\tau_i^{(1)}$ that were saved and make runs at these values of τ. Now, we possess outcomes at

$$\tau_i^{(2)} = \tau_0^{(2)} + i\Delta\tau_2, \qquad i=0,1,\ldots,N_S^{(2)}-1$$

where

$$\Delta\tau_2 = \frac{\Delta\tau_1}{2} = \frac{\Delta\tau_0}{2^2} .$$

Check to see if there are ten or more sequential $\eta(\tau_i^{(2)})$ that are identical and remove from further consideration all but the nine which are next to an opposite outcome. This procedure of setting $\Delta\tau_t = \Delta\tau_{t-1}/2$, performing additional runs and casting out all but nine of the identical sequential values of $\eta(\tau_i)$ is continued until

$$\Delta\tau_t \leq \Delta\tau$$

if one is planning to use the Probit method of analysis. For the method of analysis described in the next section the stopping rule is somewhat more complex and is given in that section.

A Method of Analysis. This method of analysis is designed primarily to acquire estimates $\hat{\bar{p}}_M(\tau_i)$ for the average probability of go $\bar{p}_T(\tau_i)$ and secondarily for an approximation $\tilde{p}(\tau)$ for the probability of go $p(\tau)$. From the desired confidence $\hat{P}$ and confidence bound $\bar{b}$, the value of M is determined by first choosing k such that

$$\int_{-k}^{k} e^{-\tau^2/2} d\tau = \hat{P}$$

and, then, choosing M to be the smallest integer greater than or equal to M' where

$$\frac{k}{2M'+1}\sqrt{(2M'+1)/4} = \bar{b}$$

or

$$M' = \frac{k^2}{8(\bar{b})^2} - \frac{1}{2} .$$

Assume now that for the specified values of I, τ_L, τ_U and $\Delta\tau$, that the run selection rule has provided a set of outcomes $\eta(\tau_i)$ where

$$\tau_i = \tau_0 + i\Delta\tau, \qquad -M \le i \le I + M-1,$$

$I \ge 1$, and $\tau_{-M} = \tau_c(t)$ from the selection rule. Using these values of $\eta(\tau_i)$, $\hat{\bar{p}}_M(\tau_i)$ is calculated by Eq. (13) for each i, $0 \le i \le I-1$.

The stopping rule for the selection rule when obtaining estimates $\hat{\bar{p}}_M(\tau_i)$ for the average probability of go $\bar{p}_T(\tau_i)$ always considers the requirement for 2M+I outcomes $\eta(\tau_i)$ for equally spaced, sequential values of τ_i with separation $\Delta\tau$. Thus, if for any integer t where $\Delta\tau_t = \frac{\Delta\tau_0}{2^t} > \Delta\tau$, one can satisfy the two conditions:

1. $\Delta\tau_{t+p} \le \Delta\tau$ and $\Delta\tau_{t+p-1} > \Delta\tau$

and

2. $\tilde{N}_S(t+p) \ge 2M+I$ and $\tilde{N}_S(t+p-1) < 2M+I$

for some positive integer p where

$$\tilde{N}_S(t+p) = 2^p N_S(t) - 2^p + 1,$$

then one no longer removes identical outcomes from further consideration but performs runs at those values

$$\tau_i(t+p) = \tau_0(t) + i\Delta\tau_{t+p}, \qquad i=0,1,\ldots,\tilde{N}_S(t+p)-1$$

for which runs have not already been performed. If these conditions are not satisfied then the procedure is terminated in the same manner as for the Probit method.

From the I values of $\hat{\bar{p}}_M(\tau_i)$, a piece-wise linear approximation $\tilde{p}(\tau)$,

$$\tilde{p}(\tau) = \begin{cases} 0 & \text{for } 0 \le \tau < \tau' \\ \dfrac{\tau - \tau'}{\tau'' - \tau'} & \text{for } \tau' \le \tau \le \tau'' \\ 1 & \text{for } \tau > \tau'', \end{cases} \qquad (19)$$

is calculated for $p(\tau)$. This linear approximation is acquired by a least square fit of $\tilde{\bar{p}}_M(\tau_i)$,

$$\tilde{\bar{p}}_M(\tau_i) = \frac{1}{2M+1} \sum_{m=-M}^{M} \tilde{p}(\tau_{i+m}), \quad 0 \le i \le I-1,$$

to $\hat{\bar{p}}_M(\tau_i)$. The least square procedure is to choose the value of i, say i=j, where $\hat{\bar{p}}_M(\tau_i) \doteq \frac{1}{2}$ and set $\tau' = \tau_{j-c}$ and $\tau'' = \tau_{j+c}$. Then

$$\tilde{p}_j(\tau_i) = \begin{cases} 0 & \text{for } i < j-c \\ \dfrac{\tau_i - \tau_{j-c}}{\tau_{j+c} - \tau_{j-c}} & \text{for } j-c \le i \le j+c \\ 1 & \text{for } i > j+c \end{cases}$$

and $\tilde{\bar{p}}_{M,j}(\tau_i)$ is calculated as

$$\tilde{\bar{p}}_{M,j}(\tau_i) = \frac{1}{2M+1} \sum_{m=-M}^{M} \tilde{p}_j(\tau_{i+m}), \quad 0 \le i \le I-1.$$

Next, calculate

$$\epsilon_{j,i}^{(c)} = \hat{\bar{p}}_M(\tau_i) - \tilde{\bar{p}}_{M,j}(\tau_i)$$

for the admissible values of i, and then calculate

$$s_j^{(c)} = \sum_{i=0}^{I-1} [\, \epsilon_{j,i}^{(c)} \,]^2.$$

By performing these calculations for c = 1,2,3,..., a sequence

$$S_j^{(1)}, S_j^{(2)}, S_j^{(3)}, \ldots$$

is generated, and the desired $\tilde{p}_j(\tau)$ curve is acquired from the choice c^* of c which minimizes $S_j^{(c)}$, i.e.,

$$S_j^{(c^*)} = \underset{c}{\text{Min}}\{S_j^{(c)}\}.$$

The preceding calculations are now repeated by replacing j by j+p to obtain a sequence of minimized $S_{j+p}^{(c_p^*)}$ for p = 0,±1,±2,..., i.e.,

$$\ldots, S_{j-2}^{(c_{-2}^*)}, S_{j-1}^{(c_{-1}^*)}, S_j^{(c^*)}, S_{j+1}^{(c_1^*)}, S_{j+2}^{(c_2^*)}, \ldots \quad .$$

The optimum solution for $\tilde{p}(\tau)$ is that $\tilde{p}_{j+p}(\tau)$ for which p minimizes the preceding sequence.

Statistical Considerations. The statement of Eq. (16) is based on the assumptions that $\bar{p}_T(\tau_i)$ is the probability of a binomial random variable given that $p(\tau)$ is the probability of a binomial random variable and that $\hat{\bar{p}}_M(\tau_i)$ is a consistent estimator of $\bar{p}_T(\tau_i)$. To show that $\bar{p}_T(\tau_i)$ is the probability of a binomial random variable we need only show that

$$\bar{q}_T(\tau_i) = \frac{1}{2T}\int_{\tau_i-T}^{\tau_i+T} q(t)dt = 1 - \bar{p}_T(\tau_i)$$

where q(t) = 1-p(t). Since

$$\bar{p}_T(\tau_i) = \frac{1}{2T}\int_{\tau_i-T}^{\tau_i+T} p(t)dt$$

then

$$1-\bar{p}_T(\tau_i) = 1 - \frac{1}{2T}\int_{\tau_i-T}^{\tau_i+T} p(t)dt = \frac{1}{2T}\int_{\tau_i-T}^{\tau_i+T} dt - \frac{1}{2T}\int_{\tau_i-T}^{\tau_i+T} p(t)dt$$

$$= \frac{1}{2T}\int_{\tau_i-T}^{\tau_i+T}[1-p(t)]dt = \frac{1}{2T}\int_{\tau_i-T}^{\tau_i+T} q(t)dt = \bar{q}_T(\tau_i).$$

To show that $\hat{\bar{p}}_M(\tau_i)$ is a consistent estimator of $\bar{p}_T(\tau_i)$ we show that $\hat{\bar{p}}_{M,N}(\tau_i)$, as defined by Eq. (12) with N replacing N′, is a consistent estimator of

$$\frac{1}{2M+1}\sum_{m=-M}^{M} p(\tau_{i+m}) \doteq \frac{1}{2T}\int_{\tau_i-T}^{\tau_i+T} P(\tau)d\tau = \bar{p}_T(\tau_i)$$

and by assuming that

a. $p(\tau)$ is continuous in τ

b. $p(\tau'') \geq p(\tau')$ for all $\tau'' \geq \tau'$

c. $T \geq 0$ is finite and $\Delta\tau = T/M$

d. $p(0) = 0$ and there exists a $\tau_U, 0 < \tau_U \leq \infty$ such that $p(\tau_U) = 1$.

The expected value $\rho_{M,N}(\tau_i)$ satisfies

$$\lim_{M\to\infty} E[\hat{\bar{p}}_{M,N}(\tau_i)] = \bar{p}_T(\tau_i)$$

for fixed T. Furthermore, for M=0

$$\lim_{N\to\infty} E[\hat{\bar{p}}_{O,N}(\tau_i)] = p(\tau_i),$$

and for N=1,

$$\lim_{M\to\infty} E[\hat{\bar{p}}_{M}(\tau_i)] = \bar{p}_T(\tau_i).$$

Thus, to show consistency, we must still show that the variance of $\hat{\bar{p}}_{M,N}(\tau_i)$ can be made sufficiently small by appropriate choices of M

and N. To simplify notation, define

$$A_j = \sum_{m=-M}^{M} \eta_j(\tau_{i+m})$$

then

$$\hat{\underline{\underline{p}}}_{M,N}(\tau_i) = \frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j$$

and

$$\sigma^2\left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right] = E\left\{ \left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right]^2 \right\} - E^2\left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right]$$

First consider the term

$$E\left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right] = \frac{1}{N(2M+1)} \sum_{j=1}^{N} E[A_j] = \frac{E[A_j]}{2M+1}$$

then

$$E^2\left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right] = \frac{E^2[A_j]}{(2M+1)^2} .$$

Next consider

$$E\left\{ \left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right]^2 \right\} = E\left\{ \left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right] \left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right] \right\}$$

$$= \left[\frac{1}{N(2M+1)} \right]^2 \left[\sum_{j=1}^{N} E(A_j^2) + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} A_j A_k \right]$$

$$= \left(\frac{1}{2M+1} \right)^2 \left[\frac{E(A_j^2)}{N} + \frac{N(N-1)}{N^2} E^2(A_j) \right]$$

Thus

$$\sigma^2\left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right] = \left(\frac{1}{2M+1} \right)^2 \left(\frac{1}{N} \right) \left[E(A_j^2) - E^2(A_j) \right]$$

Now consider

$$E(A_j) = E\left[\sum_{m=-M}^{M} \eta_j(\tau_{i+m}) \right] = \sum_{m=-M}^{M} E[\eta_j(\tau_{i+m})] = \sum_{m=-M}^{M} p(\tau_{i+m})$$

However, for $p(\tau)$ continuous and monotonic,

$$\sum_{m=-M}^{M} p(\tau_{i+m}) = (2M+1)p(\tau^*)$$

where $\tau_{i-M} \leq \tau^* \leq \tau_{i+M}$ and $E^2(A_j) = (2M+1)^2 p^2(\tau^*)$.

The term $E(A_j^2)$ is evaluated as follows

$$E(A_j^2) = E\left\{ \left[\sum_{m=-M}^{M} \eta_j(\tau_{i+m}) \right]^2 \right\}$$

$$= E\left[\sum_{m=-M}^{M} \eta_j^2(\tau_{i+m}) + \sum_{m=-M}^{M} \eta_j(\tau_{i+m}) \sum_{\substack{n=-M \\ n\neq m}}^{M} \eta_j(\tau_{i+n}) \right]$$

$$= \sum_{m=-M}^{M} p(\tau_{i+m}) + p(\tau^*) \sum_{m=-M}^{M} \sum_{\substack{n=-M \\ n\neq m}}^{M} p(\tau_{i+n})$$

since $E[\eta_j^2(\tau_{i+m})] = p(\tau_{i+m})$ and $\sum_{m=-M}^{M} 1 = 2M+1$. Write

$$\sum_{m=-M}^{M} \sum_{\substack{n=-M \\ n\neq m}}^{M} p(\tau_{i+n}) = 2M \sum_{m=-M}^{M} p(\tau_{i+m}) = 2M(2M+1)p(\tau^*);$$

and

$$E(A_j^2) = (2M+1)p(\tau^*) + 2M(2M+1)p^2(\tau^*)$$

Thus

$$E(A_j^2) - E^2(A_j) = (2M+1)[p(\tau^*) - p^2(\tau^*)]$$

and

$$\sigma^2\left[\hat{\bar{p}}_{M,N}(\tau_i) \right] = \sigma^2\left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right]$$

$$= \left(\frac{1}{2M+1} \right) \left(\frac{1}{N} \right) p(\tau^*)[1-p(\tau^*)]$$

Finally, then, for any fixed value of M, $0 \leq M < \infty$,

$$\lim_{N\to\infty} \sigma^2\left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right] = 0.$$

and, similarly, for any fixed value of N, $1 \leq N < \infty$,

$$\lim_{M\to\infty} \sigma^2\left[\frac{1}{N(2M+1)} \sum_{j=1}^{N} A_j \right] = 0.$$

These last two statements show that for M=0,

$$\hat{p}(\tau_i) = \sum_{j=1}^{N} \eta_j(\tau_i)$$

is a consistent estimator of $p(\tau_i)$, and that for N=1 and for a fixed value of T, $0 < T < \infty$,

$$\hat{\bar{p}}_M(\tau_i) = \frac{1}{2M+1} \sum_{m=-M}^{M} \eta(\tau_{i+m})$$

is a consistent estimator of $\bar{p}_T(\tau_i)$.

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DESIGNS OF EXPERIMENTS AS TELESCOPING SEQUENCES OF BLOCKS

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ABSTRACT. Sequencies of orthogonally blocked statistical designs of experiments are presented for optimum seeking. The sequences are such that observations from the first block can be used to estimate the coefficients of a simple model and then be retained and combined with observations from new blocks so that all acquired observations are used cumulatively to estimate models of successively greater generality. Such blocks are said to form a "telescoping" sequence. Specific choices were motivated by the problem of optimum seeking experiments in alloy development.

The designs consist of full and fractionally replicated two-level factorial experiments with four to eight factors. The sizes of the experiments include 8, 16, 32, and 64 treatments.

INTRODUCTION. Optimum seeking experiments have been conducted by NASA in developing improved engine materials for the supersonic transport. The use of the designs presented herewith for optimum seeking has been discussed in reference 1. In addition to optimum seeking, the designs could be used in many situations where the experimenting begins without prior knowledge of the complexity needed for the model.

The designs consist of two level fractional factorial experiments performed as sequences of blocks. The designs are to be such that the first block will be a small fraction of the full factorial, but large enough for estimating the parameters of a first degree model. Successive blocks are to be such that all acquired data can be used cumulatively to estimate models of successively greater generality, with block effects being uncorrelated with the parameter estimates. The sequences terminate in designs that give estimates of first degree and two factor interaction coefficients and the estimates are free of aliases with other second degree or lower order coefficients. Without considering blocking, Steve Webb in reference 2 applied the terms expansible and contractible to related sequences of designs.

Sequences of regular fractions were discussed in reference 3 by Cuthbert Daniel. Sequences of irregular fractions were discussed by

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Peter John in reference 4. The general subject was explored further by Sidney Adelman in reference 5.

Box and Hunter in reference 6 recommended the use of sequences of rotatable orthogonally blocked designs for optimum seeking. These properties require that the fractions be regular fractions, that is, the number of treatments is $1/2^h$ times the number of treatments in a full factorial experiment, where h is an integer. The designs to be presented are all regular fractions.

SYMBOLS.

b	number of blocks
E( )	value of ( ) if averaged over infinite number of observations
g	number of independent variables (factors)
h	fractional replicate contains $1/2^h$ times number of treatments performed in full two-level factorial experiment
i	index number for trials
j, k	index number for independent variables
f	g - h
R	resolution level
X_j	vector giving levels of x_{ij}, i = 1, . . ., n
x_{ij}	standardized level of ξ_j
y	response (observed variate)
β	unknown population parameter
ϵ	error
ξ_j	independent variable, j = 1, . . ., g
σ^2	variance of ϵ

SIZES OF EXPERIMENTS.

Degrees of Freedom for Lack of Fit. Consider the fitting of a model equation to a 2^3 full factorial experiment. The appropriate equation is as follows:

$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123} x_1 x_2 x_3 \qquad (1)$$

The equation illustrates the notation. Main effects are designated by symbols such as β_1 and β_2. Two factor interactions are represented by symbols such as β_{12}. The independent variates are represented by lower case symbols such as x_1 and x_2.

The number of treatments minus the number of parameters estimated is the degrees of freedom for lack of fit. The 2^3 experiment contains 8 treatments, but the optimum seeking begins with a first degree equation containing only four parameters, leaving four degrees of freedom for lack of fit. The final stage of optimum seeking includes the two factor interactions so that only one degree of freedom would remain for lack of fit (eq. (1)).

Some information on the lack of fit is always desirable. The degrees of freedom for lack of fit of the designs to be presented vary from 0 to 35, and designs are provided for numbers of factors varying from 4 to 8. With 9 factors the use of a regular fraction requires 128 treatments of which 66 represent degrees of freedom for lack of fit. In other words, an insistence on the use of regular fractions does not seem to be unduly extravagant unless there are 9 or more factors. The use of irregular fractions seems to be appropriate in situations involving 9 or more factors or for lesser numbers of factors, where the experimenting is very expensive, and where the relative error is known to be small.

Resolution Levels. The factorial experiment with conditions fixed at just two levels of g independent variables (factors) permits the estimation of parameters representing the grand mean over the experiment, the first-order effects of the factors, and the results of factors interacting two at a time, three at a time, and in all combinations up to g at a time. If a fraction $1/2^h$ of this experiment is performed, not all these parameters can be estimated. True response functions in physical investigations are typically smooth enough that the higher order coefficients of an approximating polynomial may be assumed to be negligible over a small enough range of the experimentation. Accordingly, only the lower

order coefficients need be estimated; however, they are allowed to be biased by (aliased with) coefficients of higher order interactions because such coefficients are assumed to be negligible.

Let the number of factors in the highest order interaction requiring estimation be e, and let the number of factors in the lowest order interaction with which it is allowed to be aliased be c; then the required resolution R of the design is defined (ref. 7) to be

$$R = e + c$$

As a minimum requirement on the first-order experiments, the coefficients will be allowed to be aliased with only the coefficients of two-factor or higher order interactions. This requires that $R = e + c = 1 + 2 = 3$. A somewhat improved design occurs if the first-order coefficients are estimated clear of two-factor interactions. This requires that $R = e + c = 1 + 3 = 4$.

For the interaction experiments, the estimates of two factor interaction coefficients should be allowed to be aliased only with higher order interaction coefficients. This requires that $R = e + c = 2 + 3 = 5$.

The design of the interaction experiment (of resolution 5) is now specified to be blocked into b blocks such that any one block will provide a design of resolution 3 for the first-degree model. As a consequence of this requirement, the experimenter may switch at any time from the method of steepest ascents to the method of local exploration by completing the $b - 1$ uncompleted blocks of the resolution 5 experiment.

Occasions could arise in which the experimenter would not wish to proceed immediately from a minimum-size first-degree design to the design for estimating all interaction coefficients. For example, a design of only eight treatments hardly provides enough information to test the validity of the first-degree model. The performance of a second block of eight treatments could lead to a much better decision. Also, the experimenter may have prior knowledge that certain interactions are negligible so that he can stop short of the experiment that estimates all two-factor interactions. For these reasons, the designs and parameter estimates are given for such intermediate size experiments.

Numbers of Factors and Block Sizes. The assumption was made that a sequence of blocks should not terminate in a total experiment that contained less than 16 treatments, that is, the assumption was made that a completed experiment containing less than 16 experimental units is too

small for any statistical assessment of validity. With 16 treatments, the smallest number of factors in the (efficient) unreplicated experiment is four, and therefore no designs were investigated having less than four independent variables.

As was shown in reference 3, the degrees of freedom efficiency of regular fractions of two level factorial experiments of resolution 5 becomes and remains poor, and the experiment sizes become enormous, if the number of factors exceeds 8. The investigation was therefore limited to 4, 5, 6, 7, and 8 factors.

The regular fractional factorial first degree experiment on four factors requires a minimum of 8 treatments, whereas the regular fractional factorial first degree experiment with eight factors requires a minimum of 16 treatments. Correspondingly, the sizes of the blocks are limited to 8 and 16 treatments.

So that the experimenter will always get results on his "standard conditions" first, the principal block will always be given as the first block.

CONSTRUCTION OF DESIGNS AND ESTIMATES OF PARAMETERS.

Defining Contrasts. The mixed usage of Yates' notation for treatments and the special notation of the present work is illustrated by table 1. The treatments are listed in the familiar Yates' notation and Yates' order in the first column. The resulting dependent variates are listed in the corresponding order in the second column. Lower case symbols like x_1, had been used for the independent variates. The full set of levels of such a variate is a column vector of plus and minus ones and is represented by the corresponding upper case symbol as shown by the column headings. A column heading showing a product means that elements from identical rows have been multiplied to produce a new column with the same number of rows.

This rule of multiplication leads to such relations as

$$(X_1X_2)(X_2X_3X_4) = X_1X_0X_3X_4 = X_1X_3X_4$$

These operations are similar to the more popular terminology in which:

$$(AB)(BCD) = AICD = ACD$$

The present usage of symbols such as β_0, β_{12}, X_0, X_1X_2 avoids such ambiguities as I standing for both the grand mean and the identity vector, and AB standing for both the interaction parameter β_{12} and

the contrast vector X_1X_2.

The general rules for sequences of blocked designs were given in reference 3. Given now are rules that are much more narrowly stated. The purpose of the narrow statement is to quickly and easily arrive at a list of treatments and aliased parameters that will be in Yates' order Thus, if the responses are listed in Yates' order then Yates' computational procedure will give estimates that will be in the order of easily identified sets of aliased parameters. Actually, this narrowly stated procedure results in no loss of generality, because the experimenter is free to assign the symbols $x_1, x_2, \ldots$ to his physical variables in any order he chooses.

Although designs are given for numbers of factors from 4 to 8 and block sizes of 8 and 16, their construction will be illustrated by only an example with 6 factors and a block size of 8. For this block size the first 8 rows of table 1 give treatment levels that can be used for the factors x_1, x_2, and x_3. The design must be completed with orthogonal levels of x_4, x_5, and x_6. For orthogonality the levels can only be levels that already occur for columns from X_1 to the product $X_1X_2X_3$. Then multiplying the elements of a new column by the elements from its equal among the old columns will result in a column of plus ones, namely, the X_0 column.

The first block is to be a $1/2^3$ replicate of the 2^6 design. The fractional replication is characterized by 2^3 defining contrasts of which 3 are independent, and the telescoping requires that some constraints be placed on the 3 independent defining contrasts. From among the columns from X_1 to the product $X_1X_2X_3$ select 3 (as yet unspecified) columns and call them U, V, and W. Then

$$X_4 = U \qquad X_5 = V \qquad X_6 = W$$

$$\underline{UX_4} = X_4^2 = X_0; \quad \underline{VX_5} = X_5^2 = X_0; \quad \underline{WX_6} = X_6^2 = X_0$$

The underlined items are the defining contrasts. Because they each contain a column not contained in the others, they are independent, and because there are three of them, they are all of the $h = 3$ independent defining contrasts. The group of defining contrasts is found by forming the products of the independent contrasts in all possible combinations:

UX_4
VX_5
WX_6
UVX_4X_5
UWX_4X_6
VWX_5X_6
$UVWX_4X_5X_6$
$UX_4UX_4 = X_0$

The fact that a sequence of telescoping designs is desired will impose some constraints on the choice of U, V, and W in terms of X_1, X_2, and X_3.

