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US Army Research Office ltr, 15 Mar 1971

ARO-D Report 70-2

PROCEEDINGS OF THE FIFTEENTH CONFERENCE ON THE DESIGN OF EXPERIMENTS IN ARMY RESEARCH DEVELOPMENT AND TESTING

Sponsored by
The Army Mathematics Steering Committee
on Behalf of
THE OFFICE OF THE CHIEF OF RESEARCH AND DEVELOPMENT

U. S. ARMY RESEARCH OFFICE-DURHAM

Report No. 70-2
July 1970

PROCEEDINGS OF THE FIFTEENTH CONFERENCE ON THE DESIGN OF EXPERIMENTS

Sponsored by the Army Mathematics Steering Committee

Host

U. S. Army Missile Command
Redstone Arsenal, Alabama

22-24 October 1969

U. S. Army Research Office-Durham
Box CM, Duke Station
Durham, North Carolina

FOREWORD

In a letter under the date of 2 November 1967, Dr. John L. McDaniel, Technical Director of the Research and Engineering Directorate at the U. S. Army Missile Command (MICOM), offered to hold the Fourteenth Conference on the Design of Experiments in Army Research, Development and Testing at his installation. Since arrangements were already underway to hold this conference in the Washington area, this invitation had to be declined by the Army Mathematics Steering Committee (AMSC), the sponsor of this series of conferences. Dr. McDaniel, when made aware of this situation, was willing for the Committee to treat his request to hold the conference as a standing invitation. Members of the AMSC were very pleased to hear this and then discussed with him the possibility of holding the Fifteenth Conference at Redstone Arsenal. These negotiations were brought to a successful conclusion; and, on 29 November 1968, Major General Charles W. Eifler issued a formal invitation to host this conference at his command on 22-24 October 1969. He appointed Dr. Siegfried Lehnigk to serve as Chairman on Local Arrangements and Mr. Raymond V. Knox to handle administrative requirements.

MICOM had already served as the host to the Ninth Conference in this series. It is interesting to note that Dr. Lehnigk, as well as Henry A. Dihm, and W. H. Ewart served as members of the Local Arrangements Committee for the Ninth Conference, as well as the Fifteenth Conference. Those in attendance at this 22-24 October meeting are much in debt to these gentlemen, as well as to many others at Redstone Arsenal, for the excellent handling of the many details connected with a meeting of this size.

Among the many highlights of the Fifteenth Conference on the Design of Experiments was the banquet talk given by Professor Oskar Morgenstern of Princeton University and the following invited speakers:

Reliability Applied to Space Flight
Dr. John E. Condon, National Aeronautics and Space Administration

Systems Reliability
Dr. Nancy R. Mann, Rocketdyne

A Probability Approach to Catastrophic Threat
Dr. Clifford J. Maloney, National Institutes of Health

The Empirical Bayes Approach to the Design and Analysis of Experiments
Professor Richard G. Krutchkoff, Virginia Polytechnic Institute

On Confidence Limits for the Performance of a System When Few Failures are Encountered
Dr. S. C. Saunders, Boeing Scientific Research Laboratories

Everyone had the opportunity to hear the above-mentioned talks, as they were given in general sessions. Unfortunately, one was not privileged to hear all of the thirty-two contributed papers. These covered a wide range of interesting statistical problems and had to be scheduled so that three talks were conducted simultaneously. Following the banquet, it was my privilege to award the Fifth Samuel S. Wilks Memorial Medal, sponsored by the American Statistical Association and the Army, to Dr. W. J. Youden. Details of this presentation are included in these Proceedings.

This conference was attended by 156 scientists; and 52 organizations were represented. Speakers and panelists came from: Boeing Scientific Research Laboratories; Cornell University; Honeywell, Inc.; Litton Systems, Inc.; National Aeronautics and Space Administration; National Institutes of Health; North Carolina State University; Princeton University; Rocketdyne; University of Alabama; University of Georgia; University of Michigan; University of Wisconsin; Vanderbilt University; Virginia Polytechnic Institute; and, 12 Army facilities.

Members of the AMSC would like to express their thanks to the many speakers, chairmen and panelists for all their efforts in behalf of this important scientific meeting. Most of the papers presented at the conference are being made available to the public through these Proceedings. The AMSC asked that copies of this manual receive wide distribution among Army laboratories and Technical Libraries.

At this time, let me express my appreciation to all members of the Program Committee (Clifford Cohen, Jr., Henry Dihm, Francis Dressel, Walter Foster, Fred Frishman, Bernard Harris, Boyd Harshbarger, Raymond Knox, Siegfried Lehnigk, H. L. Lucas, Clifford Maloney, and Herbert Solomon) for their many suggestions and advice on the selection of the speakers and the organization of the whole conference.

Frank E. Grubbs
Conference Chairman

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0845-1130	GENERAL SESSION I - Rocket Auditorium Chairman: Dr. Walter D. Foster, Biomathematics Division, Biological Laboratories, Ft. Detrick, Frederick, Maryland RELIABILITY APPLIED TO SPACE FLIGHT Dr. John E. Condon, Reliability and Quality Assurance, National Aeronautics and Space Administration, Washington, D. C. SYSTEMS RELIABILITY Dr. Nancy R. Mann, Rocketdyne, Canago Park, California
1130-1330	LUNCH - Officer's Club
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Chairman: Joseph Weinstein, Electronics Computing Laboratory, U. S. Army Electronics Command, Fort Monmouth, New Jersey

ESTIMATION OF VEHICLE PARAMETERS FOR THE GIVEN MODEL:

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John Howerton and D. Ray Campbell, Systems Evaluation Branch, Advanced Systems Laboratory, Research and Engineering Directorate, Redstone Arsenal, Alabama

A METHOD OF IMPROVING THE ESTIMATION OF VARIANCE

John Gurland, Department of Statistics, University of Wisconsin. Sponsored by the Mathematics Research Center, U. S. Army, University of Wisconsin, Madison, Wisconsin

A CLASSIFICATION OF BIVARIATE VARIANCE COMPARISONS

Clifford J. Maloney, Biometrics Section, Department of Health, Education and Welfare, National Institutes of Health, Bethesda, Maryland

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Chairman: Eugene F. Dutoit, Quality Assurance Directorate, U. S. Army Munitions Command, Picatinny Arsenal, Dover, N. J.

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Oskar M. Essenwanger, Aerophysics Branch, Physical Sciences Laboratory, Research and Engineering Directorate, Redstone Arsenal, Alabama

A STATISTICAL MODEL FOR THE ANALYSIS OF SIMULTANEOUS TWO STATION IONOSPHERIC SOUNDINGS

Erwin Biser and Richard D'Accardi, Avionics Laboratory, U. S. Army Electronics Command, Fort Monmouth, New Jersey

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G. A. Stoops and E. L. Spitznagel, Jr., Math Sciences Section, Litton Scientific Support Laboratory, Litton Systems, Inc., Fort Ord, California

1515-1545 BREAK

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Chairman: Henry Dihm, Advanced Systems Laboratory, Directorate of Research and Development, U. S. Army Missile Command, Redstone Arsenal, Alabama

Panelists:
Robert Bechhofer, Cornell University
O. P. Bruno, U. S. Army Ballistics Research & Development Center
A. C. Cohen, University of Georgia
Bernard Harris, Mathematics Research Center, U. S. Army
Boyd Harshbarger, Virginia Polytechnic Institute
H. L. Lucas, North Carolina State University

AN EMPIRICAL APPROACH TO ANALYSIS OF THE INTERACTION CHARACTERISTICS OF A SIX-COMPONENT ROCKET ENGINE TEST STAND

Aubrey W. Presson, Test Research and Analysis Branch, Test and Research and Engineering Directorate, U. S. Army Missile Command, Redstone Arsenal, Alabama

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Bernard J. Alley, U. S. Army Missile Command, Redstone Arsenal, Alabama

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Chairman: William McIntosh, U. S. Army Test and Evaluation Command, Aberdeen Proving Ground, Maryland

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R. W. Clarke, U. S. Army Watervliet Arsenal, Watervliet, New York

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R. W. Soanes, U. S. Army Watervliet Arsenal, Watervliet, New York

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Chairman: Raymond B. Schnell, U. S. Army Advanced Materiel Concepts Agency, Washington, D. C.