Defining contrasts are now to be considered for the two blocks that will constitute a 2/8 replicate. The 16 treatment levels for x_1, x_2, x_3, and x_4 are given in Yates' order by table 1. The columns of levels of x_5 and x_6 need to be identical with two of the columns from X_1 to $X_1X_2X_3X_4$ of table 1. Let these columns (as yet unspecified) be called Y and Z, that is, $X_5 = Y$, $X_6 = Z$ so that the independent defining contrasts for the 2/8 replicate are YX_5 and ZX_6. The complete group of defining contrasts is:

X_0
YX_5
ZX_6
YZX_5X_6

In the case of the 4/8 replicate, X_6 is set equal to one of the product columns of a 2^5 experiment. The defining contrast is symbolized by TX_6.

In summary, the groups of as yet, incompletely specified defining contrasts are:

1/8 replicate	2/8 replicate	4/8 replicate
X_0	X_0	X_0
UX_4	YX_5	TX_6
VX_5	ZX_6	
WX_6	YZX_5X_6	
UVX_4X_5		
UWX_4X_6		
VWX_5X_6		
$UVWX_4X_5X_6$		

Some of the constraints of the design problem are that one of the blocks of the 2/8 replicate must be identical to the 1/8 replicate, and two of the blocks of the 4/8 replicate must be identical to those of the 2/8 replicate. Thus, for example, the treatment levels of X_1, X_2, and X_3, associated with X_5 of the 2/8 replicate must have 8 points of identity with the treatment levels of X_1, X_2, and X_3 associated with X_5 in the 1/8 replicate.

These identities are achieved by setting

$$Y = V$$

or

$$Y = UVX_4$$

and also

$$Z = W$$

or

$$Z = UWX_4$$

For the 4/8 replicate, a necessary condition is that

$$T = Z$$

or that

$$T = YZX_5$$

Among the preceding constraints, desirable choices would result in TX_6 having at least 5 symbols so that the 4/8 replicate would be of resolution 5. Also, because each stage must be of resolution 3, all defining contrasts must contain at least 3 symbols. The choices of U, V, W, Y, and Z should be consistent with these objectives.

So that the first block will be a principle block (so that it will contain a treatment with all factors at their "low" levels) the defining contrasts must be negative if they contain an odd number of symbols, and positive if they contain an even number of symbols.

Suppose that $U = -X_1X_2$, $V = -X_2X_3$ and $W = X_1X_2X_3$. Multiplying the resulting defining contrasts together in all combinations gives the group for the 1/8 replicate as listed in table 8. The contrasts with the larger numbers of symbols are desirable for the 2/8 replicate. They are attained by selecting $Y = UVX_4$, and $Z = W$, and the defining contrasts for the 2/8 replicate are:

$$YX_5 = UVX_4X_5 = X_1X_3X_4X_5$$

$$ZX_6 = WX_6 = X_1X_2X_3X_6$$

$$YZX_5X_6 = UVWX_4X_5X_6 = X_2X_4X_5X_6$$

and these contrasts are listed as the 2/8 replicate in table 8. For the 4/8 replicate the choice was $T = Z$ so that

$$TX_6 = ZX_6 = WX_6 = X_1X_2X_3X_6$$

and the 4/8 replicate fails to be of resolution 5. The question arises as to whether a better choice could have been made for the defining contrasts of the 1/8 replicate.

Achievement of the highest possible resolution number at each stage of a sequence of telescoping designs would be helped if the total number of symbols in the group of defining contrasts were as large as possible. For a $1/2^h$ fraction with g factors the maximum number of symbols was given in reference 5 as

$$A = g2^{h-1}$$

For the example of six factors with blocks of size 8, this number is:

Replicate	1/8	2/8	4/8
A	24	12	6

If a resolution 5 design is to be achieved at the 4/8 replicate, then TX_6 must contain at least 5 symbols. From the preceding table, the number cannot exceed 6. The maximum total number of symbols for the 2/8 replicate is 12 so that the numbers of symbols might be distributed among the contrasts as follows:

YX_5,	ZX_6,	YZX_5X_6
3	3	5
3	4	5
3	3	6

To have a resolution 3 design for the 1/8 replicate, all 7 defining contrasts must contain at least 3 symbols, but the total number cannot exceed 24. For the telescoping, three of the 7 defining contrasts must be distributed according to one of the three preceding distributions of symbols. Considering only the upper limit of 24, the possibilities are:

(3, 3, 3, 3, 3, 4, 5)

or

(3, 3, 3, 3, 3, 3, 6)

The multiplication of two defining contrasts each containing 3 symbols could result in defining contrasts of length 2, 4, or 6. Contrasts of length 2 would violate the condition that the design must be of resolution 3. If 3 contrasts are of length three, the multiplication of all pairwise combinations results in 3 contrasts at least of length 4. Therefore the preceding combinations are not attainable, that is a telescoping sequence cannot lead from a 1/8 replicate of resolution 3 to a 4/8 replicate of resolution 5. The sequence must be continued to the full replicate.

Identification of Parameters Estimated by Yates' Contrasts. The manner in which defining contrasts can be obtained for telescoping sequences of orthogonal blocks has been illustrated. Reference 1 shows how the defining contrasts were used to determine the detailed treatments in Yates' order. Reference 1 also shows how the results of the Yates' computation are identified with the appropriate sets of aliased parameters.

In the case of the first-degree experiments, if a two-factor interaction coefficient is aliased with a single-factor coefficient (if the sum of a two-factor coefficient and a single-factor coefficient is estimated by a single contrast), then the two-factor coefficient is assumed to be zero. If a contrast does not estimate any combination of two-factor or lower order coefficients, the contrast will be given a name by listing the lowest order set of interaction coefficients that it does estimate. For example, table 17 lists a treatment bcde, and the Yates' computation would give an estimator of β_{234} in the same row. From table 15 the full set of aliased parameters can be shown to be β_{234}, $-\beta_{1245}$, β_{147}, β_{126}, $-\beta_{3457}$, $-\beta_{2356}$, β_{367}, and $-\beta_{1567}$ of which the lowest order set is β_{234}, $+\beta_{147}$, $+\beta_{126}$, $+\beta_{367}$. Those parameters, the estimates of which are confounded with block effects, will be identified by attaching an asterisk to the parameters.

PROPERTIES OF RECOMMENDED DESIGNS. The designs are identified by code numbers. For example, Plan 1/8; 7f, 8t/b; 2b means that the design is a 1/8 replicate of a full factorial experiment with 7 factors, employing 8 treatments per block, and using 2 blocks. The order of presentation of the designs (tables 2 to 29) is the order of increasing numbers of factors. For a given number of factors, a sequence of designs with blocks of 8 treatments is presented first, followed by a sequence of designs with blocks of 16 treatments. Within any sequence, the order is the order of increasing numbers of blocks. The properties of the designs are summarized in table 30 and therefore table 30 serves as a "Table of Contents" for the designs.

Use of Resolution 4 Designs in Fitting First-Order Model. In general, the use of the first-order model as a prediction equation, with coefficients estimated from an experiment, requires the assumption that all second-order parameters are zero. However, circumstances might arise where the experimenter desired an approximate first-order predicting equation and ignored the existence of possible nonzero two-factor interactions. He might then prefer a resolution 4 design to a resolution 3 design because the estimates of the first-order coefficients would not be aliased with (biased by) two-factor interactions.

Minimum-size designs of resolution 4 are shown for 4 factors by table 2, for 5 factors by table 5, and for 6 factors by table 10. Minimum-size designs of resolution 4 for 7 and 8 factors were given by Natrella (ref. 8, p. 12-18), and these designs are also given in tables 28 and 29. Unfortunately, no success was achieved in trying to include the designs of tables 28 and 29 in the telescoping sequences of 7- and 8-factor blocked designs, that is, tables 21 to 27. However, the designs of tables 28 and 29 might be used for the very first trial of a Box-Wilson procedure, when the experimenter believed that he would be so far from an optimum condition that a first-order model would be a good enough approximation.

After such a trial he could move to a new design center and then elect a design capable of being sequentially expanded by blocks into designs of higher order, that is, the designs of tables 21 or 25.

Conditions for Using Resolution 3 and Resolution 4 Designs in Estimating the Second-Order Model. If the experimenter has prior knowledge that some of the two-factor interactions are zero, he may be able to choose the labels for his factors so that the nonzero interaction parameters can be estimated from designs of less than resolution 5. The specific cases are listed:

Table 2. - Plan 1/2; 4f; 8t/b; 1b. - If one of the factors (for example x_1) does not interact with the other factors, then all the remaining interactions are estimable (table 2). If x_1 is noninteracting, the estimated parameters are β_0, β_1, β_2, β_{34}, β_3, β_{24}, β_{23}, and β_4.

Table 5. - Plan 1/2; 5f; 8t/b; 2b. - The factor believed most likely to interact with other factors should be labeled x_4 because the plan (table 5) gives unconfounded estimates of β_{14}, β_{24}, β_{34}, and β_{45}. If any one of x_1, x_2, x_3, or x_5 does not interact with the others (for example, x_1) then all the remaining two-factor interactions are estimable and the estimated parameters are β_0, β_1, β_2, β_{35}, β_3, β_{25}, β_{23}, β_5, β_4, β_{14}, β_{24}, β_{345}, β_{34}, β_{245}, $(\beta^*_{234} + \beta^*_{145})$, and β_{45}. Under previously stated assumptions, the estimates of β_{14}, β_{345}, and β_{245} are assumed to be nothing more than random error.

Table 10. - Plan 1/4; 6f; 8t/b; 2b. - If x_1 does not interact with any other factor, and if x_2 does not interact with x_4, x_5, and x_6, then the parameters estimated are as follows: β_0, β_1, β_2, β_{36}, β_3, β_{45}, β_{23}, β_6, β_4, β_{35}, β_{56}, $(\beta^*_{124} + \beta^*_{156} + \beta^*_{235} + \beta^*_{346})$, β_{34}, β_5, β_{46}, and the estimate of $(\beta_{125} + \beta_{146} + \beta_{234} + \beta_{356})$ is assumed to be random error (table 10).

Table 11. - Plan 1/2; 6f; 8t/b; 4b. - If the label x_1 had been given to the most likely noninteracting factor in the design of table 10, the performance of the two augmenting blocks of table 11 would result in a design with all interactions estimable under the minimal assumptions that β_{12}, β_{13}, and β_{16} are zero.

Table 13. - Plan 1/4; 6f; 16t/b; 1b. - Assume that there are two groups of three factors and that each factor does not interact within its group. Give the factors within one group the labels x_1, x_2, and x_6 and label the factors of the other group x_3, x_4, and x_5. Then all the nonzero two-factor interaction coefficients (one factor from each group) are

estimable and are β_{13}, β_{14}, β_{15}, β_{23}, β_{24}, β_{25}, β_{36}, β_{46}, and β_{56} (table 15).

Table 18. - Plan 1/4; 7f; 8t/b; 4b. - This plan (table 18) becomes a suitable second-order design under the assumptions that x_1 does not interact with x_3, x_4 or x_6, and that x_2, x_5, and x_7 do not interact with each other.

Table 21. - Plan 1/8; 7f; 16t/b; 1b. - This plan (table 21) estimates two-factor interactions if x_1 is noninteracting, if x_2 is noninteracting ~~with x_3, x_4, x_5, and x_7~~, and if x_5 is noninteracting with x_4 and x_6.

Table 22. - Plan 1/4; 7f; 16t/b; 2b. - This plan (table 22) estimates all two-factor interactions if any one of x_1, x_2, x_4, or x_6 does not interact with the other factors of this group.

Table 26. - Plan 1/8; 8f; 16t/b; 2b. - This plan (table 26) estimates all interactions if x_8 is noninteracting with x_1, x_2, x_3, x_5, and x_7, and if x_3 is noninteracting with x_1, x_2, x_4, and x_6. Thus the label x_8 should be given to the least interacting variable, the label x_3 should be given to the next least interacting variable, the labels x_3, x_5, and x_7 should be given to the variables least likely to interact with x_8, and the labels x_4 and x_6 should be given to the variables least likely to interact with x_3.

CHOICE OF BLOCK SIZE. The present investigation assumes that the experimenter will wish to perform a block of treatments, analyze the data, and then perform another block of treatments, and that the block effects arise during the interruption of the experimenting for analyzing data (furnaces are overhauled, instruments are newly calibrated, etc.). Under these assumptions, block sizes 8 and 16 are particularly appropriate for experiments on 4 to 8 factors. On the other hand, the physical situation could limit the experimenter to smaller block sizes. Under such limitations, other designs would have to be synthesized, and the synthesis could be done according to rules already presented.

Another reason for using small block sizes is to protect against the hazard of missing values. If through accident, the observations from one or more treatments are missing from a block, the whole block could be rerun, especially if it is small. On the other hand, only the missing treatments need be run, if the experimenter can say that no block effect will arise between the new runs and the block from which observations are missing. If the design is not severely fractionated (if the number of

treatments is significantly larger than the number of parameters estimated), methods of estimating for missing values may be used (ref. 9 or 10).

Some attributes of the proposed designs are summarized in table 30. In the case of 4 factors, all coefficients are estimable from two blocks of size 8 and a single block of size 16 is of no advantage in estimating the parameters of a second-degree model. In the case of 7 factors, the attainment of a resolution 5 design requires 64 treatments for either blocks of size 8 or size 16, so that there is no clear advantage is using blocks of size 16. With 8 factors, the minimum first-order design requires 16 treatments, and this is the only block size presented for the problem with 8 factors. In the cases of 5 and 6 factors, the choice of a block size of 8 or 16 is particularly complex.

A comparison of the number of experimental units required in experimenting with block sizes of 8 and 16 for 5 and 6 factors is given in table 31. The column headed "Total number of units required" shows that for five factors, the break-even point for the two block sizes occurs at three repetitions of the first-order experiments. For six factors, the break-even point occurs for five repetitions of the first-degree experiments. In other words, if the experimenter believes that he will perform many cycles of experimenting with the method of steepest ascents, he should use a block size of 8 because it uses a relatively smaller number of experimental units. On the other hand, the block of size 16 uses a relatively smaller number of experimental units in the method of local exploration. The block size of 16 should be used if the experimenter believes he will spend relatively few cycles of experiments with the method of steepest ascents, less than three cycles with 5 factors or less than five cycles with 6 factors.

Maximum economy could be sought with a mixed strategy. The experimenter could use the block of size 8 until his intuition told him that the first-degree model might not be appropriate. He could then switch to the block of size 16. Its greater number of degrees of freedom for "lack of fit" would provide better information about the validity of the first-degree model, and on switching to the method of local exploration, fewer experimental units would be needed to complete the interaction model than if the smaller block had been used. Thus with five factors, one or two experiments of the method of steepest ascents should be performed with the small block size followed by a switch to the larger block. With six factors, the break-even point is not reached until the fifth design center. Furthermore, two blocks of size 8 (table 10) provide a resolution 4 design, whereas the single block of size 16 (table 13) is only a resolution 3 design. With six factors, the best strategy might consist of using blocks

of size 8 (table 9) until interactions were suspected, at which point the design could be enlarged to that of table 10. If no new design center were desired, the design could then be augmented to that of table 11. If the design of table 10 had not shown significant interactions, experimenting at a new design center could continue with the design of table 9, but if significant interactions had been shown, the new experimenting should begin with the design of table 13.

CONCLUDING REMARKS. Sequences of blocked designs of experiments have been presented that are telescoping, in the sense that the first block is a design for which main effects are measurable, and that subsequent blocks, as they are added to the design, allow models of successively greater generality to be fitted to all acquired observations at each stage. The sequences terminate in designs for which all two factor interactions are measurable.

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TABLE 1. - FULL 2^4 EXPERIMENT

Treatment	Response	Matrix of independent variables															
		X_0	X_1	X_2	X_1X_2	X_3	X_1X_3	X_2X_3	$X_1X_2X_3$	X_4	X_1X_4	X_2X_4	$X_1X_2X_4$	X_3X_4	$X_1X_3X_4$	$X_2X_3X_4$	$X_1X_2X_3X_4$
(1)	y_1	+1	-1	-1	+1	-1	+1	+1	-1	-1	+1	+1	-1	+1	-1	-1	+1
a	y_2	+1	+1	-1	-1	-1	-1	+1	+1	-1	-1	+1	+1	+1	+1	-1	-1
b	y_3	+1	-1	+1	-1	-1	+1	-1	+1	-1	+1	-1	+1	+1	-1	+1	-1
ab	y_4	+1	+1	+1	+1	-1	-1	-1	-1	-1	-1	-1	-1	+1	+1	+1	+1
c	y_5	+1	-1	-1	+1	+1	-1	-1	+1	-1	+1	+1	-1	-1	+1	+1	-1
ac	y_6	+1	+1	-1	-1	+1	+1	-1	-1	-1	-1	+1	+1	-1	-1	+1	+1
bc	y_7	+1	-1	+1	-1	+1	-1	+1	-1	-1	+1	-1	+1	-1	+1	-1	+1
abc	y_8	+1	+1	+1	+1	+1	+1	+1	+1	-1	-1	-1	-1	-1	-1	-1	-1
d	y_9	+1	-1	-1	+1	-1	+1	+1	-1	+1	-1	-1	+1	-1	+1	+1	-1
ad	y_{10}	+1	+1	-1	-1	-1	-1	+1	+1	+1	+1	-1	-1	-1	-1	+1	+1
bd	y_{11}	+1	-1	+1	-1	-1	+1	-1	+1	+1	-1	+1	-1	-1	+1	-1	+1
abd	y_{12}	+1	+1	+1	+1	-1	-1	-1	-1	+1	+1	+1	+1	-1	-1	-1	-1
cd	y_{13}	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1	+1
acd	y_{14}	+1	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1
bcd	y_{15}	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1
abcd	y_{16}	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1

Table 2.[a] - PLAN 1/2; 4f; 8t/b; 1b -

R = 4

$[X_0 = X_1X_2X_3X_4.]$

Block	Treatment	Estimated effects
1	(1)	β_0
1	ad	β_1
1	bd	β_2
1	ab	$\beta_{12} + \beta_{34}$
1	cd	β_3
1	ac	$\beta_{13} + \beta_{24}$
1	bc	$\beta_{23} + \beta_{14}$
1	abcd	β_4

[a]Refs. 15 (p. 484) and 12 (p. 12-16).

TABLE 3.[a] - PLAN 1; 4f; 8t/b; 2b -

R = 5

[Block confounding, $X_1X_2X_3X_4$.]

Block	Treatment	Estimated effects (b)
1	(1)	β_0
2	a	β_1
2	b	β_2
1	ab	β_{12}
2	c	β_3
1	ac	β_{13}
1	bc	β_{23}
2	abc	β_{123}
2	d	β_4
1	ad	β_{14}
1	bd	β_{24}
2	abd	β_{124}
1	cd	β_{34}
2	acd	β_{134}
2	bcd	β_{234}
1	abcd	β^*_{1234}

[a]Refs. 3 (p. 429) and 8 (p. 12-10).
[b]Asterisk denotes confounding with blocks.

TABLE 4. - PLAN 1/4; 5f; 8t/b; 1b -

R = 3

$[X_0 = -X_2X_3X_4 = X_1X_2X_3X_5 = -X_1X_4X_5]$

Block	Treatment	Estimated effects
1	(1)	β_0
1	ae	$\beta_1 - \beta_{45}$
1	bde	$\beta_2 - \beta_{34}$
1	abd	$\beta_{12} + \beta_{35}$
1	cde	$\beta_3 - \beta_{24}$
1	acd	$\beta_{13} + \beta_{25}$
1	bc	$-\beta_4 + \beta_{23} + \beta_{15}$
1	abce	$\beta_5 - \beta_{14}$

TABLE 5. - PLAN 1/2; 5f; 8t/b; 2b -

R = 4

$[X_0 = X_1X_2X_3X_5$; block confounding, $-X_2X_3X_4]$

Block	Treatment	Estimated effects (a)
1	(1)	β_0
1	ae	β_1
2	be	β_2
2	ab	$\beta_{12} + \beta_{35}$
2	ce	β_3
2	ac	$\beta_{13} + \beta_{25}$
1	bc	$\beta_{23} + \beta_{15}$
1	abce	β_5
2	d	β_4
2	ade	β_{14}
1	bde	β_{24}
1	abd	$\beta_{124} + \beta_{345}$
1	cde	β_{34}
1	acd	$\beta_{134} + \beta_{245}$
2	bcd	$\beta^*_{234} + \beta^*_{145}$
2	abcde	β_{45}

[a]Asterisk denotes confounding with blocks.

TABLE 6. - PLAN 1; 5f; 8t/b; 4b

[Block confounding, $-X_2X_3X_4$, $-X_1X_4X_5$, $X_1X_2X_3X_5$]

Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	(1)	β_0	4	e	β_5
4	a	β_1	1	ae	β_{15}
3	b	β_2	2	be	β_{25}
2	ab	β_{12}	3	abe	β_{125}
3	c	β_3	2	ce	β_{35}
2	ac	β_{13}	3	ace	β_{135}
1	bc	β_{23}	4	bce	β_{235}
4	abc	β_{123}	1	abce	β^*_{1235}
2	d	β_4	3	de	β_{45}
3	ad	β_{14}	2	ade	β^*_{145}
4	bd	β_{24}	1	bde	β_{245}
1	abd	β_{124}	4	abde	β_{1245}
4	cd	β_{34}	1	cde	β_{345}
1	acd	β_{134}	4	acde	β_{1345}
2	bcd	β^*_{234}	3	bcde	β_{2345}
3	abcd	β_{1234}	2	abcde	β_{12345}

[a]Asterisk denotes confounding with blocks.

TABLE 7.[a] - PLAN 1/2; 5f; 16t/b; 1b - R = 5

[$X_0 = -X_1X_2X_3X_4X_5$]

Block	Treatment	Estimated effects
1	(1)	β_0
1	ae	β_1
1	be	β_2
1	ab	β_{12}
1	ce	β_3
1	ac	β_{13}
1	bc	β_{23}
1	abce	$-\beta_{45}$
1	de	β_4
1	ad	β_{14}
1	bd	β_{24}
1	abde	$-\beta_{35}$
1	cd	β_{34}
1	acde	$-\beta_{25}$
1	bcde	$-\beta_{15}$
1	abcd	$-\beta_5$

[a]Refs. 3 (p. 485) and 6 (p. 12-16).