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Elwood D. Baas, White Sands Missile Range, White Sands, New Mexico

BIOCHEMICAL ASPECTS OF FEEDBACK EFFECTS IN BIOCELLULAR SYSTEMS

George I. Lavin, Terminal Ballistic Laboratory, U. S. Army Ballistic Research and Development Center, Aberdeen Proving Ground, Maryland

Thursday, 23 October

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Chairman: David Howes, Strategy and Tactics Analysis Group, Bethesda, Maryland

Panelists:
Robert Bechhofer, Cornell University
O. P. Bruno, U. S. Army Ballistics Research & Development Center
A. C. Cohen, University of Georgia
Bernard Harris, Mathematics Research Center, U. S. Army
Boyd Harshbarger, Virginia Polytechnic Institute
H. L. Lucas, North Carolina State University

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Mary E. Blome, U. S. Army Ammunition Procurement and Supply Agency, Joliet, Illinois

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R. H. Peterson and W. C. Taylor, Army Materiel Systems Analysis Agency, Aberdeen Research and Development Center, Aberdeen Proving Ground, Maryland

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Chairman: Robert G. Stimson, Air Defense Systems Group, Office of the Chief of Staff, Washington, D. C.

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Hans Baussus von Luetzow, U. S. Army Topographic Laboratories, Fort Belvoir, Virginia

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R. E. Shannon, J. P. Ignizio, and J. L. Stimach, Research Institute, University of Alabama, Huntsville, Alabama. Sponsored by the U. S. Army Missile Command, Redstone Arsenal, Alabama

PROBABILISTIC MANPOWER PLANNING FOR THE RESEARCH AND DEVELOPMENT ORGANIZATION

Larry H. Johnson, Research and Engineering Directorate, U. S. Army Missile Command, Redstone Arsenal, Alabama

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Chairman: Brucy C. Gray, Biomathematics Division, U. S. Army Biological Laboratories, Fort Detrick, Frederick, Maryland

ANALYSIS OF FACTORIAL ARRANGEMENT IN DISCONNECTED BLOCK DESIGNS

Badrig Kurkjian and R. C. Woodall, U. S. Army Materiel Command, Harry Diamond Laboratories, Washington, D. C.

DESIGN OF FIELD TEST PROGRAMS AND STATISTICAL TECHNIQUES FOR ANALYSIS OF THE PERFORMANCE OF NAVIGATION AND POSITIONING SYSTEMS

Emil H. Jebe, Institute of Science and Technology, The University of Michigan, Ann Arbor, Michigan; and,

Ralph A. King, Department of Industrial Engineering, University of Wisconsin, Madison, Wisconsin

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A UNIFIED PROCEDURE FOR SELECTING ALTERNATE EXPERIMENTAL DESIGNS

Edwin M. Bartee, Center for Engineering Management Studies, Vanderbilt University, Nashville, Tennessee. Sponsored by the U. S. Army Missile Command, Redstone Arsenal, Alabama

1015-1045 BREAK

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Chairman: Frank E. Grubbs, U. S. Army Aberdeen Research and Development Center, Aberdeen Proving Ground, Maryland

Panelists:
Robert Bechhofer, Cornell University
O. P. Bruno, U. S. Army Ballistics Research & Development Center
A. C. Cohen, University of Georgia
Bernard Harris, Mathematics Research Center, U. S. Army
Boyd Harshbarger, Virginia Polytechnic Institute
H. L. Lucas, North Carolina State University

DETERMINING THE RELIABILITY OF AN ANTITANK MISSILE WITH SIDE THRUSTERS

R. G. Conard and N. R. Rich, Systems Evaluation Branch, Advanced Systems Laboratory, Research and Engineering Directorate, Redstone Arsenal, Alabama

TRANSMISSION OF INFRASONIC WAVES GENERATED BY LARGE MISSILE LAUNCHES

Raymond E. Lacy and C. E. Sharp, Acoustic/Seismic Communications Research Area, Institute for Exploratory Research, U. S. Army Electronics Command, Fort Monmouth, N. J.

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Chairman: Carol D. Rose, Design of Experiments Branch, U. S. Army Tank-Automotive Command, Warren, Michigan

A SUGGESTED PROCEDURE FOR ANALYZING MISSILE PERFORMANCE BY A LEAST SQUARES FIT TO A GENERALIZED LINEAR STATISTICAL MODEL AND A QUICK CHECK FOR NORMALITY OF THE DATA

Nancy R. Rich, Systems Evaluation Branch, Advanced Systems Laboratory, Research and Engineering Directorate, Redstone Arsenal, Alabama

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D. R. Barr and T. Jayachandran, Mathematical Sciences Section, Litton Scientific Support Laboratory, Litton Systems, Inc., Fort Ord, California

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Chairman: Gerhard J. Isaac, Statistics Branch, U. S. Army Medical Research and Nutrition Laboratory, Fitzsimons General Hospital, Denver, Colorado

OPTIMIZING A FOUR-PART ASSAY PROCEDURE

Walter D. Foster, Biomathematics Division, U. S. Army Biological Laboratories, Fort Detrick, Maryland

AN APPLICATION OF LINEAR PROGRAMMING TO EXPERIMENTAL DESIGN

J. Richard Moore, U. S. Army Aberdeen Research and Development Center, Aberdeen Proving Ground, Maryland

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1300-1500 GENERAL SESSION II - Rocket Auditorium

Chairman: Professor Robert M. Thrall, Department of Mathematical Sciences, Rice University, Houston, Texas

A PROBABILITY APPROACH TO CATASTROPHIC THREAT

Dr. Clifford J. Maloney, Biometric Section, Department of Health, Education and Welfare, National Institutes of Health, Bethesda, Maryland

THE EMPIRICAL BAYES APPROACH TO THE DESIGN AND ANALYSIS OF EXPERIMENTS [Part 1]

Professor Richard G. Krutchkoff, Department of Statistics, Virginia Polytechnic Institute, Blacksburg, Virginia

1500-1530 BREAK

THE EMPIRICAL BAYES APPROACH TO THE DESIGN AND ANALYSIS OF EXPERIMENTS [Part 2]

Professor Richard G. Krutchkoff

1830-1930 SOCIAL HOUR - Officer's Club

1930- BANQUET

Master of Ceremonies: Dr. John L. McDaniel, U. S. Army Missile Command, Redstone Arsenal, Alabama

Presentation of the Samuel S. Wilks Memorial Award by Dr. Frank E. Grubbs, U. S. Army Aberdeen Research and Development Center, Aberdeen Proving Ground, Maryland

Banquet Speaker:

Professor Oskar Morgenstern, Princeton University, Princeton, New Jersey

Friday, 24 October

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Chairman: Gideon A. Culpepper, Quality Control Division, White Sands Missile Range, New Mexico

THE USE OF A HYBRID COMPUTER TO EVALUATE MAN-MACHINE PERFORMANCE OF COMPLEX VEHICLE CONTROL SYSTEMS

M. L. Toivanen, Honeywell, Inc.; Bernard S. Gurman and Erwin Biser, U. S. Army Electronics Command, Fort Monmouth, New Jersey

0830-0900 TECHNICAL SESSION 11 - Conference Room

Chairman: Badrig Kurkjian, U. S. Army Materiel Command, Harry Diamond Laboratories, Washington, D. C.