TABLE 8. - DEFINING CONTRASTS, 6 FACTORS ON BLOCKS OF 8 TREATMENTS

Source	Defining contrasts		
	1/8 Replicate	1/4 Replicate	1/2 Replicate
X_4^2	$-X_1X_2X_4$		
X_5^2	$-X_2X_3X_5$		
X_6^2	$X_1X_2X_3X_6$	$X_1X_2X_3X_6$	$X_1X_2X_3X_6$
$X_4^2X_5^2$	$X_1X_3X_4X_5$	$X_1X_3X_4X_5$	
$X_4^2X_6^2$	$-X_3X_4X_6$		
$X_5^2X_6^2$	$-X_1X_5X_6$		
$X_4^2X_5^2X_6^2$	$X_2X_4X_5X_6$	$X_2X_4X_5X_6$	

TABLE 9. - PLAN 1/8; 6f; 8t/b; 1b - R = 3

$[X_0 = -X_1X_2X_4 = -X_2X_3X_5 = X_1X_2X_3X_6 = X_1X_3X_4X_5 = -X_3X_4X_6 = -X_1X_5X_6 = X_2X_4X_5X_6.]$

Block	Treatment	Estimated effects
1	(1)	β_0
1	adf	$\beta_1 - \beta_{24} - \beta_{56}$
1	bdef	$\beta_2 - \beta_{35} - \beta_{14}$
1	abe	$-\beta_4 + \beta_{12} + \beta_{36}$
1	cef	$\beta_3 - \beta_{25} - \beta_{46}$
1	acde	$\beta_{13} + \beta_{26} + \beta_{45}$
1	bcd	$-\beta_5 + \beta_{23} + \beta_{16}$
1	abcf	$\beta_6 - \beta_{15} - \beta_{34}$

TABLE 10. - PLAN 1/4; 6f; 8t/b; 2b - R=4

$[X_0 = X_1X_2X_3X_6 = X_1X_3X_4X_5 = X_2X_4X_5X_6$; block confounding, $-X_1X_2X_4]$

Block	Treatment	Estimated effects (a)
1	(1)	β_0
2	aef	β_1
2	bf	β_2
1	abe	$\beta_{12} + \beta_{36}$
1	cef	β_3
2	ac	$\beta_{13} + \beta_{45} + \beta_{26}$
2	bce	$\beta_{23} + \beta_{16}$
1	abcf	β_6
2	de	β_4
1	adf	$\beta_{14} + \beta_{35}$
1	bdef	$\beta_{24} + \beta_{56}$
2	abd	$\beta^*_{124} + \beta^*_{156} + \beta^*_{235} + \beta^*_{346}$
2	cdf	$\beta_{15} + \beta_{34}$
1	acde	β_5
1	bcd	$\beta_{125} + \beta_{146} + \beta_{234} + \beta_{356}$
2	abcdef	$\beta_{25} + \beta_{46}$

[a]Asterisk denotes confounding with blocks.

TABLE 11. - PLAN 1/2; 6f; 8t/b; 4b - R = 4

$[X_0 = X_1X_2X_3X_6$; block confounding, $-X_1X_2X_4$, $-X_2X_3X_5$, $X_1X_3X_4X_5]$

Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	(1)	β_0	4	e	β_5
3	af	β_1	2	aef	β_{15}
2	bf	β_2	3	bef	β_{25}
4	ab	$\beta_{12} + \beta_{36}$	1	abe	$\beta_{125} + \beta_{356}$
4	cf	β_3	1	cef	β_{35}
2	ac	$\beta_{13} + \beta_{26}$	3	ace	$\beta_{135} + \beta_{256}$
3	bc	$\beta_{23} + \beta_{16}$	2	bce	$\beta^*_{235} + \beta^*_{156}$
1	abcf	β_6	4	abcef	β_{56}
3	d	β_4	2	de	β_{45}
1	adf	β_{14}	4	adef	β_{145}
4	bdf	β_{24}	1	bdef	β_{245}
2	abd	$\beta^*_{124} + \beta^*_{346}$	3	abde	$\beta_{1245} + \beta_{3456}$
2	cdf	β_{34}	3	cdef	β_{345}
4	acd	$\beta_{134} + \beta_{246}$	1	acde	$\beta^*_{1345} + \beta^*_{2456}$
1	bcd	$\beta_{234} + \beta_{146}$	4	bcde	$\beta_{2345} + \beta_{1456}$
3	abcdf	β_{46}	2	abcdef	β_{456}

[a]Asterisk denotes confounding with blocks.

TABLE 12. - PLAN 1; 6f; 8t/b; 8b

[Block confounding, $-X_1X_2X_4$, $-X_2X_3X_5$, $X_1X_2X_3X_6$, $X_1X_3X_4X_5$, $-X_3X_4X_6$, $-X_1X_5X_6$, $X_2X_4X_5X_6$]

Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	1	β_0	4	e	β_5	8	f	β_6	7	ef	β_{56}
5	a	β_1	6	ae	β_{15}	3	af	β_{16}	2	aef	β^*_{156}
6	b	β_2	5	be	β_{25}	2	bf	β_{26}	3	bef	β_{256}
4	ab	β_{12}	1	abe	β_{125}	7	abf	β_{126}	8	abef	β_{1256}
7	c	β_3	8	ce	β_{35}	4	cf	β_{36}	1	cef	β_{356}
2	ac	β_{13}	3	ace	β_{135}	6	acf	β_{136}	5	acef	β_{1356}
3	bc	β_{23}	2	bce	β^*_{235}	5	bcf	β_{236}	6	bcef	β_{2356}
8	abc	β_{123}	7	abce	β_{1235}	1	abcf	β^*_{1236}	4	abcef	β_{12356}
3	d	β_4	2	de	β_{45}	5	df	β_{46}	6	def	β_{456}
8	ad	β_{14}	7	ade	β_{145}	1	adf	β_{146}	4	adef	β_{1456}
7	bd	β_{24}	8	bde	β_{245}	4	bdf	β_{246}	1	bdef	β^*_{2456}
2	abd	β^*_{124}	3	abde	β_{1245}	6	abdf	β_{1246}	5	abdef	β_{12456}
6	cd	β_{34}	5	cde	β_{345}	2	cdf	β^*_{346}	3	cdef	β_{3456}
4	acd	β_{134}	1	acde	β^*_{1345}	7	acdf	β_{1346}	8	acdef	β_{13456}
1	bcd	β_{234}	4	bcde	β_{2345}	8	bcdf	β_{2346}	7	bcdef	β_{23456}
5	abcd	β_{1234}	6	abcde	β_{12345}	3	abcdf	β_{12346}	2	abcdef	β_{123456}

[a]Asterisk denotes confounding with blocks.

TABLE 13. - PLAN 1/4; 8f; 16t/b; 1b - R = 3

$[X_0 = -X_3X_4X_5 = -X_1X_2X_6 = X_1X_2X_3X_4X_5X_6]$

Block	Treatment	Estimated effects
1	(1)	β_0
1	af	$\beta_1 - \beta_{26}$
1	bf	$\beta_2 - \beta_{16}$
1	ab	$-\beta_6 + \beta_{12}$
1	ce	$\beta_3 - \beta_{45}$
1	acef	β_{13}
1	bcef	β_{23}
1	abce	$-\beta_{36}$
1	de	$\beta_4 - \beta_{35}$
1	adef	β_{14}
1	bdef	β_{24}
1	abde	$-\beta_{46}$
1	cd	$-\beta_5 + \beta_{34}$
1	acdf	$-\beta_{15}$
1	bcdf	$-\beta_{25}$
1	abcd	β_{56}

TABLE 14.[a] - PLAN 1/2; 8f; 16t/b; 2b - R = 5

$[X_0 = X_1X_2X_3X_4X_5X_6$; block confounding, $-X_3X_4X_5]$

Block	Treatment	Estimated effects	Block	Treatment	Estimated effects (b)
1	(1)	β_0	2	ef	β_5
1	af	β_1	2	ae	β_{15}
1	bf	β_2	2	be	β_{25}
1	ab	β_{12}	2	abef	$\beta_{125} + \beta_{346}$
2	cf	β_3	1	ce	β_{35}
2	ac	β_{13}	1	acef	$\beta_{135} + \beta_{246}$
2	bc	β_{23}	1	bcef	$\beta_{235} + \beta_{146}$
2	abcf	$\beta_{123} + \beta_{456}$	1	abce	β_{46}
2	df	β_4	1	de	β_{45}
2	ad	β_{14}	1	adef	$\beta_{145} + \beta_{236}$
2	bd	β_{24}	1	bdef	$\beta_{245} + \beta_{136}$
2	abdf	$\beta_{124} + \beta_{356}$	1	abde	β_{36}
1	cd	β_{34}	2	cdef	$\beta^*_{345} + \beta^*_{126}$
1	acdf	$\beta_{134} + \beta_{256}$	2	acde	β_{26}
1	bcdf	$\beta_{234} + \beta_{156}$	2	bcde	β_{16}
1	abcd	β_{56}	2	abcdef	β_6

[a] Reduced (-8).

[b] Asterisk denotes confounding with blocks.

TABLE 15. - DEFINING CONTRASTS WITH 7 FACTORS ON BLOCKS OF 8 TREATMENTS

Source	Defining contrasts			
	1/16 Replicate	1/8 Replicate	1/4 Replicate	1/2 Replicate
x_4^2	$-x_1x_2x_4$			
x_5^2	$-x_1x_3x_5$	$-x_1x_3x_5$		
x_6^2	$-x_2x_3x_6$			
x_7^2	$x_1x_2x_3x_7$	$x_1x_2x_3x_7$		
$x_4^2x_5^2$	$x_2x_3x_4x_5$			
$x_4^2x_6^2$	$x_1x_3x_4x_6$	$x_1x_3x_4x_6$	$x_1x_3x_4x_6$	
$x_4^2x_7^2$	$-x_3x_4x_7$			
$x_5^2x_6^2$	$x_1x_2x_5x_6$			
$x_5^2x_7^2$	$-x_2x_5x_7$	$-x_2x_5x_7$	$-x_2x_5x_7$	
$x_6^2x_7^2$	$-x_1x_6x_7$			
$x_4^2x_5^2x_6^2$	$-x_4x_5x_6$	$-x_4x_5x_6$		
$x_4^2x_5^2x_7^2$	$x_1x_4x_5x_7$			
$x_4^2x_6^2x_7^2$	$x_2x_4x_6x_7$	$x_2x_4x_6x_7$		
$x_5^2x_6^2x_7^2$	$x_3x_5x_6x_7$			
$x_4^2x_5^2x_6^2x_7^2$	$-x_1x_2x_3x_4x_5x_6x_7$	$-x_1x_2x_3x_4x_5x_6x_7$	$-x_1x_2x_3x_4x_5x_6x_7$	$-x_1x_2x_3x_4x_5x_6x_7$

TABLE 16. - PLAN 1/16; 7f; 8t/b; 1b -

R = 3

[Defining contrasts given by table 15.]

Block	Treatment	Estimated effects
1	(1)	β_0
1	adeg	$\beta_1 - \beta_{24} - \beta_{35} - \beta_{67}$
1	bdfg	$\beta_2 - \beta_{14} - \beta_{36} - \beta_{57}$
1	abef	$-\beta_4 + \beta_{12} + \beta_{37} + \beta_{56}$
1	cefg	$\beta_3 - \beta_{15} - \beta_{26} - \beta_{47}$
1	acdf	$-\beta_5 + \beta_{13} + \beta_{46} + \beta_{27}$
1	bcde	$-\beta_6 + \beta_{23} + \beta_{17} + \beta_{45}$
1	abcg	$\beta_7 - \beta_{34} - \beta_{25} - \beta_{16}$

TABLE 17. - PLAN 1/8; 7f; 8t/b; 2b - R = 3

[Defining contrasts given by table 15; block confounding, $-X_1X_2X_4$.]

Block	Treatment	Estimated effects (a)
1	(1)	β_0
2	aefg	$\beta_1 - \beta_{35}$
2	bg	$\beta_2 - \beta_{57}$
1	abef	$\beta_{12} + \beta_{37}$
1	cefg	$\beta_3 - \beta_{15}$
2	ac	$-\beta_5 + \beta_{13} + \beta_{46} + \beta_{27}$
2	bcef	$\beta_{23} + \beta_{17}$
1	abcg	$\beta_7 - \beta_{25}$
2	df	$\beta_4 - \beta_{56}$
1	adeg	$\beta_{14} + \beta_{36}$
1	bdfg	$\beta_{24} + \beta_{67}$
2	abde	$\beta^*_{124} + \beta^*_{347} + \beta^*_{236} + \beta^*_{167}$
2	cdeg	$\beta_{34} + \beta_{16}$
1	acdf	$\beta_6 - \beta_{45}$
1	bcde	$\beta_{147} + \beta_{126} + \beta_{367} + \beta_{234}$
2	abcdfg	$\beta_{47} + \beta_{26}$

[a]Asterisk denotes confounding with blocks.

TABLE 18. - PLAN 1/4; 7f; 8t/b; 4b - R = 3

[Defining contrasts given by table 15; block confounding, $-X_1X_2X_4$, $-X_1X_3X_5$, $X_2X_3X_4X_5$.]

Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	(1)	β_0	4	eg	$\beta_5 - \beta_{27}$
3	af	β_1	2	aefg	β_{15}
2	bg	$\beta_2 - \beta_{57}$	3	be	$-\beta_7 + \beta_{25}$
4	abfg	β_{12}	1	abef	$-\beta_{17}$
4	cf	β_3	1	cefg	β_{35}
2	ac	$\beta_{13} + \beta_{46}$	3	aceg	$\beta^*_{135} + \beta^*_{456}$
3	bcfg	β_{23}	2	bcef	$-\beta_{37}$
1	abcg	$\beta_{123} + \beta_{246}$	4	abce	$-\beta_{137} - \beta_{467}$
2	df	β_4	3	defg	β_{45}
4	ad	$\beta_{14} + \beta_{36}$	1	adeg	$\beta_{145} + \beta_{356}$
1	bdfg	β_{24}	4	bdef	$-\beta_{47}$
3	abdg	$\beta^*_{124} + \beta^*_{236}$	2	abde	$-\beta_{147} - \beta_{367}$
3	cd	$\beta_{34} + \beta_{16}$	2	cdeg	$\beta_{345} + \beta_{156}$
1	acdf	β_6	4	acdefg	β_{56}
4	bcdg	$\beta_{234} + \beta_{126}$	1	bcde	$-\beta^*_{347} - \beta^*_{167}$
2	abcdfg	β_{26}	3	abcdef	$-\beta_{67}$

[a]Asterisk denotes confounding with blocks.

TABLE 19. - PLAN 1/2; 7f; 8t/b, 8b

[$X_0 = -X_1X_2X_3X_4X_5X_6X_7$; block confounding, $-X_1X_2X_4$, $-X_1X_3X_5$, $-X_2X_3X_6$, $X_2X_3X_4X_5$, $X_1X_3X_4X_6$, $X_1X_2X_5X_6$, $-X_4X_5X_6$]

Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	(1)	β_0	4	eg	β_5	7	fg	β_6	5	ef	β_{56}
6	ag	β_1	8	ae	β_{15}	3	af	β_{16}	2	aefg	β_{156}
2	bg	β_2	3	be	β_{25}	8	bf	β_{26}	6	befg	β_{256}
5	ab	β_{12}	7	abeg	β_{125}	4	abfg	β_{126}	1	abef	$-\beta^*_{347}$
5	cg	β_3	7	ce	β_{35}	4	cf	β_{36}	1	cefg	β_{356}
2	ac	β_{13}	3	aceg	β^*_{135}	8	acfg	β_{136}	6	acef	$-\beta_{247}$
6	bc	β_{23}	8	bceg	β_{235}	3	bcfg	β^*_{236}	2	bcef	$-\beta_{147}$
1	abcg	β_{123}	4	abce	$-\beta_{467}$	7	abcf	$-\beta_{457}$	5	abcefg	$-\beta_{47}$
8	dg	β_4	6	de	β_{45}	2	df	β_{46}	3	defg	β^*_{456}
4	ad	β_{14}	1	adeg	β_{145}	5	adfg	β_{146}	7	adef	$-\beta_{237}$
7	bd	β_{24}	5	bdeg	β_{245}	1	bdfg	β_{246}	4	bdef	$-\beta_{137}$
3	abdg	β^*_{124}	2	abde	$-\beta_{367}$	6	abdf	$-\beta_{357}$	8	abdefg	$-\beta_{37}$
3	cd	β_{34}	2	cdeg	β_{345}	6	cdfg	β_{346}	8	cdef	$-\beta_{127}$
7	acdg	β_{134}	5	acde	$-\beta_{267}$	1	acdf	$-\beta^*_{257}$	4	acdefg	$-\beta_{27}$
4	bcdg	β_{234}	1	bcde	$-\beta^*_{167}$	5	bcdf	$-\beta_{157}$	7	bcdefg	$-\beta_{17}$
8	abcd	$-\beta_{567}$	6	abcdeg	$-\beta_{67}$	2	abcdfg	$-\beta_{57}$	3	abcdef	$-\beta_7$

[a] Asterisk denotes confounding with blocks.

TABLE 20. - DEFINING CONTRASTS WITH 7 FACTORS ON BLOCKS OF 16 TREATMENTS

Source	Defining contrasts		
	1/8 Replicate	1/4 Replicate	1/2 Replicate
x_5^2	$-x_1x_4x_5$		
x_6^2	$x_1x_2x_4x_6$	$x_1x_2x_4x_6$	
x_7^2	$x_2x_3x_4x_7$		
$x_5^2x_6^2$	$-x_2x_5x_6$		
$x_5^2x_7^2$	$-x_1x_2x_3x_5x_7$	$-x_1x_2x_3x_5x_7$	
$x_6^2x_7^2$	$x_1x_3x_6x_7$		
$x_5^2x_6^2x_7^2$	$-x_3x_4x_5x_6x_7$	$-x_3x_4x_5x_6x_7$	$-x_3x_4x_5x_6x_7$

TABLE 21. - PLAN 1/8; 7f; 16t/b; 1b - R = 3

[Defining contrasts given in table 20.]

Block	Treatment	Estimated effects
1	(1)	β_0
1	aef	$\beta_1 - \beta_{45}$
1	bfg	$\beta_2 - \beta_{56}$
1	abeg	$\beta_{12} + \beta_{46}$
1	cg	β_3
1	acefg	$\beta_{13} + \beta_{67}$
1	bcf	$\beta_{23} + \beta_{47}$
1	abce	$-\beta_{57}$
1	defg	$\beta_4 - \beta_{15}$
1	adg	$-\beta_5 + \beta_{14} + \beta_{26}$
1	bde	$\beta_{24} + \beta_{16} + \beta_{37}$
1	abdf	$\beta_6 - \beta_{25}$
1	cdef	$\beta_{34} + \beta_{27}$
1	acd	$-\beta_{35}$
1	bcdeg	β_7
1	abcdfg	$\beta_{36} + \beta_{17}$

TABLE 22.[a] - PLAN 1/4; 7f; 16t/b; 2b - R = 4

[Defining contrasts given in table 20; block confounding, $-X_1X_4X_5$.]

Block	Treatment	Estimated effects	Block	Treatment	Estimated effects (b)
1	(1)	β_0	2	eg	β_5
2	afg	β_1	1	aef	β_{15}
1	bfg	β_2	2	bef	β_{25}
2	ab	$\beta_{12} + \beta_{46}$	1	abeg	$-\beta_{37}$
1	cg	β_3	2	ce	β_{35}
2	acf	β_{13}	1	acefg	$-\beta_{27}$
1	bcf	β_{23}	2	bcefg	$-\beta_{17}$
2	abcg	$-\beta_{57}$	1	abce	$-\beta_7$
2	df	β_4	1	defg	β_{45}
1	adg	$\beta_{14} + \beta_{26}$	2	ade	$\beta^*_{145} + \beta^*_{256}$
2	bdg	$\beta_{24} + \beta_{16}$	1	bde	$\beta_{245} + \beta_{156}$
1	abdf	β_6	2	abdefg	β_{56}
2	cdfg	β_{34}	1	cdef	$-\beta_{67}$
1	acd	$\beta_{134} + \beta_{236}$	2	acdeg	$-\beta_{247} - \beta_{167}$
2	bcd	$\beta_{234} + \beta_{136}$	1	bcdeg	$-\beta_{147} - \beta_{267}$
1	abcdfg	β_{36}	2	abcdef	$-\beta_{47}$

[a] [redacted].

[b] Asterisk denotes confounding with blocks.

TABLE 23. - PLAN 1/2; 7f; 16t/b; 4b - R = 5

$[X_0 = -X_3X_4X_5X_6X_7$; block confounding, $-X_1X_4X_5$, $-X_2X_5X_6$, $X_1X_2X_4X_6]$

Block	Treatment	Estimated effects	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	(1)	β_0	2	eg	β_5	4	fg	β_6	3	ef	β_{56}
3	a	β_1	4	aeg	β_{15}	2	afg	β_{16}	1	aef	β_{156}
4	b	β_2	3	beg	β_{25}	1	bfg	β_{26}	2	bef	β^*_{256}
2	ab	β_{12}	1	abeg	β_{125}	3	abfg	β_{126}	4	abef	β_{1256}
1	cg	β_3	2	ce	β_{35}	4	cf	β_{36}	3	cefg	$-\beta_{47}$
3	acg	β_{13}	4	ace	β_{135}	2	acf	β_{136}	1	acefg	$-\beta_{147}$
4	bcg	β_{23}	3	bce	β_{235}	1	bcf	β_{236}	2	bcefg	$-\beta_{247}$
2	abcg	β_{123}	1	abce	β_{1235}	3	abcf	β_{1236}	4	abcefg	$-\beta_{1247}$
3	dg	β_4	4	de	β_{45}	2	df	β_{46}	1	defg	$-\beta_{37}$
1	adg	β_{14}	2	ade	β^*_{145}	4	adf	β_{146}	3	adefg	$-\beta_{137}$
2	bdg	β_{24}	1	bde	β_{245}	3	bdf	β_{246}	4	bdefg	$-\beta_{237}$
4	abdg	β_{124}	3	abde	β_{1245}	1	abdf	β^*_{1246}	2	abdefg	$-\beta_{1237}$
3	cd	β_{34}	4	cdeg	$-\beta_{67}$	2	cdfg	$-\beta_{57}$	1	cdef	$-\beta_7$
1	acd	β_{134}	2	acdeg	$-\beta_{167}$	4	acdfg	$-\beta_{157}$	3	acdef	$-\beta_{17}$
2	bcd	β_{234}	1	bcdeg	$-\beta_{267}$	3	bcdfg	$-\beta_{257}$	4	bcdef	$-\beta_{27}$
4	abcd	β_{1234}	3	abcdeg	$-\beta_{1267}$	1	abcdfg	$-\beta_{1257}$	2	abcdef	$-\beta_{127}$

[a]Asterisk denotes confounding with blocks.

TABLE 24. - DEFINING CONTRASTS WITH 8 FACTORS

ON BLOCKS OF 16 TREATMENTS

Source	Defining contrasts		
	1/16 Replicate	1/8 Replicate	1/4 Replicate
X_5^2	$-X_1X_4X_5$		
X_6^2	$X_1X_3X_4X_6$	$X_1X_3X_4X_6$	
X_7^2	$X_2X_3X_4X_7$		
X_8^2	$-X_2X_3X_8$	$-X_2X_3X_8$	
$X_5^2X_6^2$	$-X_3X_5X_6$		
$X_5^2X_7^2$	$-X_1X_2X_3X_5X_7$	$-X_1X_2X_3X_5X_7$	$-X_1X_2X_3X_5X_7$
$X_5^2X_8^2$	$X_1X_2X_3X_4X_5X_8$		
$X_6^2X_7^2$	$X_1X_2X_6X_7$		
$X_6^2X_8^2$	$-X_1X_2X_4X_6X_8$	$-X_1X_2X_4X_6X_8$	$-X_1X_2X_4X_6X_8$
$X_7^2X_8^2$	$-X_4X_7X_8$		
$X_5^2X_6^2X_7^2$	$-X_2X_4X_5X_6X_7$	$-X_2X_4X_5X_6X_7$	
$X_5^2X_6^2X_8^2$	$X_2X_5X_6X_8$		
$X_5^2X_7^2X_8^2$	$X_1X_5X_7X_8$	$X_1X_5X_7X_8$	
$X_6^2X_7^2X_8^2$	$-X_1X_3X_6X_7X_8$		
$X_5^2X_6^2X_7^2X_8^2$	$X_3X_4X_5X_6X_7X_8$	$X_3X_4X_5X_6X_7X_8$	$X_3X_4X_5X_6X_7X_8$

TABLE 25. - PLAN 1/16; 8f; 16t/b; 1b - R = 3

[Defining contrasts given in table 24.]