EXPERIMENTAL DESIGN CONSIDERATIONS IN VALIDATING A METHOD OF MODELING A MAN-ORGANIZED SYSTEM

B. B. Lukens and R. A. Brown, Research Institute, University of Alabama, Huntsville, Alabama. Sponsored by the U. S. Army Missile Command, Redstone Arsenal, Alabama

0830-0900 TECHNICAL SESSION 12 - Control Room

Chairman: Alan S. Galbraith, Mathematics Division, Army Research Office-Durham, Durham, North Carolina

AN INVESTIGATION OF THE EFFECT OF SOME PRIOR DISTRIBUTIONS ON BAYESIAN CONFIDENCE INTERVALS FOR ATTRIBUTE DATA

Alan W. Benton, Surveillance and Reliability Division, Aberdeen Research and Development Center, Aberdeen Proving Ground, Maryland

0905-0930 Rocket Auditorium

OPEN MEETING OF THE AMSC SUBCOMMITTEE ON PROBABILITY AND STATISTICS

Chairman: Dr. Walter D. Foster, Biomathematics Division, U. S. Army Biological Laboratories, Fort Detrick, Maryland

0930-1000 BREAK

1000-1200 GENERAL SESSION III - Rocket Auditorium

Chairman: Dr. Joseph Bluhm, U. S. Army Mechanics and Materiel Research Center, Watertown, Massachusetts

TECHNIQUES FOR CONSTRUCTING MUTUALLY ORTHOGONAL LATIN SQUARES

Professor W. T. Federer, Department of Plant Breeding and Biometry, Cornell University, Ithaca, New York. Presently at the Mathematics Research Center, U. S. Army, University of Wisconsin, Madison, Wisconsin

ON CONFIDENCE LIMITS FOR THE PERFORMANCE OF A SYSTEM WHEN FEW FAILURES ARE ENCOUNTERED

Dr. S. C. Saunders, Boeing Scientific Research Laboratories, Seattle, Washington

CLOSING OF THE CONFERENCE

Dr. Frank E. Grubbs, Chairman of the Conference

1200-1300 LUNCH - Cafeteria - Building 7101

1300-1500 TOUR

****

ALTERNATE PAPER

The following paper will be presented if there is a cancellation in the program:

A PROBABILITY MODEL FOR THE ASSESSMENT OF HUMAN INCAPACITATION FROM PENETRATING MISSILE WOUNDS

William P. Johnson and William J. Bruchey, Jr., Vulnerability Laboratory, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland

****

PROGRAM COMMITTEE

Clifford Cohen, Jr.	Bernard Harris
Henry Dihm	Raymond Knox
Francis Dressel (Secretary)	Siegfried Lehnigk
Walter D. Foster	H. L. Lucas
Fred Frishman	Clifford Maloney
Boyd Harshbarger	Herbert Solomon

Frank E. Grubbs (Chairman)

RELIABILITY APPLIED TO SPACE FLIGHT

John E. Condon
NASA Headquarters
Washington, D. C.

This month, October, marks the eleventh anniversary of NASA. Reflecting on NASA's accomplishments during the past eleven years, I feel we can point with pride to an outstanding record of success. Our record of mission success during these eleven years is over 75%, topped by a manned flight record of outstanding success in the Mercury, Gemini and Apollo programs.

The superlatives have been exhausted in describing the success and significance of Apollo - particularly Apollo 11. I suspect that many of you are keenly interested in knowing how we have attained the level of reliability so vital to the success of the Apollo program. I have given a great deal of thought to this subject during the past three months and regretfully - though not unexpectedly - have not found a simple, concise answer to this question. There are many factors which have contributed to the reliability of Apollo and thus it is not possible to single out any one factor as being all encompassing. However, there are two areas which, in my view, are worthy of special attention:

1. major attention by top management to the reliability of Apollo hardware; and,

2. emphasis, through all phases of the program, on the engineering aspects of reliability.

I will devote my remarks to the latter of these two points following some brief comments on the former.

The effective attainment of reliable space hardware requires the attention of all members of program/project team coupled with strong management support. This has been a key factor in the success of Apollo as top management has actively participated in key milestone reviews which are so important to the successful performance of the system. To illustrate this point, the following are examples of key Apollo milestone reviews.

> <u>Critical Design Review</u>. The purpose of this review is to formally review the design of the Contract End Item when the design is essentially complete. The review is intended to precede the release of engineering for manufacture. Among other things, this review established the integrity of the design by review of analytical and test data, and reliability apportionment and analysis available at that particular point in time.

Certification of Flight Worthiness. The purpose of this milestone is to certify that each flight stage and module is a complete and qualified item of hardware prior to shipment and is accompanied by adequate and accurate supporting documentation. Through this review the Apollo Program Director is informed of any deficiencies prior to shipment of the stage or module. This review certifies, for example, that:

1. acceptance, qualification and reliability tests have been successfully completed and meet the specification requirements;

2. departures from specification and drawing requirements have been approved by Material Review Boards;

3. critical hardware failures have been analyzed and corrected.

Flight Readiness Review (FRR). This is a two part review scheduled for each mission by a joint letter signed by the Program Director and the Mission Director. The purpose of the Program Director's FRR is to determine that the space vehicle hardware and launch complex are ready to commence the mission period. This includes consideration of the checkout and qualification status of all hardware, the summary of failures and disposition thereof, with particular emphasis on failures that have occurred during the pre-launch and checkout phase, and all modifications, deviations and waivers. The purpose of the Mission Director's FRR is to make a judgment for initiating the mission period and committing the deployment of world-wide forces to support the mission. Upon satisfactory completion of the Flight Readiness Review the mission period will commence.

The active participation of top management in these reviews gives emphasis to their importance, helps ensure that all factors which influence the successful performance of the hardware have received proper attention, and results in a "team" approach to system reliability.

The nature of NASA systems - highly complex, small quantity, R&D systems - requires that we concentrate on the engineering aspects of reliability rather than the analytical aspects, particularly at the system and major subsystem levels. In this regard, I would like to discuss the following:

1. adequacy of design for mission requirements;

2. identification and control of failure modes;

3. testing; and,

4. identification and correction of all failures.

We place heavy emphasis on the design review function and require our contractors, as part of their reliability program, to have a design review program. Contractors are required to establish and conduct a formal program of planned, scheduled and documented design reviews at the system, subsystem and component levels. These reviews are comprehensive critical audits of all pertinent aspects of the design of the hardware and software and are conducted at major program milestones beginning in the feasibility stage. Participation in these design reviews should be inter-organizational including competent personnel from such areas as design, fabrication, test, reliability assurance, quality assurance, and parts applications. In this way, inter-disciplinary engineering competence is brought to bear on all aspects of hardware design so as to identify and eliminate potential problems. NASA personnel may participate in these design reviews as deemed necessary. Each design review must be documented and the contractor's reliability organization is responsible for follow-up action to ensure that all recommendations are satisfactorily completed. An effective design review program pays high dividends through the early identification and elimination of problems which would manifest themselves at a later time when correction may be more costly.

Also, as an integral part of the early design phase, we require the contractor to develop analyses to determine possible modes of failure and their effects on mission objectives and crew safety. These analyses are conducted at the system, subsystem and component levels. Each potential failure is considered in terms of its probability of occurrence and is categorized as to probable effect on mission success; e.g., loss of life of crew member, mission termination, launch scrub or delay, etc. These analyses, generally referred to as Failure Mode, Effect and Criticality Analyses (FMEA) have the following important applications:

1. determining the need for redundancy, fail-safe design and derating;

2. determining the need to select parts and components of higher reliability;

3. identifying single failure points and reducing such to acceptable levels of risk;

4. supporting reliability predictions and assessments;

5. supporting system safety and hazard analyses;

6. assuring that test programs are responsive to known and suspected potential failure modes;

7. establishing allowing operating times or cycles; and,

8. determining operational contingency plans.

Of particular importance in our manned flight program is the use of FMEA's to identify single failure points which could adversely effect crew safety and mission objectives.

NASA places strong emphasis on testing throughout all phases of hardware development and fabrication. We require the contractor to develop an integrated test program which will evaluate all aspects of system performance capability to the extent practical. In terms of reliability considerations we expect the testing program to be directed towards:

1. verifying the capability of the design;

2. evaluating the susceptibility of the design and hardware to failures;

3. identifying unexpected interactions among components and assemblies;

4. identifying failure modes which reflect defects in materials, workmanship and fabrication processes; and,

5. obtaining failure rate and other reliability data.

To the extent practical, tests are planned using statistical design-of-experiment techniques and are conducted under environmental conditions and for time periods commensurate with mission conditions.