Block	Treatment	Estimated effects
1	(1)	β_0
1	aef	$\beta_1 - \beta_{45}$
1	bgh	$\beta_2 - \beta_{38}$
1	abefgh	$\beta_{12} + \beta_{67}$
1	cfgh	$\beta_3 - \beta_{28} - \beta_{56}$
1	acegh	$\beta_{13} + \beta_{46}$
1	bcf	$-\beta_8 + \beta_{23} + \beta_{47}$
1	abce	$-\beta_{57} - \beta_{18}$
1	defg	$\beta_4 - \beta_{15} - \beta_{78}$
1	adg	$-\beta_5 + \beta_{14} + \beta_{36}$
1	bdefh	$\beta_{24} + \beta_{37}$
1	abdh	$-\beta_{68} - \beta_{25}$
1	cdeh	$\beta_{34} + \beta_{16} + \beta_{27}$
1	acdfh	$\beta_6 - \beta_{35}$
1	bcdeg	$\beta_7 - \beta_{48}$
1	abcdfg	$\beta_{58} + \beta_{17} + \beta_{26}$

TABLE 26. - PLAN 1/8; 8f; 16t/b; 2b - R = 3

[Defining contrasts given in table 24; block confounding, $-X_1X_4X_5$]

Block	Treatment	Estimated effects	Block	Treatment	Estimated effects (a)
1	(1)	β_0	2	eg	β_5
2	afg	β_1	1	aef	$\beta_{15} + \beta_{78}$
1	bgh	$\beta_2 - \beta_{38}$	2	beh	β_{25}
2	abfh	β_{12}	1	abefgh	$-\beta_{37}$
1	cfgh	$\beta_3 - \beta_{28}$	2	cefh	β_{35}
2	ach	$\beta_{13} + \beta_{46}$	1	acegh	$-\beta_{27}$
1	bcf	$-\beta_8 + \beta_{23}$	2	bcefg	$-\beta_{58} - \beta_{17}$
2	abcg	$-\beta_{57} - \beta_{18}$	1	abce	$-\beta_7$
2	df	β_4	1	defg	β_{45}
1	adg	$\beta_{14} + \beta_{36}$	2	ade	$\beta^*_{145} + \beta^*_{356} + \beta^*_{478}$
2	bdfgh	β_{24}	1	bdefh	$-\beta_{67}$
1	abdh	$-\beta_{68}$	2	abdegh	$-\beta_{347} - \beta_{568} - \beta_{167}$
2	cdgh	$\beta_{34} + \beta_{16}$	1	cdeh	$\beta_{345} + \beta_{156} + \beta_{678}$
1	acdfh	β_6	2	acdefgh	β_{56}
2	bcd	$-\beta_{48}$	1	bcdeg	$-\beta_{147} - \beta_{367} - \beta_{458}$
1	abcdfg	β_{26}	2	abcdef	$-\beta_{47}$

[a]Asterisk denotes confounding with blocks.

TABLE 27.[a] - PLAN 1/4; 8f; 15t/b; 4b - R = 5

[Defining contrasts given in table 24; block confounding, $-X_1X_4X_5$, $X_1X_3X_4X_6$, $-X_3X_5X_6$.]

Block	Treatment	Estimated effects	Block	Treatment	Estimated effects (b)	Block	Treatment	Estimated effects (b)	Block	Treatment	Estimated effects (b)
1	(1)	β_0	2	eg	β_5	4	fh	β_6	3	efgh	β_{56}
3	agh	β_1	4	aeh	β_{15}	2	afg	β_{16}	1	aef	β_{156}
1	bgh	β_2	2	beh	β_{25}	4	bfg	β_{26}	3	bef	β_{256}
3	ab	β_{12}	4	abeg	$-\beta_{37}$	2	abfh	$-\beta_{48}$	1	abefgh	$-\beta_{367} - \beta_{458}$
4	cg	β_3	3	ce	β_{35}	1	cfgh	β_{36}	2	cefh	$\beta^*_{356} + \beta^*_{478}$
2	ach	β_{13}	1	acegh	$-\beta_{27}$	3	acf	β_{136}	4	acefg	$-\beta_{267}$
4	bch	β_{23}	3	bcegh	$-\beta_{17}$	1	bcf	β_{236}	2	bcefg	$-\beta_{167}$
2	abcg	$-\beta_{57}$	1	abce	$-\beta_7$	3	abcfgh	$-\beta_{567} - \beta_{348}$	4	abcefh	$-\beta_{67}$
3	dh	β_4	4	degh	β_{45}	2	df	β_{46}	1	defg	$\beta_{456} + \beta_{378}$
1	adg	β_{14}	2	ade	β^*_{145}	4	adfgh	$-\beta_{28}$	3	adefh	$-\beta_{258}$
3	bdg	β_{24}	4	bde	β_{245}	2	bdfgh	$-\beta_{18}$	1	bdefh	$-\beta_{158}$
1	abdh	$-\beta_{68}$	2	abdegh	$-\beta_{347} - \beta_{568}$	4	abdf	$-\beta_8$	3	abdefg	$-\beta_{58}$
2	cdgh	β_{34}	1	cdeh	$\beta_{345} + \beta_{678}$	3	cdfg	$\beta_{346} + \beta_{578}$	4	cdef	β_{78}
4	acd	β_{134}	3	acdeg	$-\beta_{247}$	1	acdfh	$-\beta^*_{238}$	2	acdefgh	β_{178}
2	bcd	β_{234}	1	bcdeg	$-\beta_{147}$	3	bcdfh	$-\beta_{138}$	4	bcdefgh	β_{278}
4	abcdgh	$-\beta_{457} - \beta_{368}$	3	abcdeh	$-\beta_{47}$	1	abcfg	$-\beta_{38}$	2	abcdef	$-\beta_{467} - \beta_{358}$

[a] ~~Rel. = 1 (g = 24).~~

[b] Asterisk denotes confounding with blocks.

TABLE 28.[a] - PLAN 1/8; 7f; 16t/b; 1b - R = 4

$$[X_0 = X_1X_2X_3X_5 = X_2X_3X_4X_6 = X_1X_2X_4X_7 = X_1X_4X_5X_6 = X_3X_4X_5X_7 = X_1X_3X_6X_7 = X_2X_5X_6X_7]$$

Block	Treatment	Estimated effects
1	(1)	β_0
1	aeg	β_1
1	befg	β_2
1	abf	$\beta_{12} + \beta_{35} + \beta_{47}$
1	cef	β_3
1	acfg	$\beta_{13} + \beta_{25} + \beta_{67}$
1	bcg	$\beta_{23} + \beta_{15} + \beta_{46}$
1	abce	β_5
1	dfg	β_4
1	adef	$\beta_{14} + \beta_{56} + \beta_{27}$
1	bde	$\beta_{24} + \beta_{36} + \beta_{17}$
1	abdg	β_7
1	cdeg	$\beta_{26} + \beta_{34} + \beta_{57}$
1	acd	$\beta_{134} + \beta_{126} + \beta_{237} + \beta_{245} + \beta_{356} + \beta_{467} + \beta_{157}$
1	bcdf	β_6
1	abcdefg	$\beta_{45} + \beta_{16} + \beta_{37}$

[a]Refs. 12 (p. 486) and 18 (p. 12-18).

TABLE 29.[a] - PLAN 1/16; 8f; 16t/b; 1b - R = 4

$$[X_0 = X_1X_2X_3X_5 = X_2X_3X_4X_6 = X_1X_2X_4X_7 = X_1X_3X_4X_8 = X_1X_4X_5X_6 = X_3X_4X_5X_7 = X_2X_4X_5X_8 = X_1X_3X_6X_7 = X_1X_2X_6X_8 = X_2X_3X_7X_8 = X_2X_5X_6X_7 = X_3X_5X_6X_8 = X_1X_5X_7X_8 = X_4X_6X_7X_8 = X_1X_2X_3X_4X_5X_6X_7X_8$$

Block	Treatment	Estimated effects
1	(1)	β_0
1	aegh	β_1
1	befg	β_2
1	abfh	$\beta_{12} + \beta_{35} + \beta_{47} + \beta_{68}$
1	cefh	β_3
1	acfg	$\beta_{13} + \beta_{25} + \beta_{48} + \beta_{67}$
1	bcgh	$\beta_{23} + \beta_{15} + \beta_{46} + \beta_{78}$
1	abce	β_5
1	dfgh	β_4
1	adef	$\beta_{14} + \beta_{27} + \beta_{38} + \beta_{56}$
1	bdeh	$\beta_{24} + \beta_{36} + \beta_{17} + \beta_{58}$
1	abdg	β_7
1	cdeg	$\beta_{34} + \beta_{26} + \beta_{18} + \beta_{57}$
1	acdh	β_8
1	bcdf	β_6
1	abcdefgh	$\beta_{45} + \beta_{16} + \beta_{37} + \beta_{28}$

[a]Refs. 12 (p. 486) and 18 (p. 12-18).

TABLE 30. - ATTRIBUTES OF RECOMMENDED DESIGNS

Table	Replication	Factors, g	Treatments per block	Number of blocks	Resolution, R	Number of two-factor interactions, g(g - 1)/2	Number of estimable two-factor interactions (a)
2	1/2	4	8	1	4	6	0
3	Full	4	8	2	5	6	6
4	1/4	5	8	1	3	10	0
5	1/2	5	8	2	4	10	4
6	Full	5	8	4	5	10	10
7	1/2	5	16	1	5	10	10
9	1/8	6	8	1	3	15	0
10	1/4	6	8	2	4	15	0
11	1/2	6	8	4	4	15	9
12	Full	6	8	8	5	15	15
13	1/4	6	16	1	3	15	9
14	1/2	6	16	2	5	15	15
16	1/16	7	8	1	3	21	0
17	1/8	7	8	2	3	21	0
18	1/4	7	8	4	3	21	11
19	1/2	7	8	8	5	21	21
21	1/8	7	16	1	3	21	1
22	1/4	7	16	2	4	21	15
23	1/2	7	16	4	5	21	21
25	1/16	8	16	1	3	28	0
26	1/8	8	16	2	3	28	11
27	1/4	8	16	4	5	28	28
28	1/8	7	16	1	4	21	0
29	1/16	8	16	1	4	28	0

[a]Only unconfounded two-factor interaction estimators are counted.

TABLE 31. - COMPARISON OF TOTAL TREATMENTS (EXPERIMENTAL UNITS) REQUIRED WHEN FIRST BLOCK IS PERFORMED TO ESTIMATE FIRST-ORDER MODEL AT STATED NUMBER OF DESIGN CENTERS AND INTERACTION EXPERIMENT IS PERFORMED ONLY AT FINAL DESIGN CENTER

Factors	Design centers for first-order model	Treatments for first-order model		Treatments for completion of interaction model		Total number of units required	
		Blocks of size 8	Blocks of size 16	Blocks of size 8	Blocks of size 16	Blocks of size 8	Blocks of size 16
5	1	8	16	24	0	32	16
5	2	16	32	24	0	40	32
5	3	24	48	24	0	48	48
5	4	32	64	24	0	56	64
6	1	8	16	56	16	64	32
6	2	16	32	56	16	72	48
6	3	24	48	56	16	80	64
6	4	32	64	56	16	88	80
6	5	40	80	56	16	96	96
6	6	48	96	56	16	104	112

ON A CLASS OF NONPARAMETRIC TESTS FOR INTERACTIONS IN FACTORIAL EXPERIMENTS*

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1. Summary and Introduction. This paper deals with a class of permutationally distribution-free aligned rank order tests for interactions in factorial experiments replicated in complete blocks. The asymptotic power-efficiencies of the proposed tests with respect to the classical analysis of variance test are also studied.

Nonparametric analysis of variance tests, available in the literature, mostly relate to one way or two way (without interaction) layouts. Though the approach of Lehmann (1964) (see also Puri and Sen (1966)) can be adapted to construct tests for interactions in factorial experiments, the necessity of avoiding incompatibility of the unadjusted estimates as well as of estimating some functional of the parent distribution (appearing in the expression for the dispersion matrix of the estimators) makes such tests only asymptotically distribution-free and somewhat tedious to apply. In the present paper, the theory of aligned rank order tests based on Chernoff-Savage (1958) type of statistics, developed in Sen (1968), is further extended to provide suitable tests for interactions in factorial layouts with equal number of observations per cell. Under certain permutational invariance arguments the nonparametric structure of the proposed tests is established. These tests are also free from the other two difficulties mentioned earlier. Further, using a generalization of Chernoff-Savage (1958) theorem on the asymptotic normality of rank order statistics to aligned observations, the asymptotic power-efficiencies of the proposed tests (along with certain bounds are studied.

2. Preliminary notions. We shall consider in detail only the case of replicated two factor experiments with one observation per call and indicate briefly the theory

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for the case of several factors and/or observations per cell. The chance variable Y_{ijk} associated with the yield of the plot placed in the ith replicate and receiving the combination of the jth variety (or level of the first factor) and the k-th treatment (or level of the second factor), is expressed, in accordance with the usual fixed-effects model, as

$$(2.1) \qquad Y_{ijk} = \mu_i + \nu_j + \tau_k + \eta_{jk} + U_{ijk}; \quad i=1,\ldots,n;\ j=1,\ldots,p;\ k=1,\ldots,q,$$

where $\mu_1,\ldots,\mu_n$ stand for the replication-effects, $\nu_1,\ldots,\nu_p$ for the variety-effects, $\tau_1,\ldots,\tau_q$ for the treatment-effects, $\eta_{11},\ldots\eta_{pq}$ for the variety x treatment interactions, and U_{ijk}'s are the residual error components. In (2.1), we may put

$$(2.2) \qquad \sum_{j=1}^{p} \nu_j=0,\ \sum_{k=1}^{q} \tau_k=0,\ \sum_{k=1}^{q} \eta_{jk}=0,\ j=1,\ldots,p,\ \text{and}\ \sum_{j=1}^{p} \eta_{jk}=0,\ k=1,\ldots,q.$$

It is assumed that $(U_{i11},\ldots,U_{ipq})$, $i=1,\ldots,n$ are independent and identically distributed stochastic vectors having a common continuous (joint) cumulative distribution function (cdf) $F(x_{11},\ldots,x_{pq})$ which is symmetric in its pq arguments; this includes the conventional assumption of independence and identity of distributions of all the npq error components as a particular case. We want to test

$$(2.3) \qquad H_o:\ \underset{\sim}{T} = (\eta_{jk}) = \underset{\sim}{0}^{pxq},$$

against the set of alternatives that $\underset{\sim}{T}$ is non-null. By means of the following intra-block transformations, we eliminate the nuisance parameters ν_j's and τ_k's. Let us consider the p x q matrices

$$(2.4) \qquad \underset{\sim}{Y}_i = (Y_{ijk}),\ \underset{\sim}{U}_i = (U_{ijk}),\ \underset{\sim}{Z}_i = (Z_{ijk}) \text{ and } \underset{\sim}{E}_i = (E_{ijk}),\ i=1,\ldots,n,$$

where we define

(2.5) $$\underset{\sim}{Z}_i = (\underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{\ell}_p\underset{\sim}{\ell}_p')\underset{\sim}{Y}_i(\underset{\sim}{I}_q - \frac{1}{q}\underset{\sim}{\ell}_q\underset{\sim}{\ell}_q'), \quad i=1,\ldots,n;$$

(2.6) $$\underset{\sim}{E}_i = (\underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{\ell}_p\underset{\sim}{\ell}_p')\underset{\sim}{U}_i(\underset{\sim}{I}_q - \frac{1}{q}\underset{\sim}{\ell}_q\underset{\sim}{\ell}_q'), \quad i=1,\ldots,n,$$

$\underset{\sim}{I}_t$ being the identity matrix of order t and $\underset{\sim}{\ell}_t$ the (row) t-vector having all the t elements equal to unity, $t \geq 1$. Then from (2.1) through (2.6), we obtain that

(2.7) $$\underset{\sim}{Z}_i = \underset{\sim}{T} + \underset{\sim}{E}_i, \quad i=1,\ldots,n.$$

In the sequel, we shall work with the nuisance parameter-free model (2.7). Also, we will only consider the case when $p, q \geq 3$. If either of them is less than 3, the situation simplifies as follows. Suppose $q=2$, $p > 2$, then from (2.1) and (2.2) we have $\eta_{j1} = -\eta_{j2} = \eta_j$ (say), $j=1,\ldots,p$; thus (2.3) reduces to H_o^*: $\eta_1=\ldots=\eta_p = 0$. Again from (2.5) and (2.6), we have

(2.8) $Z_{ij1} = -Z_{ij2} = Z_{ij}$ (say), $e_{ij1} = -e_{ij2} = e_{ij}$ (say) for all $i=1,\ldots,n$.

It follows from lemma 3.1 [to be proved in section 3] that $(e_{i1},\ldots,e_{ip})$ are symmetric dependent or interchangeable random variables for all $i=1,\ldots,n$. Consequently, based on the set of observations $\{Z_{ij}, j=1,\ldots,p; i=1,\ldots,n\}$, the problem of testing H_o in (2.3) reduces to that of testing the interchangeability of $(Z_{i1},\ldots,Z_{ip})$ (for all $i=1,\ldots,n$), against shift alternatives. As such, the results of Sen (1968) will directly apply, and the details are omitted. If $p=q=2$, we have $\eta_{11} = -\eta_{12} = -\eta_{21} = \eta_{22} = \eta$ (say), and

(2.9) $$Z_{i11} = -Z_{i12} = -Z_{i21} = Z_i \text{ (say)}, \quad i=1,\ldots,n,$$

and it is also easily seen that the distribution of Z_i is symmetric about η. So the problem of testing H_o in (2.3) reduces to that of testing the symmetry (about zero)

of the distribution of Z_i; this is the well known one sample location problem, and hence, is not discussed.

3. The basic permutation principle. We define $\underset{\sim}{U}_i$ and $\underset{\sim}{E}_i$ $(i=1,\ldots,n)$ as in (2.4) and (2.6), and let $F^*(\underset{\sim}{E})$ be the joint cdf of E_i for $i=1,\ldots,n$. Let $\underset{\sim}{j} = (j_1,\ldots,j_p)$ be any permutation of $(1,\ldots,p)$ and $\underset{\sim}{J}$, the set of all possible $(p!)$ permutations, so that $\underset{\sim}{j} \in \underset{\sim}{J}$. Also let

$$\underset{\sim}{I}_p(\underset{\sim}{j}) = (\delta_{ij_k})_{i,k=1,\ldots,p}, \tag{3.1}$$

where δ_{rs} is the usual Kronecker delta. Now for any $\underset{\sim}{j} \in \underset{\sim}{J}$, $\underset{\sim}{I}_p(\underset{\sim}{j})$ is non-singular and has a unique reciprocal $\underset{\sim}{I}_p(\underset{\sim}{j}^*)$ (say), which also belongs to the set $\{\underset{\sim}{I}_p(\underset{\sim}{j}): \underset{\sim}{j} \in \underset{\sim}{J}\}$. Further, it is easily seen that if $\underset{\sim}{I}_p(\underset{\sim}{j}_1)$ and $\underset{\sim}{I}_p(\underset{\sim}{j}_2)$ be defined as in (3.1) for $\underset{\sim}{j}_1 \in \underset{\sim}{J}$, $\underset{\sim}{j}_2 \in \underset{\sim}{J}$, then $\underset{\sim}{I}_p(\underset{\sim}{j}_1)\underset{\sim}{I}_p(\underset{\sim}{j}_2)$ also belongs to $\{\underset{\sim}{I}_p(\underset{\sim}{j}): \underset{\sim}{j} \in \underset{\sim}{J}\}$. Thus the set $\{\underset{\sim}{I}_p(\underset{\sim}{j}): \underset{\sim}{j} \in \underset{\sim}{J}\}$ forms a finite group of elementary transformations. It can also be verified that

$$\underset{\sim}{I}_p(\underset{\sim}{j})(\underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{\ell}\underset{\sim}{\ell}'\underset{\sim}{p}\underset{\sim}{p}')\underset{\sim}{I}_p(\underset{\sim}{j}^*) = \underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{\ell}\underset{\sim}{\ell}'\underset{\sim}{p}\underset{\sim}{p}' \quad \text{if} \quad \underset{\sim}{I}_p(\underset{\sim}{j})\underset{\sim}{I}_p(\underset{\sim}{j}^*) = \underset{\sim}{I}_p. \tag{3.2}$$

Similarly, let $\underset{\sim}{k} = (k_1,\ldots,k_q)$ be any permutation of $(1,\ldots,q)$; the set of all possible $(q!)$ permutations is denoted by $\underset{\sim}{K}$. A second group of elementary transformation matrices is then defined by

$$\{\underset{\sim}{I}_q(\underset{\sim}{k}): \underset{\sim}{k} \in \underset{\sim}{K}\} \quad \text{where} \quad \underset{\sim}{I}_q(\underset{\sim}{k}) = (\delta_{ik_\ell})_{i,\ell=1,\ldots,q}. \tag{3.3}$$

Let us then define a finite group $\mathcal{G}_1$ of transformations $\{g_i(\underset{\sim}{j},\underset{\sim}{k}): \underset{\sim}{j} \in \underset{\sim}{J}, \underset{\sim}{k} \in \underset{\sim}{K}\}$ by

$$\begin{aligned} \underset{\sim}{E}_i(\underset{\sim}{j},\underset{\sim}{k}) &= g_i(\underset{\sim}{j},\underset{\sim}{k})\underset{\sim}{E}_i \\ &= \underset{\sim}{I}_p(\underset{\sim}{j})\underset{\sim}{E}_i\underset{\sim}{I}_q'(\underset{\sim}{k}), \quad \underset{\sim}{j} \in \underset{\sim}{J}, \ \underset{\sim}{k} \in \underset{\sim}{K}, \text{ for } i=1,\ldots,n. \end{aligned} \tag{3.4}$$

Finally, the group of all the $(p!q!)^n$ transformations in (3.4) is denoted by $\mathcal{G}_n^*$ i.e.,

$$\mathcal{G}_n^* = (\mathcal{G}_1, \ldots, \mathcal{G}_n). \tag{3.5}$$

As before, we denote the cdf's of $\underset{\sim}{U}_i$ and $\underset{\sim}{E}_i$ by F and F*, respectively. Let now $\mathcal{F}$ be the class of all pq-variate continuous cdf's for which the pq variates are interchangeable. By definition (in section 2), $F \in \mathcal{F}$.