The final area to be discussed is that of failure reporting and corrective action. We expect all failures and nonconformances to be identified, analyzed and effective correction action taken - we cannot tolerate unexplained failures or ineffective corrective action in our space programs. We specifically require our contractors to employ a controlled system for identification, reporting, analysis, correction and prevention of recurrence of all nonconformances and suspected nonconformance of a functional nature which occur throughout the contract period. Some of the requirements which the system must satisfy are as follows:

1. it shall cover hardware, certain software, the interfaces between hardware and software and the interfaces between hardware or software and test or operational personnel;

2. it shall cover all nonconformances or suspected nonconformances of a functional nature such as:

a. unusual condition occurring in test or handling which are suspected to have an effect on the hardware;

b. transient malfunctions and suspected malfunctions; and,

c. notable deviations from previous performance - parameter drift.

3. It shall provide for investigation of each reported failure by an engineering analyses, followed, where appropriate, by laboratory analysis of failed hardware. Such investigation shall be adequate to assess causes, mechanisms, and potential effects of the failure and serve as a basis for decisions on the most efficient remedial and preventive actions;

4. it shall provide for a review of the technical closeout decision on each reported failure by higher levels of technical management commensurate with the criticality category of the failure involved; and,

5. closeout action shall be considered complete when:

   a. remedial actions have been <u>accomplished</u>;

   b. necessary preventive design and software changes have been devised and <u>accomplished</u>;

   c. necessary design or computer program changes have been <u>verified in test</u>;

   d. effectivity of preventive actions have been established;

   e. change has been made in existing identical items of hardware to which the change is pertinent; and,

   f. closeout documentation has been signed by proper management authority.

Such a system may seem unnecessarily extensive but experience has shown that it is necessary and pays high dividends.

In conclusion, I would like to point out that a significant portion of our reliability problems are due to nonelectronic parts and components. Such items as valves, fittings, seals, actuators, etc., continue to receive major attention as we strive to attain the levels of reliability necessary for mission success.

As we look to the future we will be striving to decrease, significantly, our cost per pound of payload, the complexity of our systems will continue to increase and, thus, our need for strong emphasis on the engineering aspects of reliability will not abate.

COMPUTER-AIDED SELECTION OF PRIOR DISTRIBUTIONS FOR GENERATING MONTE CARLO CONFIDENCE BOUNDS ON SYSTEM RELIABILITY*

Nancy R. Mann
Rocketdyne
Canoga Park, California

ABSTRACT. A description is given of results of preliminary investigations (by a group at North American Rockwell Corporation) related to the Monte Carlo generation of lower confidence bounds on the reliability of a logically complex system. In calculating system confidence bounds by use of a Monte Carlo procedure, one must generate the distribution of each independent subsystem reliability, given the life-test failure data for that subsystem. Therefore, an assumption of a specified a priori distribution for each subsystem reliability is implicit in the procedure.

In order that clues may be obtained as to optimum prior assumptions to be used in calculating Monte Carlo bounds for a complex system, the model has been restricted to a series system wherein each independent subsystem has exponentially distributed failure time and prototypes of each subsystem are tested until a fixed (but not necessarily the same for each subsystem) number of failures occurs. For this model, optimum (uniformly most accurate unbiased) exact classical confidence bounds on the reliability $R(t_m)$ at a specified mission time t_m are available, although not easily calculated (Lentner, M. M and Buehler, R. J., 1963. J. Amer. Statist. Assoc. 58, 670-677 and El Mawaziny, A. H., 1965. Unpublished doctoral dissertation, Iowa State University). Computer programs for calculating the optimum classical bounds and the Bayesian Monte Carlo bounds were written, and a means of numerically comparing various forms of prior distributions against an optimum standard was thus provided. *One prior distribution widely used in obtaining Monte Carlo and general Bayesian exact lower confidence bounds on system reliability is thereby shown numerically to yield bounds which are conservative in the classical sense for this series-system model. Another suggested prior distribution is shown to give bounds which are usually conservative but under certain conditions are liberal, and hence not truly confidence bounds. Moreover, it is demonstrated by a combination of numerical and analytical results, that for a series system containing more than one independent subsystem*

*This research was sponsored by the Mathematics and Statistics Panel of the Aerospace and Systems Group (A&SG) of North American Rockwell Corporation and funded under the Internal Research and Development program of the Executive Offices of A&SG.