LEMMA 3.1 *If* $F \in \mathcal{F}$, *F* is* $\mathcal{G}_1$*-invariant.*

PROOF. On defining $\underset{\sim}{U}_i$, $\underset{\sim}{I}_p(\underset{\sim}{j})$ and $\underset{\sim}{I}_q(\underset{\sim}{k})$ as in (2.4), (3.1) and (3.3), we let

$$\underset{\sim}{U}_i(\underset{\sim}{j},\underset{\sim}{k}) = \underset{\sim}{I}_p(\underset{\sim}{j})\underset{\sim}{U}_i\underset{\sim}{I}_q'(\underset{\sim}{k}), \quad \underset{\sim}{j} \in \underset{\sim}{J}, \ \underset{\sim}{k} \in \underset{\sim}{K}. \tag{3.6}$$

Since $F \in \mathcal{F}$, it remains invariant under row (or column) permutations. Hence, $\underset{\sim}{U}_i(\underset{\sim}{j},\underset{\sim}{k})$ has the same cdf F for all $\underset{\sim}{j} \in \underset{\sim}{J}$, $\underset{\sim}{k} \in \underset{\sim}{K}$. Now, from (2.6), (3.2) and (3.4), we obtain that

$$\begin{aligned}
\underset{\sim}{E}_i(\underset{\sim}{j},\underset{\sim}{k}) &= \underset{\sim}{I}_p(\underset{\sim}{j})\underset{\sim}{E}_i\underset{\sim}{I}_q'(\underset{\sim}{k}) \\
&= \underset{\sim}{I}_p(\underset{\sim}{j})(\underset{\sim}{I}_p - \tfrac{1}{p}\underset{\sim}{\ell}_p\underset{\sim}{\ell}_p')\underset{\sim}{U}_i(\underset{\sim}{I}_q - \tfrac{1}{q}\underset{\sim}{\ell}_q\underset{\sim}{\ell}_q')\underset{\sim}{I}_q'(\underset{\sim}{k}) \\
&= \underset{\sim}{I}_p(\underset{\sim}{j})(\underset{\sim}{I}_p - \tfrac{1}{p}\underset{\sim}{\ell}_p\underset{\sim}{\ell}_p')\underset{\sim}{I}_p(\underset{\sim}{j}^*)\underset{\sim}{I}_p(\underset{\sim}{j})\underset{\sim}{U}_i\underset{\sim}{I}_q'(\underset{\sim}{k})\underset{\sim}{I}_q(\underset{\sim}{k}^*)(\underset{\sim}{I}_q - \tfrac{1}{q}\underset{\sim}{\ell}_q\underset{\sim}{\ell}_q')\underset{\sim}{I}_q'(\underset{\sim}{k}) \\
&= (\underset{\sim}{I}_p - \tfrac{1}{p}\underset{\sim}{\ell}_p\underset{\sim}{\ell}_p')\underset{\sim}{U}_i(\underset{\sim}{j},\underset{\sim}{k})(\underset{\sim}{I}_q - \tfrac{1}{q}\underset{\sim}{\ell}_q\underset{\sim}{\ell}_q')
\end{aligned} \tag{3.7}$$

Thus, the invariance of the cdf of $\underset{\sim}{U}_i(\underset{\sim}{j},\underset{\sim}{k})$ (under $\mathcal{G}_{i1}$) implies the invariance of the cdf $\underset{\sim}{E}_i(\underset{\sim}{j},\underset{\sim}{k})$ under $\mathcal{G}_{i1}$. Hence the lemma.

Let now Z_n^* be the npq-dimensional (Euclidean) space of the sample point $\underset{\sim}{Z}_n^* = (\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n)$. Then the finite group $(\mathcal{G}_n^*)$ of transformations in (3.4) and (3.5) maps the sample space onto itself, and under H_o in (2.3), the distribution

of $\underset{\sim}{Z}_n^*$ is $\mathcal{G}_n^*$-invariant. Thus, proceeding as in Hoeffding (1952, pp. 169-170), we may prove the existence of similar size α $(0 < \alpha < 1)$ tests for H_o in (2.3), valid for all $F \in \mathcal{F}$. These tests are essentially conditional tests based on the consideration of equiprobable all possible row and column permutations of the matrices $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n$. Such a conditional test is termed a permutation test. In this paper, we shall study permutation tests based on a celebrated class of rank order statistics due to Chernoff and Savage (1958).

4. Formulation of the rank order tests. Let $c(u)$ be 1, $\frac{1}{2}$ or 0 according as u is $>$, $=$ or <0, and let

$$(4.1) \qquad R_{ijk} = \tfrac{1}{2} + \sum_{r=1}^{n}\sum_{s=1}^{p}\sum_{t=1}^{q} c(Z_{ijk}-Z_{rst}),\ i=1,\ldots,n;\ j=1,\ldots,p;\ k=1,\ldots,q;$$

by virtue of the assumed continuity of F, ties among Z_{ijk}'s may be ignored, in probability. We define a sequence of real numbers $\underset{\sim}{S}_N = (S_{N1}, \ldots, S_{NN})$, where

$$(4.2) \qquad S_{N\alpha} = J_N(\alpha/(N+1)),\ 1 \le \alpha \le N;$$

the function J_N is defined as in Chernoff and Savage (1958) and is assumed to satisfy the regularity conditions of theorem 1 of Chernoff and Savage (1958). Let

$$(4.3) \qquad \underset{\sim}{T}_N = (T_{N,jk});\ T_{N,jk} = \frac{1}{n}\sum_{i=1}^{n} S_{NR_{ijk}} \text{ for } j=1,\ldots,p,\ k=1,\ldots,q.$$

Also, let

$$(4.4) \qquad \underset{\sim}{T}_N^* = (\underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{\ell}_p\underset{\sim}{\ell}_p')\underset{\sim}{T}_N(\underset{\sim}{I}_q - \frac{1}{q}\underset{\sim}{\ell}_q\underset{\sim}{\ell}_q') = (T_{N,jk}^*),$$

$$S^{*}_{NR_{ijk}} = S_{NR_{ijk}} - \frac{1}{p}\sum_{j=1}^{p} S_{NR_{ijk}} - \frac{1}{q}\sum_{k=1}^{q} S_{NR_{ijk}} + \frac{1}{pq}\sum_{j=1}^{p}\sum_{k=1}^{q} S_{NR_{ijk}}, \tag{4.5}$$

for $i=1,\ldots,n$; $j=1,\ldots,p$, $k=1,\ldots,q$. Then, from (4.3), (4.4) and (4.5), we obtain

$$T^{*}_{N.jk} = \frac{1}{n}\sum_{i=1}^{n} S^{*}_{NR_{ijk}}, \quad j=1,\ldots,p, \ k=1,\ldots,q. \tag{4.6}$$

We denote by $\mathcal{P}_N$, the permutational (conditional) probability measure induced by the $(p!q!)^n$ equally likely transaformations in $\mathcal{G}^{*}_{n}$, defined by (3.4) and (3.5). Then, by simple arguments it follows that

$$E(\underset{\sim}{T}^{*}_{N} \mid \mathcal{P}_N) = \underset{\sim}{0}^{pxq}. \tag{4.7}$$

Also, let

$$\sigma^2(\mathcal{P}_N) = \frac{1}{n(p-1)(q-1)} \sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{q} (S^{*}_{NR_{ijk}})^2, \tag{4.8}$$

and let $\otimes$ stand for the symbol for Kronecker product of two square matrices. Then by routine computations, we obtain that

$$n\, E(\underset{\sim}{T}^{*}_{N} \otimes \underset{\sim}{T}^{*}_{N} \mid \mathcal{P}_N) = (\underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{\ell}_p \underset{\sim}{\ell}'_p)\otimes(\underset{\sim}{I}_q - \frac{1}{q}\underset{\sim}{\ell}_q \underset{\sim}{\ell}'_q)\sigma^2(\mathcal{P}_N). \tag{4.9}$$

Thus considering the generalized inverse of the pq x pq matrix in (4.9) and employing it to construct a quadratic form in the elements of $\underset{\sim}{T}^{*}_{N}$, we derive the following test statistic

$$\mathcal{L}_N = [n/\sigma^2(\mathcal{P}_N)] \sum_{j=1}^{p}\sum_{k=1}^{q} (T^{*}_{N,jk})^2, \tag{4.10}$$

which is analogous to the classical parametric test based on the variance ratio criterion [cf. Scheffé (1959)].

For small values of n, p and q, the exact permutation distribution of $\mathcal{L}_N$ can be obtained by considering the $(p!q!)^n$ (conditionally) equally likely row and column permutations of the matrices $\underset{\sim}{S}_N^{(i)} = (S_{NA_{ijk}})$, $i=1,\ldots,n$. This procedure becomes prohibitively laborious for large values of n, p or q. For this reason, we consider the following large sample approach.

Let us denote the marginal cdf of Z_{ijk} by $F^*_{[jk]}(x)$ and the joint cdf of $(Z_{ijk}, Z_{ij'k'})$ by $F^*_{[jk;j'k']}(x,y)$ for all $j,j'=1,\ldots,p$; $k,k'=1,\ldots,q$. Let then

$$H(x) = \frac{1}{pq} \sum_{j=1}^{p} \sum_{k=1}^{q} F^*_{[jk]}(x); \tag{4.11}$$

$$H_{10}(x,y) = \frac{1}{qp(p-1)} \sum_{k=1}^{q} \sum_{j' \neq j=1}^{p} F^*_{[jk;j'k]}(x,y); \tag{4.12}$$

$$H_{01}(x,y) = \frac{1}{pq(q-1)} \sum_{j=1}^{p} \sum_{k \neq k'=1}^{q} F^*_{[jk;jk']}(x,y); \tag{4.13}$$

$$H_{11}(x,y) = \frac{1}{p(p-1)q(q-1)} \sum_{j \neq j'=1}^{p} \sum_{k \neq k'=1}^{q} F^*_{[jk;j'k']}(x,y). \tag{4.14}$$

We denote by $J(u) = \lim_{N \to \infty} J_N(u)$: $0 < u < 1$, and define

$$\delta^2 = \int_0^1 J^2(u)du - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J[H(x)]J[H(y)]d[H_{10}(x,y) + H_{01}(x,y) - H_{11}(x,y)]. \tag{4.15}$$

Then, proceeding as in the proof of theorem 4.2 of Puri and Sen (1966) and omitting the details, we obtain the following theorem.

THEOREM 4.1 Under the conditions of theorem 1 of Chernoff and Savage (1958), $[\sigma^2(\mathcal{P}_N) - \delta^2]$ converges in probability to zero.

Now, if we assume that

$$(4.16) \qquad P\{[J[H(Z_{jk})]-J[H(Z_{j'k})]-J[H(Z_{jk'})]+J[H(Z_{j'k'})]] = \text{Constant}\} < 1,$$

for at least one pair of $j \neq j'$ and $k \neq k'$, then as in theorem 4.1 of Sen (1966b), it can be shown that δ^2, defined by (4.15), is strictly positive. (4.16) will be termed, in the sequel, as the non-degeneracy condition of the cdf F^*. The main theorem of this section is the following.

THEOREM 4.2 Under the conditions of theorem 1 of Chernoff and Savage (1958), the permutation distribution of $\mathcal{L}_N$ converges to a chi-square distribution with $(p-1)(q-1)$ degrees of freedom (d.f.).

PROOF. By virtue of (4.7), (4.9) and (4.10), it suffices to show that for any non-null $\underset{\sim}{A} = (a_{jk})$, $Y_n = n^{\frac{1}{2}} \sum_{j=1}^{p} \sum_{k=1}^{q} a_{jk} T^*_{N,jk}$ converges in law (under $\mathcal{P}_N$) to a normal distribution as $n \to \infty$. Now, using (4.4), (4.5), (4.6) and the first two conditions of theorem 1 of Chernoff and Savage (1958), we write

$$(4.17) \qquad Y_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} Y_{ni} + o_p(1); \quad Y_{ni} = \sum_{j=1}^{p} \sum_{k=1}^{q} a^*_{jk} J\left(\frac{R_{ijk}}{N+1}\right), \quad i=1,\ldots,n,$$

where a^*_{jk}'s are linear functions of a_{jk}'s and they satisfy the constraints that $\sum_{k=1}^{q} a^*_{jk} = 0$, $j=1,\ldots,p$ and $\sum_{j=1}^{p} a^*_{jk} = 0$, $k=1,\ldots,q$. We note that under $\mathcal{P}_N$, Y_{ni} can have only $p!q!$ (conditionally) equally likely values obtained by permuting the rows and columns of the matrix $\underset{\sim}{R}_i = (R_{ijk})$ $j=1,\ldots,p$, $k=1,\ldots,q$, and $Y_{n1},\ldots,Y_{nn}$ are all stochastically independent (under $\mathcal{P}_N$). Thus, it readily follows that $E(Y_{n_i} \mid \mathcal{P}_N) = 0$, and

$$E(Y_{ni}^2 \mid \mathcal{P}_N) = \sum_{j=1}^{p}\sum_{k=1}^{q} (a_{jk}^*)^2 \cdot \frac{1}{(p-1)(q-1)} \left\{ \sum_{j=1}^{p}\sum_{k=1}^{q} J^2\left(\frac{R_{ijk}}{N+1}\right) - \right.$$

(4.18)

$$\left. \frac{1}{p}\sum_{k=1}^{q}\left(\sum_{j=1}^{p} J\left(\frac{R_{ijk}}{N+1}\right)\right)^2 - \frac{1}{q}\sum_{j=1}^{p}\left(\sum_{k=1}^{q} J\left(\frac{R_{ijk}}{N+1}\right)\right)^2 + \frac{1}{pq}\left(\sum_{j=1}^{p}\sum_{k=1}^{q} J\left(\frac{R_{ijk}}{N+1}\right)\right)^2 \right\} .$$

Thus, by routine analysis, it follows as in theorem 4.1 that

$$\frac{1}{n}\sum_{i=1}^{n} E(Y_{ni}^2 \mid \mathcal{P}_N) \to \delta^2 \sum_{j=1}^{p}\sum_{k=1}^{q} (a_{jk}^*)^2 > 0, \tag{4.19}$$

where δ^2 is defined by (4.15) and is positive by (4.16). Further, using the growth condition of theorem 1 of Chernoff and Savage (1958), it follows that

$$\frac{1}{n}\sum_{i=1}^{n} E(|Y_{ni}|^{2+\delta} \mid \mathcal{P}_N) < \infty \text{ , for some } \delta > 0. \tag{4.20}$$

Consequently, using the Berry-Esseen theorem (cf. [4 , p. 288]), the asymptotic normality of Y_n follows from (4.19) and (4.20). Hence the theorem.

By virtue of theorem 4.2, an asymptotically size $\alpha (0 < \alpha < 1)$ test for the hypothesis of no interaction may be proposed as follows. If

$$\mathcal{L}_N \begin{cases} \geq \chi^2_{(p-1)(q-1),\alpha}, & \text{reject } H_o \text{ in (2.3)}, \\ < \chi^2_{(p-1)(q-1),\alpha}, & \text{accept } H_o, \end{cases} \tag{4.21}$$

where $\chi^2_{t,\alpha}$ is the upper $100\alpha\%$ point of a chi-square distribution having t d.f. .

5. Asymptotic efficiency of the test based on T_N. It can be easily shown that the test in (4.19) is consistent for any non-null $\underset{\sim}{\tau} = (\tau_{jk})$. For the study of

the asymptotic efficiency of the test based on $\mathcal{L}_N$, we shall therefore consider the following sequence of Pitman-alternatives, specified by

$$(5.1) \qquad H_N: \ \underset{\sim}{T} = \underset{\sim}{T}_N = N^{-\frac{1}{2}} \underset{\sim}{\Delta}, \ \underset{\sim}{\Delta} = (\Delta_{jk}),$$

where Δ_{jk}'s are real and finite and they satisfy

$$(5.2) \qquad \sum_{j=1}^{p} \Delta_{jk} = 0, \ k=1,\ldots,q; \ \sum_{k=1}^{q} \Delta_{jk} = 0 \text{ for } j=1,\ldots,p.$$

Thus, under $\{H_N\}$, the cdf of $\underset{\sim}{Z}_i$ (defined by (2.5),) is specified by $F^*(\underset{\sim}{x} - N^{-\frac{1}{2}}\underset{\sim}{\Delta})$, (where $\underset{\sim}{x}$ is a p x q matrix), and $F^*(\underset{\sim}{x})$ is invariant under the row or column permutations of $\underset{\sim}{x}$. Thus, the univariate marginal cdf $F^*_{[jk]}(x)$ (of $Z_{ijk}-N^{-\frac{1}{2}}\Delta_{jk}$) is independent of (j,k) and is denoted by $H(x)$ [cf. (4.11)]. Similarly, the bivariate marginal cdf $F^*_{[jk,\,jk']}(x,y)$ will be independent of $(j, k\neq k')$ and is denoted by $H_{01}(x,y)$ [cf. (4.13)], $F^*_{[jk,\,j'k]}(x,y)$ will be independent of $(j\neq j', k)$ and is denoted by $H_{10}(x,y)$ [cf. (4.12)], and $F^*_{[jk;j'k']}(x,y)$ will be independent of $(j\neq j', k\neq k')$ and is denoted by $H_{11}(x,y)$ [cf. (4.14)].

Now, for arbitary $F^*(\underset{\sim}{x})$, the asymptotic normality of $N^{\frac{1}{2}}(\underset{\sim}{T}_N - \underset{\sim}{\mu}_N)$ (where $\underset{\sim}{\mu}_N = (\mu_{N,jk})$, $\mu_{N,jk} = \int_{-\infty}^{\infty} J[H(x)]dF^*_{[jk]}(x)$, $j=1,\ldots,p$, $k=1,\ldots,q$) can be proved along the same line as in the proof of theorem 5.1 of Sen (1967) (with direct-extension to the matrix case). We shall specifically consider the case when $\{H_N\}$ holds. For this, we define

$$(5.3) \qquad B(H) = \int_{-\infty}^{\infty} \frac{d}{dx} J[H(x)]dH(x),$$

$$(5.4) \qquad A^2 = \int_0^1 J^2(u)du - [\int_0^1 J(u)du]^2,$$

$$(5.5)\qquad \rho_{ij} = \frac{1}{A^2}\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J[H(x)]J[H(y)]\,dH_{ij}(x,y) - \Big(\int_0^1 J(u)\,du\Big)^2\Big],$$

for $(i,j) = (0,1)$, $(1,0)$ and $(1,1)$, where H_{ij}'s are defined earlier. Then, by the same technique as in theorem 5.1 of Sen (1967), we obtain that under $\{H_N\}$,

$$(5.6)\qquad n^{\frac{1}{2}}E(\underset{\sim}{T}_N^*|H_N) = [(pq)^{-\frac{1}{2}}B(H)]\,\underset{\sim}{\Delta} + o(1)$$

$$(5.7)\qquad n\,E(\underset{\sim}{T}_N^* \otimes \underset{\sim}{T}_N^*|H_N) = (\underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{\ell}\underset{\sim}{\ell}'_p)\otimes(\underset{\sim}{I}_q - \frac{1}{q}\underset{\sim}{\ell}\underset{\sim}{\ell}'_q)A^2(1-\rho_{10}-\rho_{01}+\rho_{11})+o(1),$$

where $B(H)$, A^2, ρ_{ij}'s are defined in (5.3), (5.4) and (5.5). Again, using (4.15), theorem 4.1 and some routine computations, it follows that under $\{H_N\}$ in (5.1)

$$(5.8)\qquad \sigma^2(\mathcal{P}_N) \xrightarrow{p} A^2(1-\rho_{10}-\rho_{01}+\rho_{11}).$$

(5.6), (5.7), (5.8) and the asymptotic normality of $n^{\frac{1}{2}}\,\underset{\sim}{T}_{N,jk}$ lead to the following theorem.

THEOREM 5.1 If (i) $\{H_N\}$ in (5.1) holds, (ii) the conditions of theorem 1 of Chernoff and Savage (1958) hold and (iii) the conditions of lemma 7.2 of Puri (1964) hold, $\mathcal{L}_N$, defined by (4.10), has asymptotically a non-central chi-square distribution with $(p-1)(q-1)$ d.f. and the non-centrality parameter

$$(5.9)\qquad \Delta_{\mathcal{L}_N} = \Big[\frac{1}{pq}\sum_{j=1}^{p}\sum_{k=1}^{q}\lambda_{jk}^2\Big]\Big[B^2(H)/A^2(1-\rho_{10}-\rho_{01}+\rho_{11})\Big].$$

Referring back to the model in (2.1), let σ_u^2 be the variance of U_{ijk} and ρ_u be the correlation between any two U_{ijk}'s belonging to the same block. Let $F_{(p-1)(q-1),(n-1)(pq-1)}$ be the classical analysis of variance ratio test statistic for testing H_o in (2.3) when the parent cdf is assumed to be normal. Then, it can

be shown that under $\{H_N\}$ in (5.1), $Q_N = (p-1)(q-1)\, F_{(p-1)(q-1),(n-1)(pq-1)}$ has asymptotically a non-central chi-square distribution with $(p-1)(q-1)$ d.f. and the non-centrality parameter

$$\Delta_Q = [\frac{1}{pq} \sum_{j=1}^{p} \sum_{k=1}^{q} \lambda_{jk}^2]/\sigma_u^2(1-\rho_u). \tag{5.10}$$

Let now σ_e^2 be the variance of e_{ijk}, defined by (2.6). Then, after some simplifications, we obtain from (2.6) that

$$\sigma_e^2 = [(p-1)(q-1)/pq]\sigma_u^2(1-\rho_u). \tag{5.11}$$

Consequently, from theorem 5.1, (5.10) and (5.11), we arrive at the following.

THEOREM 5.2 When the conditions of theorem 5.1 hold, the asymptotic relative efficiency (A.R.E.) of the $\mathcal{L}_N$-test with respect to the classical analysis of variance test is given by

$$e(\{\mathcal{L}_N\}, \{Q_N\}) = \frac{pq}{(p-1)(q-1)(1-\rho_{10}-\rho_{01}+\rho_{11})} \left[\sigma_e^2 B^2(H)/A^2\right]. \tag{5.12}$$

We note that σ_e^2 is the variance of the cdf H, and hence the second factor on the right hand side of (5.12) resembles the usual efficiency factor for the well-known Chernoff-Savage (1958) type of test statistics. Also, it follows from lemmas 4.4 and 4.5 of Sen (1968) that

$$\rho_{10} \geq -1/(p-1), \quad \rho_{01} \geq -1/(q-1), \tag{5.13}$$

where the equality sign holds iff $J = H^{-1}$ (apart from an additive constant). Thus, from (5.12) and (5.13), we obtain that

(5.14) $$e(\{\mathcal{L}_N\},\{Q_N\}) \geq \frac{1}{\left(1 - \frac{1}{pq}\left[1-(p-1)(q-1)\rho_{11}\right]\right)} \cdot \left[\sigma_e^2 B^2(H)/A^2\right] .$$

This leads to the following corollary.

COROLLARY 5.2.1 A sufficient condition for $e(\{\mathcal{L}_N\},\{Q_N\})$ to be at least as large as $[\sigma_e^2 B^2(H)/A^2]$ is that $\rho_{11} < 1/(p-1)(q-1)$.

We shall now consider two special $\mathcal{L}_N$-statistics, namely, the Normal Scores and Wilcoxon scores statistics. In the first case, $J_N(\frac{\alpha}{N+1})$, defined by (4.2), is the expected value of the α-th smallest observation of a sample of size N from a standard normal distribution, for $\alpha=1,\ldots,N$. In this case, it is well-known [cf. Chernoff and Savage (1958)] that $\sigma_e^2 B^2(H)/A^2$ is greater than or equal to 1, where the equality sign holds only when H is also normal. Thus, the minimum A.R.E. of the normal scores test with respect to the classical analysis of variance test is equal to $1/(1 - \frac{1}{pq}\left[1-(p-1)(q-1)\rho_{11}\right]) \geq \frac{1}{2-(\frac{1}{p}+\frac{1}{q})} > \frac{1}{2}$. On the other hand, for the class of parent cdf's for which $\rho_{11} \leq 1/(p-1)(q-1)$, the normal scores test will be at least as efficient as the Q_N-test. In particular, if H(x) is normal, $\rho_{11} = 1/(p-1)(q-1)$ and $\sigma_e^2 B^2(H)/A^2 = 1$, so that the normal scores test and the Q_N-test become asymptotically power equivalent. For Wilcoxon scores, $J_N(\frac{\alpha}{N+1}) = \frac{\alpha}{N+1}$; $1 \leq \alpha \leq N$. In this case, $[\sigma_e^2 B^2(H)/A^2]$ is known to be greater than or equal to 0.864 for all H. Consequently, the A.R.E. of the Wilcoxon scores test with respect to the Q_N-test is bounded below by

(5.15) $$0.864/\left(1 - \frac{1}{pq}\left[1-(p-1)(q-1)\rho_{11}\right]\right) \geq 0.432 .$$

For normal F, it is known that

(5.16) $$\rho_{10} = \frac{6}{\pi}\sin^{-1}\left(\frac{-1}{2(p-1)}\right),\ \rho_{01} = \frac{6}{\pi}\sin^{-1}\left(\frac{-1}{2(q-1)}\right),\ \rho_{11} = \frac{6}{\pi}\sin^{-1}\left(\frac{1}{2(p-1)(q-1)}\right),$$

and hence from (5.12), we obtain that for normal distributions the A.R.E. of the Wilcoxon-scores test with respect to the Q_N-test is given by

$$(5.17) \qquad \frac{3pq}{\pi(p-1)(q-1)\left[1+\frac{6}{\pi}\left(\mathrm{Sin}^{-1}\frac{1}{2(p-1)}+\mathrm{Sin}^{-1}\frac{1}{2(q-1)}+\mathrm{Sin}^{-1}\frac{1}{2(p-1)(q-1)}\right)\right]}.$$

The following table illustrates the numerical values of (5.17).

q \ p	2	3	4	5	6	7	8	9	10	15	20	∞
2	.955	.966	.965	.963	.962	.961	.960	.960	.959	.958	.957	.955
3		.975	.974	.972	.971	.961	.970	.970	.969	.968	.968	.966
4			.972	.971	.970	.970	.969	.969	.968	.967	.966	.965
5				.970	.969	.968	.968	.967	.967	.966	.965	.963
6					.969	.967	.966	.966	.966	.964	.964	.962
7						.966	.966	.965	.965	.964	.963	.961
8							.965	.964	.964	.963	.962	.960
9								.964	.964	.962	.962	.960
10									.963	.962	.961	.959
15										.961	.960	.958
20											.959	.957
∞												.955

Thus, the efficiency is bounded below by $3/\pi$ and may be as high as 0.975.

If we have more than one observation per cell, we may still work with the aligned observations obtained by making adjustments for row, column and grand means. The permutation argument is essentially the same (with p and q replaced by pr and qr, respectively, r being the number of observations per cell). For more than two factors, summing over a subset of factors, we may arrive at the desired nuisance parameter-free model and proceed as in this paper. For brevity, the details are omitted.

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ON THE p-RANK OF THE DESIGN MATRIX OF A DIFFERENCE SET

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ABSTRACT. Let A be the incidence matrix of a block design constructed from a relative difference set. Let r_p be the rank mod p of A where p is a prime. In this paper we find inequalities for r_p and determine r_p completely in some cases and in particular when A is the incidence matrix of the hyperplanes of a projective or Euclidean geometry. An inequality for the p-rank of arbitrary balanced incomplete block designs is also obtained.

INTRODUCTION. A difference set (v,k,λ,h) in a group G of order v relative to a subgroup H of order h is a set of k elements $g_1, \ldots, g_k$ of G such that the equation

$$g_i g_j^{-1} = g,$$

has exactly λ solutions for all $g \notin H$ and no solution for $g \in H$, $g \neq 1$. If $h = 1$ then the set is called a difference set v,k,λ.

The blocks

$$B_g = (g_1 g, \ldots, g_k)$$

form a group divisible design with parameters $\lambda_1 = 0$, $\lambda_2 = \lambda$, which for $h = 1$ reduces to a balanced incomplete block design.

Relative difference sets were first introduced by R.C. Bose (1942). The general definition given here is due to A.T. Butson (1963).

The use of the incidence matrix of such a design, the design matrix of the difference set, for short, as a check matrix of an error correcting code using a majority rule decoding procedure was first proposed by L.D. Rudolph in a master thesis (1964). These codes were extensively studied and practically implemented by E.J. Weldon, Jr. (1966, 1967).

In all these codes the alphabet consists of residues mod. p, a prime, or more generally of the elements of a finite field with p^s elements, which we shall

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denote by $GF(p^\varepsilon)$. It is therefore of great practical importance as well as of theoretical interest to find the rank mod p of such a design matrix which we shall sometimes call the p-rank of the difference set.

In section 1 of this paper we shall prove a theorem which for Abelian groups and for $(p, v) = 1$ gives an upper bound for this p-rank and which in certain cases determines it completely.

In the last three section we shall determine the p-rank for the incidence matrices of the hyperplanes of $EG(m,q)$ and $PG(m,q)$ (the m-dimensional Euclidean and projective geometries over $GF(q)$ which can in fact also be constructed as design matrices of difference sets. This p-rank has previously been obtained in special cases by E. J. Weldon (1967) and MacWilliams (1966). The formula proved in this paper has however already been conjectured by Rudolph (1967).

Section 1. Let G be an Abelian group and $\mathfrak{R}$ the group ring of G over a field F, whose characteristic is prime to the order v of G. We shall extensively use the characters χ of G and $\mathfrak{R}$ and in particular the relations

$$(1) \qquad \sum_{\chi} \chi(g) = \begin{cases} v & \text{for } g = 1, \\ 0 & \text{for } g \neq 1 \end{cases},$$

$$(2) \qquad \sum_{g} \chi(g) = \begin{cases} v & \text{for } \chi = \chi_1, \\ 0 & \text{for } \chi \neq \chi_1 \end{cases}.$$

If $A = \sum_g a_g g$ then

$$(3) \qquad \frac{1}{v} \sum_{\chi} \chi(A)\, \chi(g^{-1}) = a_g .$$

The notation is explained and formulas (1), (2), (3) are derived in Mann (1965 pp 73-75). Note however that we are here writing the groups multiplicatively.

Let

$$A = \sum_g a_g g$$

be an element of R. We associate with A the matrix $(a_{gg^*}{}^{-1})$, whose rows and columns are labeled by the group elements. We wish to find the rank of $(a_{gg^*}{}^{-1})$, which we shall also call the rank of A.

To this purpose let

$$(\chi(g))$$

be a matrix whose rows are labeled by the v characters χ and whose columns are labeled by the v group elements g. The entry in row χ and column g is $\chi(g)$.

We have

$$(4) \qquad (\chi(g))(a_{gg^*}{}^{-1})(\chi(g^{-1}))^T = v\,\mathrm{diag}(\chi(A)) .$$

To prove this relation we apply (2) to the element in row χ and column χ' of the l. h. s. of (4) and obtain

$$\sum_g \sum_{g^*} a_{gg^*}{}^{-1}\chi(g)\chi'(g^{*-1}) = \sum_{g^*}\sum_g a_g \chi(g)\chi(g^*)\chi'(g^{*-1}) = \begin{cases} v\chi(A) & \text{for } \chi = \chi' \\ 0 & \text{otherwise.} \end{cases}$$

This proves (4). Setting $A = 1$ in (4) we see that the matrix $(\chi(g))$ is non singular. Hence we have

Theorem 1. Let $A = \sum a_g g$ be an element of the group ring of an Abelian group G of order v over a field F whose characteristic is prime to v. Then the rank of A is equal to the number of characters χ of G such that $\chi(A) \neq 0$.

Note that in theorem 1 the coefficients a_g are in F and $\chi(A)$ is an element of $F(\alpha)$ where α is a vth root of unity over F. We can however apply theorem 1 also to the group ring R of G over the domain J of integers. To this purpose consider the field $R(\zeta_v)$ where R is the field of rationals and ζ_v a primitive vth

root of unity over R. Let $\mathfrak{J}(\zeta_v)$ be the domain of integers of $R(\zeta_v)$. Let $f(x)$ be an irreducible factor mod p of the cyclotomic polynomial of order v. Then since $1, \zeta_v, \ldots, \zeta_v^{\varphi(v)-1}$ is an integral basis for $\mathfrak{J}(\zeta_v)$ we know (see Mann 1955 theorem 8.1) that the ideal $(f(\zeta_v), p)$ of $\mathfrak{J}(\zeta_v)$ is a prime ideal divisor $\mathfrak{P}$ of p. Every prime ideal divisor of p can be written in this form and $f(\zeta_v) \equiv 0(\mathfrak{P})$. Similarly if α is a root of $f(x)$ over $GF(p)$ then $f(\alpha) = 0$ and α is a primitive vth root of unity. Moreover the mapping $\zeta_v \longleftrightarrow \alpha$ is an isomorphism τ mapping the residues mod $\mathfrak{P}$ into $GF(p)(\alpha)$. Let $\mathfrak{R}_p$ be the groupring of G over $GF(p)$. Then the mapping $\zeta_v \longleftrightarrow \alpha$ maps every character χ of $\mathfrak{R}$ into a character χ_p of $\mathfrak{R}_p$ in such a way that $\tau(\chi(A)) = \chi_p(A)$ for every $A = \sum a_g g$. In particular

$$\chi(A) \equiv 0(\mathfrak{P}) \longleftrightarrow \chi_p(A) = 0 .$$

Hence we have

Theorem 2. Let $A = \sum_g a_g g$ be an element of the groupring of an Abelian group G of order v over the integers. Let $(v, p) = 1$. The p-rank of the matrix $(a_{gg^*}-1)$ is equal to the number of characters χ such that

$$\chi(A) \not\equiv 0 \ (\mathfrak{P})$$

where $\mathfrak{P}$ is a fixed prime ideal divisor of p in the field of vth roots of unity.

For any set S in G we shall write

$$S(t) = \sum_{g \in S} g^t$$

and $S = S(1)$.

Let D be a difference set relative to the subgroup H of G. Then by definition

$$DD(-1) = k - \lambda H + \lambda G . \tag{6}$$

If χ is any character of G then $\overline{\chi}(g) = \chi(g^{-1})$ is also a character and $\chi = \overline{\chi}$ if and only if χ is of order 1 or 2 that is to say $\chi(g) = \pm 1$ for all g. If $\chi(D)\chi(D(-1)) = \chi(D)\overline{\chi}(D) = 0$ then at most one of $\chi(D)$, $\overline{\chi}(D)$ is distinct from 0 but both are zero if $\chi(D) = \overline{\chi}(D)$. Let t be the number of elements of order 2 in G, t_1 the number of elements of order 2 in G/H and set $v_1 = v/h$. We have to consider the following cases:

	number of χ,	number of χ of order 1 or 2	$\chi(D)\,\overline{\chi}(D)$
$\chi(H) = h,\ \chi(G) = v$	1	1	k^2
$\chi(H) = h,\ \chi(G) = 0$	$v_1 - 1$	t_1	$k-\lambda h$
$\chi(H) = 0,\ \chi(G) = 0$	$v - v_1$	$t - t_1$	k

From this we get

Corollary 2.1. Let D be a v, k, λ, h relative difference set. Let $(p, v) = 1$ and $v_1 = v/h$. Let t be the number of elements of order 2 in G, t_1 that in G/H. Let r_p be the p-rank of D then

$$v_1 - 1 \le r_p \le \frac{v+v_1-2-t+t_1}{2} \quad \text{if } k \equiv 0\ (p),\ k-\lambda h \not\equiv 0\ (p),$$

$$v-v_1+1 \le r_p \le \frac{2v-v_1+1-t_1}{2} \quad \text{if } k \not\equiv 0\ (p),\ k-\lambda h \equiv 0\ (p),$$

$$r_p \le \frac{v-t-1}{2} \quad \text{if } k \equiv 0\ (p),\ k-\lambda h \equiv 0\ (p),$$

$$r_p = v \quad \text{if } k \not\equiv 0\ (p),\ k-\lambda h \not\equiv 0\ (p).$$

Moreover r_p is equal to its upper bound if $(\chi(D), \chi D(-1), p) = 1$ for all non principal χ.

The last condition is always fulfilled if $k \not\equiv 0\ (p^2)$, $k-\lambda h \not\equiv 0\ (p^2)$ because p has no multiple factors in $R(\zeta_v)$ since $(p, v) = 1$.

In the case of a difference set v, k, λ one finds: If $k - \lambda \equiv 0(p)$ $k - \lambda \not\equiv 0 (p^2)$ then $r_p = \frac{v+1}{2}$ if $k \not\equiv 0(p)$ and $\frac{v-1}{2}$ if $k \equiv 0(p)$. (v must be odd otherwise $k - \lambda$ is a square) .

Another difference set v, k, λ in which corollary 2.1 gives the p-rank for all p is the difference set D consisting of the quadratic residues mod q where q is a prime and $q \equiv 3(4)$. In this case

$$\chi(D) = \sum_{r \neq 0,\, r \equiv x^2 (q)} \zeta_q^r \,, \quad \chi(D) + \bar{\chi}(D) = -1$$

for every nonprincipal character χ. If $(q+1)/4 \equiv 0(p)$ then $r_p = \frac{q+1}{2}$.

If p is odd and $q \equiv -\gamma^2 (p)$ and $A = D + \frac{\gamma+1}{2}$ we have, choosing $\gamma \equiv 1(2)$,

$$AA(-1) = \frac{1}{4}[(q + \gamma^2 + (q-1+2\gamma)G] \,.$$

Hence for every nonprincipal character χ we have

$$\chi(A)\, \chi(A(-1)) \equiv 0(p) \,.$$

On the other hand

$$(\chi(A), \chi(A(-1)), p) = (\gamma, p) = 1$$

and this means that one and only one of $\chi(A), \chi(A(-1))$ is divisible by a fixed prime divisor $\mathfrak{p}$ of p. Also

$$\chi_1(A) \begin{cases} \not\equiv 0(p) & \text{for } \gamma \neq 1 \,, \\ \equiv 0(p) & \text{for } \gamma = 1 \,. \end{cases}$$

Now if $M(A)$, $M(D)$ denote the matrices of A and D respectively we have

$$M(A) = M(D) + \frac{\gamma+1}{2} I$$

where I is the identity. Hence

$$\text{p-rank}\,(M(D) + \frac{\gamma+1}{2} I) = \begin{cases} \frac{q+1}{2} & \text{if } \gamma \not\equiv 1(p), \\ \frac{q-1}{2} & \text{if } \gamma \equiv 1(p) \,. \end{cases}$$

The above result was communicated to one of the authors by A. M. Gleason.

For difference sets D with parameter values v, k, λ, where $k - \lambda \equiv 0 (p^2)$ the parameters v, k, λ do not necessarily determine the p rank of D. For instance the difference set 31, 15, 7 constructed from the hyperplanes of PG(2, 4) by the method of section 2 has by theorem 3 the 2-rank 6, while the quadratic residue difference set with the same parameters has as we have shown the 2-rank 16.

Section 2.

Let $q = p^s$, p a prime and let $v = q^m - 1$. Let E be the field GF(q) and let α denote a primitive vth root of unity over E. Then $\alpha \in GF(q^m)$ and is a generator of the multiplicative group of $GF(q^m)$.

The minimal polynomial $f(x)$ of α over E is of degree m. In fact

$$f(x) = \prod_{t=0}^{m-1} (x - \alpha^{q^t}) . \tag{7}$$

$1, \alpha, \ldots, \alpha^{m-1}$ is a basis of $GF(q^m)$ over GF(q). Hence for $0 \leq j \leq v-1$ we have

$$\alpha^j = \sum_{i=0}^{m-1} y_{ji} \alpha^i \tag{8}$$

The coordinates of the vector

$$y_j = (y_{j0}, \ldots, y_{jm-1})$$

will be called the coordinates of α^j. The set of these vectors may be regarded as the non-zero points of a Euclidean geometry EG(m, q) over E.

The points whose coordinates satisfy a non-homogeneous linear equation

$$\sum_{i=0}^{m-1} t_i x_i = t_m , \quad t_i \in E , \quad t_m \neq 0 \tag{9}$$

are the q^{m-1} points of a hyperplane of EG(m, q). The exponents j of the corresponding powers of α form a difference set D mod v relative to the subgroup generated by $(q^m - 1)/(q-1)$ (Bose 1942).

Let $\underline{\theta} = (\theta_0, \theta_1, \ldots, \theta_{v-1})$ be the vector defined by

$$\theta_j = 1 \qquad \text{if } j \in D,$$

$$\theta_j = 0 \qquad \text{otherwise.}$$

We shall say that $\underline{\theta}$ is the incidence vector of a Euclidean hyperplane (briefly an E. H. vector). Every hyperplane of E. G. (m, q) which does not contain the origin corresponds to an E. H. vector and every incidence vector is a cyclic permutation of $\underline{\theta}$. We consider the circulant matrix whose first row is $\underline{\theta}$ and shall determine its p - rank in section 4 . (This matrix is in fact the design matrix of D of section 1.)

Let $r = (q^m - 1)/(q-1)$. We have $\alpha^r = \omega \in E$ and $\alpha^{r+j} = \omega \sum_{i=0}^{m-1} y_{ji} \alpha^i$. Thus the coordinates of the points $1, \alpha, \ldots, \alpha^{r-1}$ represent all the points of a projective geometry, PG(m-1, q).

The α^j, $0 \leq j \leq r-1$ whose coordinates satisfy a linear homogeneous relation

$$\sum_{i=0}^{m-1} t_i y_i = 0 \ , \quad t_i \in E \tag{10}$$

are the points of a hyperplane of PG(m-1, q). The corresponding values of j form a difference set mod($q^m - 1)/(q-1)$. (Singer 1938, also Mann 1965 Theorem 6. 1) We wish to determine the p-rank of the design matrix N of this difference set.

In order to be able to apply the same arguments to the projective and Euclidean case we shall consider (q-1) replications of the incidence vector of the difference set.

Let P. D. be the set of exponents j, $0 \leq j \leq v-1$ such that the coordinates of α^j satisfy the equation (10) clearly $j \in$ P. D. iff $j + r \in$ P. D .

Let $\underline{\theta}^* = (\theta^*_0, \ldots, \theta^*_{v-1})$ be the incidence vector of P.D. defined by

$$\theta^*_j = 1 \qquad \text{if } j \in \text{P. D.},$$

$$\theta^*_j = 0 \qquad \text{if } j \notin \text{P. D.}$$

We shall say that $\underline{\theta}^*$ is a P. H. (projective hyperplane) vector. The vector $\underline{\theta}^*$ consists of q-1 replications of the same incidence vector of the projective hyperplane defined by (10). Every projective hyperplane of PG(m-1, q) is represented in this way by a P. H. vector and every P. H. vector is a cyclic permutation of θ^*.

Let $M(\underline{\theta}^*)$ be the circulant matrix with first row $\underline{\theta}^*$. The first r rows of $M(\underline{\theta}^*)$ consist of q-1 repetitions of the design matrix N and the p-rank of $M(\underline{\theta}^*)$ is clearly equal to the p-rank of N. Hence instead of the p-rank of N we shall determine the p-rank of $M(\underline{\theta}^*)$.

Section 3. We now consider again equation (8) and form the matrix

$$Q = \begin{matrix} y_{00} & \cdots & y_{v-10} \\ \vdots & & \vdots \\ y_{0m-1} & \cdots & y_{v-1\,m-1} \end{matrix}$$

We consider Q as the check matrix of a code C. The matrix Q has rank m since the first m columns of Q form a unit matrix. Hence the code with check matrix Q has rank v - m. If f(x) is the irreducible polynomial for α over E and

$$f(x) = b_0 + b_1 x + \ldots + b_m x^m$$

then (8) shows that the v dimensional vector

$$(b_0, b_1, \ldots, b_m, 0, \ldots, 0)$$

and all its cyclic permutations are vectors of the code with check matrix Q. But these are precisely v-m independent vectors. Hence f(x) is the generator

polynomial of the code C. Polynomials in the sense used here are residue classes mod x^v-1 and must always be reduced mod. x^v-1. Let

$$g(x) = b_m + b_{m-1}x + \ldots + b_0 x^m$$

which has α^{-1} as a root and let

(11) $$(x^v-1)/g(x) = h(x) = h_0 + h_1x + \ldots + h_{v-m}x^{v-m}.$$

We have $g(x)h(x) \equiv 0(x^v-1)$ hence

$$\begin{aligned} b_0h_0 + b_1h_1 + \ldots + b_m h_m &= 0 \\ b_0h_1 + b_1h_2 + \ldots \; b_m h_{m+1} &= 0 \\ &\vdots \\ b_1h_0 + \ldots \qquad\quad + b_m h_{m-1} &= 0 \quad . \end{aligned}$$

This shows that the v dimensional vector

$$(h_0, h_1, \ldots, h_{v-m}, 0 \ldots 0)$$

and all its cyclic permutations are orthogonal to the vectors of C. Hence the code generated by h(x) is in the code orthogonal to C and since its degree is v-m it generates the whole code orthogonal to C that is to say the code generated by Q.

To every linear form

$$\sum_{i=0}^{m-1} a_i y_i$$

corresponds a v dimensional vector

$$\underline{u} = (u_0, \ldots, u_{v-1})$$

where

$$u_j = \sum_{0}^{m-1} a_i y_{ji} \quad .$$

The vector $\underline{u}$ is in the code generated by Q hence

$$u(x) = \sum_{j=0}^{v-1} u_j x^j \equiv 0(h(x)) \quad .$$

On the other hand if

$$u(x) = \sum_{i=0}^{v-1} u_i x^i \equiv 0(h(x))$$

then $(u_0, \ldots, u_{v-1})$ is in the code generated by Q. This shows that all E. H. vectors

$$\underline{\theta} = (\theta_0, \ldots, \theta_{v-1})$$

can be obtained by setting for some $u(x) = s(x)h(x)$

$$\theta_j = 1 \quad \text{if } u_j = t_m \neq 0$$

$$\theta_j = 0 \quad \text{otherwise.}$$

Similarly all P. H. vectors $\underline{\theta}^*$ can be obtained by setting for some $u(x) = s(x)\, h(x)$

$$\theta_j^* = 1 \quad \text{if } u_j = 0$$

$$\theta_j^* = 0 \quad \text{otherwise.}$$

A moments reflection will show that

$$D(x) = \sum_{j=0}^{v-1} \theta_j x^j = \sum_{j=0}^{v-1} x^j - \sum_{j=0}^{v-1} (u_j - t_m)^{q-1} x^j \tag{12}$$

if $\underline{\theta}$ is an E.H.vector and

$$D(x) = \sum_{j=0}^{v-1} \theta_j^* x^j = \sum_{j=0}^{v-1} x^j - \sum_{j=0}^{v-1} u_j^{(q-1)} x^j \tag{13}$$

if $\underline{\theta}^*$ is a P. H. vector.

By theorem 1 the rank of the design matrix of D equals the number of vth roots of unity which are not roots of $D(x)$ since the residue ring $E[x]/(x^v-1)$ is isomorphic to the group ring of a cyclic group of order v.

For any polynomial $f(x)$ we shall say that β is a non root of $f(x)$ if and only if β is a vth root of unity and $f(\beta) \neq 0$. We will determine the rank of D by choosing $u(x)$ so that it will be possible to find the non roots of $D(x)$ from formulas (12) or (13) respectively.

Since $g(x) = (x^v-1)/h(x)$ is prime to $h(x)$ we can determine $e(x)$ so that

(14) $$e(x) = \sum_{i=0}^{v-1} e_i x^i \equiv 1 \ (g(x)) \equiv 0 \ (h(x)) .$$

We then have if β denotes a vth root of unity

(15) $$e(\beta) = 1 \quad \text{if } g(\beta) = 0$$
$$e(\beta) = 0 \quad \text{if } g(\beta) \neq 0$$

moreover the vector $e = (e_0, e_1, \ldots, e_{v-1})$ is a vector of the row space of Q .

Section 4. We first prove two lemmas.

Lemma 1. Let $g(x)$ be any divisor of $x^v - 1$ and let $h(x) = (x^v - 1)/g(x)$. Let $(v, p) = 1$, $q = p^s$. Let β denote generically a vth root of unity over G. F. (q) . Let

$$a(x) = \sum_{i=0}^{v-1} a_i x^i \equiv 0 (h(x))$$

then

(16) $$\sum_{g(\beta)=0} a(\beta) \beta^{-i} = v a_i .$$

Proof: From the inversion formula (3) we have

$$\sum_{\beta} a(\beta) \beta^{-i} = v a_i .$$

The polynomial $x^v - 1$ has no double roots over G. F. (q) . Hence $h(\beta) = 0$ if and only if $g(\beta) \neq 0$. But $a(\beta) = 0$ if $h(\beta) = 0$ and this yields (16) .

Formula (16) is essentially due to Mattson and Solomon (1961) .

Lemma 2. Let G be an Abelian group of order v over a field F whose characteristic is prime to v . For each g let

$$v a_g = \sum_{\chi} l_\chi \chi(g^{-1}) .$$

Let $A = \sum_{g \in G} a_g g$ then

$$\chi(A) = l_\chi$$

Proof: Setting $\bar{\chi}(g) = \chi(g^{-1})$ we have on account of (2)

$$\chi(A) = \sum_{g \in G} a_g \chi(g) = \frac{1}{v} \sum_{g \in G} \sum_{\chi'} \ell_{\chi'} \bar{\chi}'(g) \chi(g)$$

$$= \frac{1}{v} \sum_{\chi'} \ell_{\chi'} \sum_{g} \bar{\chi}'\chi(g) = \ell_{\chi} .$$

This proves lemma 2.

Corollary. Let β generically denote a vth root of unity over a field F whose characteristic is prime to v. Put

$$va_i = \sum_{\beta} \ell_{\beta} \beta^{-i} , \quad i = 0, \ldots, v-1 .$$

Let $f(x) = \sum a_i x^i$ then

$$f(\beta) = \ell_{\beta} .$$

The corollary follows if we apply lemma 2 to the group ring of a cyclic group of order v over a field F.

In particular the non roots of $f(x)$ are precisely those vth roots of unity β for which $\ell_{\beta} \neq 0$.

Lemma 1 applied to $e(x)$ of (14) gives

$$ve_i = -e_i = \sum_{g(\beta)=0} \beta^{-i} = \sum_{t=0}^{m-1} (\alpha^i)^{q^t} .$$

From (12) and (13) we get choosing $t_m = -1$ in (12)

$$\theta_i = 1 - (1 - \sum_{t=0}^{m-1} \alpha^{iq^t}) q-1$$

$$\theta_i^* = 1 - (\sum_{t=0}^{m-1} \alpha^{iq^t}) q-1 .$$

By the corollary to lemma 2 α^{-j} will be a non-zero of

$D(x) = \sum_{i=0}^{\nu-1} \theta_i x^i$ if and only if the coefficient of γ^j in the expansion of

(17) $$\psi(\gamma) = 1 - (1 - \sum_{t=0}^{m-1} \gamma^{q^t})^{q-1}$$

is not 0. Similarly, α^{-j} will be a non-zero of $D(x) = \sum \theta_i^* x^i$ if and only if γ^j occurs in the expansion of

(18) $$1 - (\sum_{t=0}^{m-1} \gamma^{q^t})^{q-1}$$

with a non-zero coefficient.

We shall carry out the calculations for the Euclidean case in detail. The projective case can be treated in a similar way.

We may write

(19) $$(1 - \sum_{t=0}^{m-1} \gamma^{q^t})^{q-1} = (1 - \sum_{t=0}^{m-1} \gamma^{q^t})^{(p-1)(1 + p + \ldots + p^{s-1})} .$$

The exponents occuring in the multinomial expansion of $(1 - \sum_{t=0}^{m-1} \gamma^{q^t})^{p-1}$ are all of the form

(20) $$j_1 q^{t_1} + \ldots + j_e q^{t_e}$$

where $0 \leq t_1 < \ldots < t_e \leq m-1$ and $j_1 + \ldots + j_e \leq p-1$. Moreover the coefficients of these powers of γ are all prime to p. Two exponents of this type are distinct if $t_1, \ldots, t_e$ or $j_1, \ldots, j_e$ are distinct. Moreover every exponent of type (20) occurs. Hence (17) becomes

(21) $$\psi(\gamma) = 1 - \sum c_i \gamma^{\tau_1} \sum c_i \gamma^{p\tau_i} \ldots \sum c_i \gamma^{p^{s-1}\tau_i}$$

$$= 1 - \sum c_{i_1} \ldots c_{i_s} \gamma^{\tau_{i_1} + p\tau_{i_2} + \ldots + p^{s-1}\tau_{i_s}}$$

where the sums are extended over all numbers τ_i of the form (20) and the c_i are not 0. Hence we finally get

(22) $$\psi(\gamma) = \sum z_\sigma \gamma^\sigma$$

where the sum is extended over all $\sigma > 0$ of the form

$$(23) \qquad \sigma = \sum_{i=0}^{s-i} \sum_{j=0}^{m-i} t_{ij} p^i q^j , \quad \sum_j t_{ij} \leq p-1 \qquad i=0, \ldots, s-1 .$$

and $z_\sigma \neq 0$.

Let $Q_m(p-1)$ be the number of partitions of all non-negative numbers $\leq p-1$ into m non-negative summands. The number of numbers of the form (23) is the number of terms in 22 and is given by

$$(Q_m(p-1))^s - 1$$

and this is the number of non-roots of (12) . A similar argument shows that the non-roots of $D^*(x)$ in (13) are given by the element 1 and by all α^{-j} such that

$$(24) \qquad j = \sum_{i=0}^{s-1} \sum_{j=0}^{m-1} t_{ij} p^i q^j , \quad \sum_j t_{ij} = p-1 .$$

The number of non roots of $D^*(x)$ in (13) is therefore

$$(P_m(p-1))^s + 1$$

where $P_m(p-1)$ is the number of partitions of $p-1$ into m summands.

It is well known that

$$P_m(t) = \binom{m+t-1}{t} = \binom{m+t-1}{m-1} ,$$

$$Q_m(t) = \binom{m+t}{t} = \binom{m+t}{m} .$$

Hence we have (note that the projective geometry considered was (m-1) dimensional)

Theorem 3. The p-rank of the incidence matrix of the hyperplanes of a m-dimensional Euclidean or projective geometry over $GF(p^s)$ is

$$\binom{m+p-1}{m}^s + \epsilon$$

where $\epsilon = +1$ for the projective and $\epsilon = -1$ for the Euclidean geometry.

If $\rho = \sum t_i q^i \qquad 0 \leq t_i < q$

then we put

$$c_q(\rho) = \sum t_i \ .$$

It is not difficult to verify from (23) that

(25) $$c_q(\sigma p^i) \leq (q-1)$$

for all i and that (23) represents all numbers $< q^m$ which satisfy 25. Similarly (24) represents all j such that $0 < j \leq q^m$ and

(26) $$c_q(p^i j) = q-1$$

for all values of i.

Section 5. A part of theorem 2 can be generalized to balanced incomplete block designs. We shall prove

Theorem 4. Let A be the incidence matrix of a balanced incomplete block design with parameters v, k, λ and let $n = k - \lambda$. Let p be a divisor of n. Then the p-rank of A is at most $(v+\epsilon)/2$ where $\epsilon = 0$ if $k \equiv 0 (p)$ and $\epsilon = 1$ otherwise.

We have

(27) $$AA^T = n I + \lambda J \equiv \lambda J (p),$$

where J is a $v \times v$ matrix all of whose elements are 1.

If B, C are square matrices of order v over any field then (Mac Duffee (1933), chapter I Corollary 8.3)

(28) $$\text{rank}(B) + \text{rank}(C) \leq v + \text{rank}(BC) \ .$$

We may consider A as a matrix over $GF(p)$. The matrix J has rank 1. Hence the rank of the right side of (27) is ϵ as defined in theorem 4 and theorem 4 follows.

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After completion of this paper it came to the authors' attention that Theorem 3 had already been obtained for projective geometries by I.M. Goethals and P. Delsart. (On a class of Majority logic decodable codes, forthcoming IEEE Trans. on Information theory.) Their methods are however quite different from those presented here.

SOME STATISTICAL METHODS IN MACHINE INTELLIGENCE RESEARCH

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ABSTRACT. About a dozen examples are given of the use of statistical methods in research on machine intelligence, mostly, though not all, previously known, but not previously brought together. The topics include the application of rationality to the research as a whole; the trading of immediate gain for information; adaptive control without the identification of a model, by using smoothing techniques; phoneme recognition using distinctive features and their derivatives; the compiling of dictionaries; "botryology" or concept formation by clump-finding; information retrieval; medical diagnosis; game playing and its relationship to theorem proving; design of an alphabet or of a vocabulary; and artificial neural networks. Among the statistical themes that are emphasized are the estimation of probabilities; the use of amounts of information and of evidence as substitutes for utility when utility is difficult to estimate; decision trees; "evolving" probabilities; and maximum, minimum, and minimax entropy in diagnosis. In this survey of methods it has been necessary at several points to make do with references to the literature.

I. INTRODUCTION. This paper is concerned with examples of statistical methods in machine intelligence research and is not much concerned with non-statistical methods. I believe that some of the ideas are new.

One meaning of "intelligence" is the ability to adapt to a wide variety of circumstances in the attainment of some goal such as self-preservation. In practice this will always involve many subgoals. This definition involves both powers of perception and intellectual activity. I think we have gone further in the mechanization of the intellect than of perception. Spiders and bees seem to have better powers of perception than any machines to date at least in their powers of pattern recognition. It is not clear whether perception should be regarded as an attribute of intelligence but I shall do so.

The work on machine intelligence is an attempt to extend the use of computers into fields where humans and many animals are still supreme, especially into apparently and actually non-numerical fields, roughly describable as "information processing." Elementary information processing could be defined as what can be done using punched cards, sorters, collators and the like. Machine intelligence might then be roughly equated to advanced information processing. Some people would insist that the programs or machines must be adaptive. The subject is still in its infancy: as Oliver Selfridge remarked "Artificial intelligence remains tainted with artificiality."

One aspect of intelligence is judgment. You say that a person has used judgment when you don't know how he arrived at some opinion (19). This is especially true when one is talking about one's own judgment. This could be

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called the "Elementary-my-dear-Watson" effect. One approach to machine intelligence is to discover how judgments are made and then to simulate them. Machine intelligence research is therefore closely related to experimental psychology. That is why there is a society, founded in the U.K., called "A is B", meaning "Artificial Intelligence and the Simulation of Behaviour." About a third of the members are experimental psychologists.

Some examples of work on machine intelligence are:

Machine translation and, more generally, "Computational Linguistics."
Some aspects of information retrieval.
Game playing.
Theorem proving.
Musical composition and the graphic arts. [See (44), which book will be based on an exhibition organized by the Institute for Contemporary Arts.]
Probability estimation.
Classification in general.

Included in classification is "pattern recognition of which there are two kinds (i) the recognition that an already specified pattern is present (properly called "pattern recognition"), (ii) the specification of new patterns, which is also called the "theory and practice of clumps" or "botryology," from the Greek βοτρυς, a cluster of grapes. There are already 27 words beginning with "botry" in Funk and Wagnall's English dictionary so one more won't do any harm. A good name is important: there would be fewer professors of history if it were called "what happened."

Examples of classification are the recognition of printed and handwritten characters, speech recognition including the categorization of phonemes, the classification and recognition of fingerprints, medical diagnosis, and numerical taxonomy. Rutowitz (50) gives a short survey.

Apart from the simulation of thought processes, there has also been some work on the simulation of neural networks (for example, (4, 5, 11, 49)). This work is also related to the theory of reliable machines made of unreliable components (for example, (9, 40)), and borders on the assembly and subassembly theories of mind (24, 34, 38).

It is possible to regard all statistical methods as an attempt to mechanize intelligence, since they are concerned with the reduction of judgment to calculation as far as possible. Perhaps machine intelligence is mainly concerned with <u>new kinds</u> of applications of statistical methods.

An excellent introduction to machine intelligence research is (39).

II. <u>EXAMPLES</u>.

<u>Example</u> (1). As a first example of the application of statistical methods, let's consider the application of the principle of rationality to the work on machine intelligence. The principle of rationality is the recommendation to maximize expected utility. Let p be the probability that

an "ultraintelligent machine" can be built for cost C, where by definition an ultraintelligent machine is better at every intellectual activity than any man; and let the value of this machine if it can be built be u. Then it is easily seen that $|p\ u| > C$ for almost all C, even if p is small. I have put the moduli signs in here because, although it is clear that u is large it is not clear whether it is positive or negative.

Good's Second Law is that when getting advice from consultants on whether to undertake some project, it is important to get two different consultants, one to estimate the probability of success and the other to estimate the value if the project is successful. If a single consultant is asked to judge whether to spend an amount C his answer is too much tied up with his own reputation. If he thinks p is not large he might advise against the expenditure in order to protect himself, regardless of the size of u.

I think this elementary point is important and often overlooked. It shows that a little rationality can go a long way.

A division of responsibility between judge and jury is familiar in law courts, but the jury is usually expected to return a definite verdict instead of an estimated probability. It can also fail to reach agreement, of course. The term of imprisonment of a suspect ought to depend officially on the probability of guilt. Perhaps some day everyone will have to pass an examination in the philosophy of probability before sitting on a jury, just as drivers of cars in the United States have to take a written test.

Example (ii). The two-armed bandit. This problem apparently originated in connection with the choice between two medical treatments (53). It is relevant to adaptive control. Before discussing it I must first refer to "dynamic programming."

When electronic computers were fairly new, "programming" became a vogue word and therefore the expressions "linear programming", "mathematical programming", and "dynamic programming" were introduced although they are more logically called "linear planning", "mathematical planning", and "dynamic (mathematical) planning" respectively since none of them has any necessary connection with machine programming. Richard Bellman, who originated the expression "dynamic programming" agrees with this remark. The improved terminology enables one to speak for example of the programming of dynamic planning.

Dynamic planning is concerned with decision situations in which the current best decision cannot be conveniently worked out without working back from the future. One has a decision tree which is often stochastic and the payoff depends at least partly on where one ultimately ends up on the tree. For example, in the game of chess, the strategy of the entire game really depends on analysis of the end game. This sheds light on the appropriate strategy for the middle game and that in its turn sheds light on the opening strategy. Thus dynamic planning is in some respects hundreds of years old.

A good example of the use of dynamic planning is for the two-armed bandit problem (3, 45, 46, 53, 54, 55, 56). In this problem we have a gambling

machine with two arms or handles, we put in a stake and we can choose which of the two arms to pull. Associated with each arm there is an unknown fixed physical probability that we shall receive a certain fixed reward, the same reward for both arms. (There was an electronic two-armed bandit at Rand Corporation some years ago.) The question is, what is the best strategy? There are various forms of this problem depending on whether the game is of finite or infinite duration. If it is of infinite duration, it is more realistic to discount the future at some rate although the infinite game has also been considered without a discounting factor. When the game is infinite and there is no discounting factor, the object is to win in the largest possible fraction of time in the long run. For this game the solution is the following intuitively obvious one: Since there will ultimately be a very high probability that we know which is the handle with the higher probability of a payoff, we should pull this handle in a proportion of cases tending to one. The other handle must be pulled in a proportion of cases tending to zero but nevertheless in a number of cases tending to infinity. This form of the problem is not of much practical interest, but, with discounting of the future, it is a rather good model of a typical situation in which we have to decide whether to go for short-term gains or to pay for additional information. It is easy to express the problem of finding an optimal solution in terms of some mathematical equations which, however, have never been solved explicitly. I have discussed this problem several times with Dr. Michie of Edinburgh. About seven years ago, he suggested that the information should be measured in terms of Fisher's definition of amount of information with some suitable choice of units, in order that the information could be interpreted as a cash value. However we refuted this and we proposed that expected amount of information in Shannon's sense or else expected weight of evidence might be better.* This we have not yet refuted although in principle it would be quite easy to do so, if the assumption is wrong, by means of a computer program. Michie did write a program in 1960, for solving the dynamic planning equations numerically, but it is not yet quite flexible enough to deal with this particular conjecture. To be more specific, the conjecture is that the long-term financial value of an act is the sum of its immediate expected financial value plus an amount proportional to the expected amount of information or to the expected weight of evidence. (Compare (36)). The expectations can be worked out provided that we assume some initial distrubution for the physical probabilities p and q.

The two-armed bandit problem occurs when one is trying to decide whether to adopt a certain medical treatment when there are two treatments to choose between. The problem can of course be generalized to a Hindu-god bandit having n arms or even a continuous infinity.

The infinite game with discounting of the future is a simple model for the strategy of scientific research, or even of adaptive behaviour generally, and it is relevant to certain types of adaptive control strategy, as in the next example.

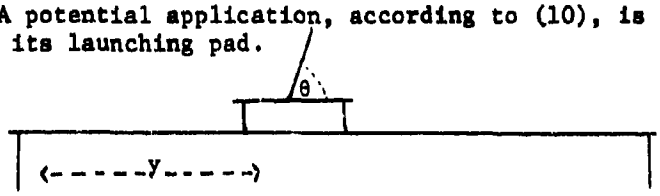
In the application of the two-armed bandit problem to the choice of a medical drug we are unfortunately involved with the ethics of experimenting

*The amount of information and the weight of evidence concerning H provided by E are defined as $I(H:E) = \log [P(E|H)/P(E)]$ and $W(H:E) = \log [P(E|H)/P(E|\text{not } H)]$ (see (14, 16) and references therein).

on people. It would be possible though perhaps impracticable to draw lots in order to select the patients to be given the treatment currently thought to be the less effective. This might be fair and would satisfy the statistician's requirement for randomization.

Example (iii). Adaptive control: a non-identifying approach. (The pole-balancer.)

A classical model of a control system is $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, \underset{\sim}{u}, t)$, where $\underset{\sim}{x}$ is the state variable (vector), $\underset{\sim}{u}$ the control variable, f a known function, and t is time. There is also a loss function or loss functional. In adaptive control, f is not usually entirely known, and $\underset{\sim}{u}$ is chosen either in the light of previous "runs" or in the light of the current run or both. Non-adaptive control is rather like "dead reckoning" in navigation and so too is adaptive control when it does not depend on the current run. A well-known simple example is the pole-balancing problem in which we have a pole hinged to the top of a cart which runs along a finite straight track, with a cliff-edge at each end. Our objective is to balance the pole for as long as possible without falling over the edge of the cliff, to stay alive as long as possible so to speak. A potential application, according to (10), is to the balancing of a rocket on its launching pad.



Suppose we measure y and θ at discrete moments of time. At each such moment we can apply a "bang-bang" control in which a constant force is applied to the cart either to the left or the right at our choice. We think of the state of the system as a point in phase-space, with four coordinates say $\underset{\sim}{x} = (y, \dot{y}, \theta, \dot{\theta})$. (Strictly speaking, phase-space uses positions and momenta.) Our "strategy" can be defined as a function from points $\underset{\sim}{x}$ in phase-space to controls u which take the values L and R. In this example u is a scalar. The "cost" of our strategy can be defined in various ways, for example as a decreasing function of the life-time of the system.

Even in the theory of adaptive control it is usually considered necessary to identify the dynamics of the system (see, for example, (10)). But a juggler can balance a stick without explicitly knowing any dynamics, so it must be possible to do the same with a machine. It might be expensive of course. Dr. Donald Michie of Edinburgh proposed dividing the phase-space into a small number (namely 5 x 5 x 3 x 3) of discrete cells or "boxes," and recording only which box the phase point is in at any moment rather than its exact coordinates. Time is taken as discrete.

Each run provides information of the form

$$(\underset{\sim}{x}_1, u_1), \quad (\underset{\sim}{x}_2, u_2), \ldots$$

where each u_i is L or R (left or right). The idea is to use each run for learning an improved "strategy" (see below) for the control u as a function of $\underline{x}$. This learning might take place between runs, during runs or both. You learn how to live as long as possible by experience gained in previous incarnations. A strategy is a function from $\underline{x}$ to u since we assume that only the positions and velocities are relevant, i.e. there is no hysteresis in the system, or if there is it is allowed for by weighting the past exponentially. [Barnard, 1959, made a useful suggestion about weighting the past. He suggested that if the current behaviour of a system changes by an unusually large amount, then the past should temporarily be discounted at an increased rate.]

We can define a strategy by imagining a little demon in each box. A record is made by each demon whose box has been used, corresponding to each of its uses. This record states whether the bang-bang control was L or R on each occasion and also states the weighted average of life-times of the runs, corresponding separately to L decisions and R decisions.

If the parameters of the system are unvarying, then given a large enough sample it would ultimately become clear to each little demon whether L or R was probably the better decision for him. Actually he could never be quite certain and should occasionally make the apparently less good choice merely in order to gather information, as in the two-armed bandit problem.

The mean life-time, or rather a decreasing function of it, is not a very good cost function. To use it is too much like trying to teach someone (or a machine) to play chess by discouraging any move in a game that he happened to lose. It is far more efficient to make use of sub-goals for the purpose of choosing positive and negative reinforcements. (Compare (19).) If scores can be associated with the various cells or boxes, then a score can be associated with the entire path, this time using a discounting of the future.

Another point is this. If the dimensionality of the problem is much more than 4 (which is the number of dimensions of phase space in the pole-balancing problem), the number of cells or boxes is apt to be extremely large, and it will become difficult or impracticable to take a large enough sample. In this case two different modifications of Michie's approach are possible.

(i) Suppose we can make use of spacial continuity. Then each demon can make use of the statistics acquired by surrounding demons, giving weights that tail off according to the distance away of the other demons.

(ii) We can ignore continuity but treat the various cells by some extension of a treatment of multidimensional contingency table, when estimating probabilities (23).

If in method (ii) we were to categorize the life-times also into say only two categories (above and below some threshold varying with the state of the game but the same for all cells in any one run), then the data would reduce to a multidimensional contingency table 2 x 2 x 5 x 5 x 3 x 3,

and the methods of (23) could be directly applied. I shall therefore refer to this work now in greater detail.

Example (iv). Estimation of probabilities in multidimensional contingency tables. Suppose that a man is teaching a machine to recognize patterns such as letters of the alphabet, phonemes, diseases, or fingerprints. For diseases the information would be fed to the machine by punching up cards from long questionnaires. Thus for each object the machine has a list of attributes and also the name of the class to which the object belongs, as supplied by the instructor. I shall assume that each attribute is descrete, such as yes-no, and has no natural ordering or, if it has, the ordering can be ignored. This is true for the twelve "distinctive features" of phonemes, due to Roman Jakobson, such as voiced/unvoiced, strident/mellow, consonantal/nonconsonantal. Actually at least some of these features can be expressed quantitatively and there are reasons for thinking that we should also record the signs of their derivatives with respect to time. This would increase the dimensionality of the problem still further (29). We would be working in a discretized phase-space of at most 24 dimensions.

Each object provides one entry in a multidimensional contingency table. Owing to the high dimensionality, the frequency in most of the cells will be 0 or 1. There is then a problem of estimating the probability of each cell. If we can do this we can obtain the likelihoods of the various letters, diseases, or crooks, on any future occasion, corresponding to any set of attributes. For phonemes we should also take into account polyphonemic statistics and similarly in medical diagnosis the history of the set of symptoms of a patient is relevant.

An approach to this problem of estimating probabilities is to use the principle of maximum entropy, that is, to maximize $-\Sigma\, p_i \log p_i$ subject to various linear constraints. These linear constraints are obtained by taking marginal totals in a small enough number of dimensions to obtain adequately large frequencies. Even without this, the principle generates null hypotheses for consideration. For example, in two dimensions it generates the null hypothesis of independence of rows and columns, a null hypothesis that every statistician would entertain on grounds of simplicity and conventionality. For a $2 \times 2 \times \ldots \times 2 = 2^m$ table, for which there is only one degree of freedom when all the marginal totals are known, it generates the hypothesis that the product of the probabilities on the black cells is equal to the product on the white ones, when the table is regarded as a multidimensional chessboard, that is, the highest-order interaction vanishes. (For $m = 3$ this hypothesis was proposed as natural by Fisher. (See (2).) The equation is of degree $2^m - 1$, but it always has exactly one positive solution.

For a $d_1 \times d_2 \times \ldots \times d_m$ table with all rth order marginal totals given, the principle of maximum entropy generates the null hypothesis that all rth order and higher-order interactions vanish. This is true for more than one definition of interaction. One definition is the discrete multidimensional Fourier transform of the logarithms of the probabilities, but Goodman (33) showed that a real but slightly more complicated definition could be used

without upsetting the results. When the d's are all 2, the discrete Fourier transform is real and agrees with the definition (59) of interactions used in factorial experiments, as pointed out in (17). The result for any contingency table can be expressed in terms of all the "embedded binary cubes." Note that if we accept a null hypothesis we are in a position to smooth the observations, that is, to "improve" them.

To allow for the finiteness of the sample a reasonable procedure is to maximize some linear combination of the entropy and the log-likelihood (23). This is equivalent to selecting the p_i's at the mode of the final (posterior) distribution if the initial density is proportional to $\Pi\, p_i^{-kp_i}$.

I think $k = 1$ is adequate, but that better would be a density of the form $\int_0^1 \Pi\, p_i^{-kp_i}\, \psi(k)\, dk$ by analogy with the work on Bayesian significance tests for multinomial distributions (31). (This density is a function of the entropy.)

Example (v). In work on mechanical translation it is necessary to make special-purpose and general-purpose dictionaries. Various problems of the following kind should then arise: what is the coverage of the dictionary, that is, what is the probability that the next word met will be one that is already in the dictionary? And what would be the coverage if the sample on which the dictionary was based were doubled? These questions can be answered by means of the theory of the sampling of species (15, 32). For example, if n_r is the number of distinct words represented r times in the sample (that is, if n_r is the frequency of the frequency r), then the coverage is approximately $1 - n_1/N$ if n_1 is large, where N is the sample size. In fact n_1 always is large in practice, however large the sample. The expected coverage if the sample size is doubled is approximately $1 - (n_1 - 2n_2 + 3n_3 - \ldots)/N$. One of the basic ideas in this theory was due to Turing (private communication, 1940): its logic is extremely similar to that of the empirical Bayes method and some of the smoothing techniques of the species-sampling problem can be carried over into the empirical Bayes method for other problems.

This statistical problem does not of course go to the heart of mechanical translation but its solution should be known to all workers in this field since the compilation of dictionaries is expensive and should be organized rationally.

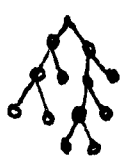
Example (vi). Botryology, for example, in Information Retrieval. Given computers of very great speed and capacity there are prospects of automatic indexing of documents, an operation that normally requires rather high-grade effort and is expensive. The index terms do not need to be existing words: a clump of related words can be regarded as an index term. One point in making use of clumps (or clusters) is to overcome the difficulty arising from synonyms. Sometimes the discovery of such a clump will suggest

the desirability of inventing a new word. Given an incidence matrix of words and documents, a variety of botryological procedures have been suggested (see, for example, (21) and its references (6,47,52,58)), and some of them have been tried on small collections of documents, such as a few hundred. Most procedures suggested have involved a preliminary calculation of a relevance or relatedness matrix, at least resembling a correlation or covariance matrix of words or of documents. I think it is better ((24), pp. 52-54; (26), pp. 120 and 124) to work directly with the document-word incidence matrix, in order to cut down on the amount of calculation. This will be especially worth while when dealing with a sparse incidence matrix, which is the usual situation. For an arbitrary real rectangular matrix there is an interative procedure for obtaining the "singular vectors" which is analogous to a well-known method for obtaining eigenvectors of a square symmetric matrix. It can be used for component analysis (57). An elaboration of it has been suggested for hierarchical botryology, together with a significance test (26). The process should give clumps of index terms and associated or conjugate clumps of documents. Similarly if we have an incidence matrix of symptoms and people, we can look for clumps of symptoms and conjugate clumps of people. If the botryological calculations are sucessful we should discover new diseases or complaints, or at least syndromes, together with the people who suffer from them. (A syndrome is a collection of correlated symptoms whose causal relationship is often poorly understood.)

Botryology can be regarded as the science of concept formulation. A concept can often be thought of as a clump of previously existing concepts.

Example (vii). Speech recognition without tuition.

Different people use different phonemes and this is a source of difficulty for any speech recognition machine. But even without tuition a machine might be able to categorize the phonemes of a given speaker botryologically. I shall suppose that the distinctive-feature approach is used, possibly with time derivatives, so that each speech sound will be represented by a binary vector in m dimensions, where $12 \leq m \leq 24$. A stretch of speech is to be converted by the machine into a sequence of say n such vectors. Many of these vectors will represent transitions between phonemes since we cannot assume that the problem of segmentation of the phonemes can be solved at the start. The machine now has a binary matrix B with m rows and n columns. This can be treated by a method which I call "crude convergence" which is an iterative method of maximizing $x'By$ where x and y are binary vectors (25), p. 120). After convergence we could extract the two quarter-matrices, corresponding respectively to the positive and to the negative components of these vectors, and repeat the process. In this manner we might be able to obtain a dichotomous dendroidal categorization of the type shown in the diagram. A slight generalization would allow polytomies.



An alternative and more classical approach, which however would probably involve far more calculation, would be to start with the correlation or covariance matrix of the n original vectors. In any case the "transition phonemes" would mostly be too rare to be relevant, and those that were not rare might deserve to be called phonemes. To finish off the job would be a problem like the solution of a simple substitution cipher, but many of the phonemes would be given only probabilistically.

It would be interesting to try this process both on human languages and on the sounds made by dolphins and whales, which are linguistic for all we know. Of course with unknown languages the transformation of the speech into a sequence of phonemes is only a small step in the solution, but a necessary one.

Example (viii). Medical diagnosis. If we can solve the probability estimation problems we can of course apply Bayes' theorem in order to do automatic medical diagnosis. If it is too difficult to obtain a really good Bayesian model we can use a less good one and then interpret the resulting Bayesian log-factors or weights of evidence as orthodox non-Bayesian statistic. This is an example of the Bayes/non-Bayes compromise. (See, for example, (25).)

Assuming a Bayesian model how do we choose between two "facets" for the eliciting of a datum? This question, raised by Card (8), may be regarded as a special case of that of how to design an experiment. More generally we might wish to decide between a number of facets and a number of treatments. Theoretically we should use the principle of rationality. But utilities are often difficult to judge, so we might instead use measures of information, evidence, or corroboration as if they were utilities (see, for example, (14), p. 72; (16, 20, 37)).

The various possible diseases or complaints can be regarded as hypotheses, H_1, H_2, H_3,..., but these unfortunately are not necessarily mutually exclusive. (The same complication arises in chemical analysis.) At any moment let us suppose we have current probabilities $P(H_1) = p_1, \ldots, P(H_m) = p_m$. If we elicit a datum E these probabilities change to $P(H_1 \mid E) = q_1, \ldots, P(H_m \mid E) = q_m$. A reasonable criterion of how well off we are in our diagnostic work is the entropy $-\Sigma\ q_1 \log q_1$. The smaller the entropy the closer we are to a complete diagnosis. So a possible criterion for the selection of the facet is to arrange to minimize the expected entropy, assuming of course that the expected cost (in time, effort, and danger to the patient) is the same for the various alternative selections. (Otherwise we must allow for this cost.)

It is interesting to note that it makes sense to maximize entropy when estimating probabilities, but to minimize its expectation when planning an experiment to obtain revised estimates of the probabilities. This seems analogous to the fact that the physical entropy of isolated systems tends to a maximum (the Second Law of Thermodynamics) whereas in the evolution of

living systems, entropy tends to decrease. This has been called the Fourth Law of Thermodynamics (28). Negative entropy, which includes food and social order, can be regarded as a physical expression for utility, at least as an approximation. Life fights a game against Death with negentropy as the prize.

More generally, when planning an experiment for which we intend to estimate probabilities by maximizing entropy we could try to minimize the expected maximum entropy: that is to minimax the entropy. It is as if we were playing a game against Nature, where we try to maximize utility interpreted as negentropy, and Nature tries to minimize it. Minimaxing of expected loss (i.e. maximinning of expected utility) was proposed as a statistical principle by Abraham Wald and has been defended not as rational but as prudent by R.B. Braithwaite. As far as I know the suggestion of minimaxing entropy is new and, since it implies that Death rather than Nature is an opponent, I think it makes better sense than minimaxing expected loss.

The principle of minimizing (or minimaxing) expected entropy can be derived from another principle, that of maximizing (or maximinning) expected amount of information. Suppose that we have several hypotheses $H_1, H_2, \ldots, H_m$ (typically H_i) and we wish to select an experimental set-up for which the possible results are $E_1, E_2, \ldots, E_n$ (typically E_j). The expected amount of information from the experiment is

$$\mathcal{E}_{i,j} I(H_i:E_j) = \mathcal{E}_{i,j} \log \frac{P(H_i \mid E_j)}{P(H_i)}$$

$$= \mathcal{E}_{i,j} \log P(H_i \mid E_j) - \mathcal{E}_i \log P(H_i),$$

where the colon denotes "provided by."

The second term does not depend on the experimental set-up. So maximizing $\mathcal{E}_{i,j} I(H_i : E_j)$ is equivalent to maximizing $-\mathcal{E}_j$ [entropy of $(H_1, \ldots, H_m)$ conditional on E_j], that is, it is equivalent to minimizing the expected final entropy of $(H_1, \ldots, H_m)$ by appropriate selection of the experimental set-up $(E_1, \ldots, E_n)$.

Information is not an absolute measure of utility and should not be used if we have a better measure. Alternatives are degrees of corroboration and especially weight of evidence (for example, (14, 20)). We might then try to maximize the expected weight of evidence (the vinculum denotes negation):

$$\mathcal{E}_{i,j} W(H_i : E_j) = \mathcal{E}_{i,j} \log \frac{P(E_j \mid H_i)}{P(E_j \mid \overline{H}_i)} = \mathcal{E}_{i,j} \log \frac{O(H_i \mid E_j)}{O(H_i)}$$

$$= \mathscr{E}_{i,j} \log O(H_i \mid E_j) - \mathscr{E}_i \log O(H_i).$$

The second term is again independent of the experimental set-up. So the maximization of $\mathscr{E}_{i,j} W(H_i : E_j)$ is equivalent to minimizing $\mathscr{E}_j$ ["odds entropy" of the hypotheses conditional on the experimental set-up]

$$-\mathscr{E}_j \{ -\Sigma_i P(H_i \mid E_j) \log O(H_i \mid E_j) \} .$$

Another possibility is the expected logarithm of the "repeat rate":

$$\mathscr{E}_j \log \Sigma_i [P(H_i \mid E_j)]^2 .$$

Information has the formal advantage over (weight of) evidence that, owing to an additional additive property, the principle of maximizing expected information is consistent when applied to a pair of completely independent problems. (The logarithm of the repeat rate is also additive.) But since neither information nor evidence is exactly a utility, this formal advantage of information over evidence is not decisive, and my view is that maximizing expected weight of evidence is better at least when there are only two hypotheses, and especially when the initial odds are difficult to estimate. It breaks down when the weight of evidence is infinite, positive or negative, but this is rare. Even when bacilli have been taken from a patient's blood and have satisfied twenty criteria the weight of evidence is apt to be only of the order of 6 bans (a Bayes factor of 10^6), and anyway (as Dr. Card remarked in conversation) the patient might really be only a carrier of the suspected disease. Nevertheless, in the acquisition of evidence, there is sooner or later a law of diminishing returns. An advantage in using expected weight of evidence as a pseudo-utility is that it is independent of the initial odds of the "null" hypothesis, which can often be judged only within a fairly wide interval. It is therefore a relevant measure until we are confident that enough evidence has already been acquired, say until one of the diseases is at least 100 to 1 on. Similarly, when we use the principle of minimum entropy in the design of an experiment and have difficulty in ascribing sharp probabilities to the hypotheses, it is prudent to ascribe those values of the probabilities, within the intervals in which they are judged to lie, in such a manner as to maximize our estimate of the entropy. This proposal is another form of the principle of minimaxing expected entropy, closely related to but not identical with the principle mentioned before. Another two candidates for maximization in the design of an experiment are (writing v for "or",– for "not", : for "provided by", | for "given", and / for "as against"):

$$\Sigma_{i,i'} P(H_i \vee H_{i'}) [P(H_i \mid H_i \vee H_{i'}) \mathscr{E}_j \{W(H_i/H_{i'} : E_j) \mid H_i\}$$

$$+ P(H_{i'} \mid H_i \vee H_{i'}) \mathscr{E}_j \{W(H_{i'} / H_i : E_j) \mid H_{i'}\}]$$

$$= \sum_{i'} \mathcal{E}_{i,j} W(H_i / H_{i'} : E_j)$$

and

$$\sum_{i,i'} P(H_i \vee H_{i'}) [\mathcal{E}_j \{W(H_i / H_{i'} : E_j) \mid H_i\}$$

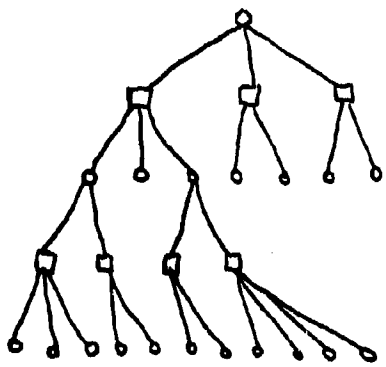
$$+ \mathcal{E}_j \{W(H_{i'} \mid H_i) : E_j \mid H_{i'}\}]$$

$$= 2 \sum_{i,i',j} P(H_i) [P(E_j \mid H_i) - P(E_j \mid H_{i'})] \log \frac{P(E_j \mid H_i)}{P(E_j \mid H_{i'})} \cdot$$

We could here give additional weight to the (i, i') term if it is especially important to distinguish between hypotheses (diseases) i and i'. By trial and error we might be able to decide what measure is best to use.

In the above discussion I have ignored the hierarchical nature of many pattern detection or diagnostic processes. These also produce statistical problems associated with probabilistic decoding or regeneration (see, for example, (24), pp. 37, 38, 57, 62, and 77).

Let us think of medical diagnosis as analogous to chemical analysis. What we have is a stochastic design tree as in the diagram.



A diagnostic decision tree.

Each round node denotes a set of data and each square node denotes a facet and a cost. Associated with each set of data is a probability vector of all possible diagnoses. If one of these probabilities exceeds say 0.99 we have won, that is, we have completed the diagnosis. Or we could measure the value of an endpoint by the negentropy of the probability vector, or by one of the other measures mentioned above. If we can estimate all the probabilities sharply, then our optimal method of diagnosis would be performed by iterative "expectimaxing" (see the next section), but if not then we could instead use

iterative maximinning as in a game with an opponent.

Medical and chemical diagnosis are but two examples of the problem of recognition in general. We could clearly set up a model of the general recognition process as a stochastic recognition tree, with iterative expectimaxing or minimaxing of the entropy as the basis of the optimal strategy, while holding in mind that expectimaxing the utilities would in principle be better if the utilities could be estimated. In some military applications, in which the objects we wish to recognize are camouflaged, we might wish to maximin the expected utilities when fighting a clever opponent.

Example (ix). Game-playing and theorem-proving (see for example, Good 1968, Newell et al, 1959 where further references will be found). In the Borel-von Neumann theory of games a "game of perfect information" is described as "trivial." But in normal English usage, chess is far from being a trivial game, and this might seem to show that, as far as chess is concerned, the von Neumann theory of games is of rather trivial application. But properly interpreted it does have an application, because in practice chess is a game involving an element of luck (12). Personally I should define a non-trivial game as one that is so complicated that its optimal strategy cannot be definitely established and whose analysis therefore must depend on "evolving probabilities". (See below.) This definition could be used whether or not the game is in principle one of perfect information. In this sense a non-trivial game has some analogy with classical statistical mechanics.

Generation 1
" 2
" 3
" 4
" 5
" 6

Consider an analysis tree starting with a position π_0. We must have some rule for terminating the analysis at various positions π which are endpoints of the tree. This is because the tree would usually be too large if every variation were analyzed, although the number of possible games of chess is admittedly not more than $10^{30,000}$, if the game is drawn when fifty consecutive moves are played on each side without a capture or pawn move (13). If we decide where to prune the tree and can evaluate the "evolving" expected utility (see below) of each end-point, then we can work backwards by iterative maximinning to all other points on the tree, and thus decide what move to make in position π_0.

In order to save space I shall discuss theorem-proving at the same time as game-playing. In theorem-proving, at any moment we have a collection of

mathematical propositions. The collection of propositions at any moment is analogous to a chess position, and the transformation rules are analogous to the moves of the game. If there is a particular theorem that we are trying to prove, then if we reach a "position" which includes that theorem as one of its propositions we have "won the game." If there is no particular theorem we are trying to prove then the pay-off can be measured by whether we get interesting or useful propositions. The quantification of these aspects is of course far from being formalized.

In theorem-proving we do not have an opponent, so instead of (iterative) maximinning we use "expectimaxing" as Michie calls it.

Neither in chess nor in theorem-proving do we necessarily have a tree: there can be closures so that we are dealing with a linear oriented graph. But even for a graph we can number the "generations" according to their distance from π_0, the position currently under discussion.

Suppose we have some measure for the turbulence of a position, which is inversely related to its quiescence. A quiescent position is one in which there are no obvious lines that urgently require analysis. We must also be able to measure the superficial, shifting, or evolving probabilities of a win, draw or loss at each position, for a game. These are the kind of probabilities that change in the light of further thought without new empirical information. For example, the evolving probability that the millionth digit of π is a 7 is 0.1 until we have completed the calculations. ((14), p. 49). Evolving probabilities are not strictly consistent. In practical affairs most probabilities are of this kind. (Compare (22).) For theorem-proving we must have a measure of how close we are to the required theorem, or else an evolving expected utility of each move.

The decision of whether to regard π as an endpoint depends on

(a) The depth of π from π_0, more precisely on the probabilistic depth $- \log P(\pi \mid \pi_0)$, where $P(\pi \mid \pi_0)$ is the probability that we shall reach π from π_0. The effective depth of the whole tree could be defined as $-\Sigma_\pi P(\pi \mid \pi_0) \log P(\pi \mid \pi_0)$ summed over all end-points of the tree. This is an incomplete entropy since $\Sigma_\pi P(\pi \mid \pi_0) < 1$. The value of storing an analysis of π_0 perhaps depends largely on $P(\pi_0)$ times the effective depth of this analysis.

(b) The turbulence of π.

(c) The obviousness of the outcome at π.

(d) The size of the analysis tree as a whole. (The thresholds which help to determine the tree size need adjustment in the light of a pilot analysis.)

(e) The time left on our clock and on the opponent's clock.

More precisely, (a), (b), and (c) could be allowed for by guessing

$$P(\pi \mid \pi_0) . \mathcal{E} \mid U(\pi \mid \$) - U(\pi) \mid ,$$

where U (π) is the superficial utility of π and U (π | \$) is the utility of π in the light of say a dollar's worth of analysis.

Evaluation of quiescent positions. Every chess player is taught the approximate values of the pieces at an early stage, P = 1, B = N = 3, R = 5, Q = 9 1/2. These are not the only features of a position but they will serve as an example. I believe these values are proportional to the weights of evidence in favor of winning rather than losing, a pawn being worth about 7 decibans in master chess (19). A machine can make use of a linear evaluation function $a_P n_P + a_B n_B + \ldots$, where n_P, a_B,... are the numbers of pawns, bishops, etc., and the coefficients a_P, a_B,... are to be determined. These coefficients can vary in the light of the machine's experience. In other words the machine can learn optimal values of the coefficients. (For example, (19).) The machine can optimize the coefficients even without an instructor by analyzing positions and minimaxing, and then choosing the coefficients so as to maximize the correlation of the direct evaluation of positions and the "analyzed evaluation." (51).

If the various pawns, etc., are given separate identities, the machine could discover, for example, that centre pawns are more valuable than the side pawns. If quadratic terms are included the machine can discover that two bishops are worth more than bishop and knight. In other words, with quadratic terms the machine can form new concepts. Any such new concept can be added to a basic list of concepts as a single item. In this manner there is the possibility of higher-level concepts being formed in later experience. It is likely to be too expensive to use cubic terms from the start. A minimal concept could be defined as a quadratic term in an evaluation function or as an interaction between known causative agents.

Before leaving the discussion of game-playing and theorem-proving I have one further comment. As mentioned, theorem-proving involves getting from one point of an oriented linear graph to another. But many of the steps are reversible and it can pay to work both forwards and backwards. In fact it can be proved under certain assumptions that for very difficult problems the number of steps required if two-ended working is used, is apt to be about the square root of the number required when working forward only (27).

Example (x). Design of an alphabet of letters or phonemes for a known language, and the choice of a vocabulary for teaching, or for the design of an artificial language for machines or men. Usually such designs are arrived at purely intuitively and historically, but they could be given a statistical basis, at least in part.

These designs should allow for the following things, for all of which statistical data would be relevant (29):

(i) the rate of transmission of information;
(ii) the cost of learning the alphabet or vocabulary;
(iii) the cost of errors arising out of confusion of symbols that are not adequately distinct (60);
(iv) historical facts which influence (i), (ii), and (iii);
(v) generality of communication for example, it is useful if the alphabets used for various languages are the same or similar;

(vi) (for ordinary alphabets) the relationship of the alphabet to phonemes;
(vii) the cost of compiling dictionaries, and of making reference to them when the vocabulary is too large to be completely learned;
(viii) the cost of asking for explanations of terms;
(ix) the cost of errors arising from guessing meanings when dictionaries are not available or one is not willing to refer to one, or one is unwilling to ask for an explanation.

Example (xi). Artificial neural networks. (See for example (5, 24, 35, 42, 43). Many further references will be found in these.) An unlimited supply of statistical problems can be generated by considering artificial neural networks containing some random or pseudorandom features, but I shall not have space to discuss these. One example is the construction of reliable circuits using unreliable components. This is relevant to an understanding of the brain since real neurons are unreliable at least in the sense that we lose many thousands of them everyday (7). Another example of artificial neural networks is the class of machines called "perceptrons."(49).

Then there is the assembly theory of the brain due to Hebb (34) and Milner (38) and the modification known as the subassembly theory which I have speculated about (24). One of the functions of the sub-assembly modification is to aid the understanding of the unconscious mind as well as the conscious mind. These theories are all intended to be speculative and suggestive and it is a challenging problem to formulate them with enough precision to be able to make predictions and physical models. Many of the problems here will perhaps be too difficult to solve other than by very expensive simulation. In order to raise enough funds for such work it might therefore be necessary to rely on inconclusive arguments.

Conclusions. Machine intelligence research in a wide variety of fields should make use of statistical methods and especially methods of probability estimation; the principle of rationality (maximization of expected utility); the use of amounts of information and "weights of evidence" as substitutes for utility when utility is difficult to estimate; decision trees such as those occurring in game-playing; "evolving probabilities"; and maximum, minimum, and minimax entropy.

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