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there exists no prior distribution for subsystem reliability which is independent of the data and which yields the optimum lower bounds. Other numerical results related to the selection of optimum methods for generating the bounds and evaluation of certain approximate methods are described.

BACKGROUND AND APPROACH

Review of Pertinent Literature. If it is possible to determine confidence bounds on system reliability solely from the testing of the subsystems of which the system is comprised, saving of expensive system testing can be effected. It may, in fact, sometimes be infeasible to test the system as a whole. Furthermore, this method of obtaining system confidence bounds can be used for exploratory system design.

The subject of confidence bounds for system reliability from subsystem testing is one about which much has been written, but not a great deal is known. Consider a series system in which the failure times of k independent subsystems are exponentially distributed; i.e., for T a random variable representing failure time, Prob $(T>t) = R(t) = \exp(-\lambda t)$, $t \geq 0$, $\lambda > 0$. Suppose n_j prototypes of the jth subsystem, are subjected to life test and the life test is terminated at the time of the r_jth ordered failure, $j=1,2,\ldots,k$. For this special model, there exist optimum (uniformly most accurate unbiased)[1] exact[2] confidence bounds on the reliability $R(t_m)$ at time t_m, the probability that the system will survive at least until time t_m. [See Lentner and Buehler (27) and El Mawaziny (12)]. No such optimum bounds have been found for a model which is equivalent to this exponential-failure-time series-system model, except for the fact that total test time t_j rather than number of failures r_j is specified for the life test of jth subsystem, $j=1,2,\ldots,k$, and number of failures is the observable random variable. For either the fixed-time or fixed-number of failures model, optimum exact confidence

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[1]The definitions of uniformly most accurate and unbiased confidence bounds are as given by E. Lehman (26). They are as follows: A confidence bound $\underline{\theta}(X)$ satisfying $P_\theta\{\underline{\theta}(X) \leq \theta\} \geq 1-\alpha$ for all θ and for all $\theta' < \theta$, $P_\theta\{\underline{\theta}(X) \leq \theta'\} =$ minimum is a uniformly most accurate lower confidence bound for θ at level $1-\alpha$. A family of lower confidence bounds at level $1-\alpha$ is said to be unbiased if $P_\theta\{\underline{\theta}(X) \leq \theta'\} \leq 1-\alpha$ for all $\theta' < \theta$ for all θ.

[2]A lower confidence bound at level $1-\alpha$ is said to be exact if $P_\theta\{\underline{\theta}(X) \leq \theta\} = 1-\alpha$ for all θ.

bounds have not been derived for cases in which either failure time has other than an exponential distribution (or can be converted by a transformation of the data to an exponential distribution) or the system is other than an independent series system.

Another much used failure model, often called the "attribute" model, is one in which only pass-fail binomially distributed data are collected for each independent subsystem. For this model, optimum exact confidence bounds on reliability (or probability of successful operation) of a series system have been derived [see Buehler (6)], but the problem of actually constructing such optimum bounds has not been completely solved [see Lipow (31), Lipow (32), Lloyd and Lipow (33), Steck (48), and Schick (43)]. If a Poisson approximation to the binomial distribution is applicable, then results of Harris (22) provide optimal exact bounds on the reliability of an independent series system for the attribute, model if one randomizes appropriately in obtaining the bounds. One would expect the Poisson approximation to the binomial distribution to apply when the number of prototypes of each subsystem tested is large and the probability of failure for each subsystem is small. There appears to be some question, however, [see Garner (19)] as to whether the approximation loses its applicability as the number of subsystems increases.

Many approximate and non-optimal exact confidence bounds on system reliability have been derived. There have been several approximate confidence bounds on system reliability at time t_m derived for the exponential fixed-number-of-failures model wherein the independent subsystems for a series system. Some of the papers containing these derivations were written prior to the publication of the derivation of the optimum bounds [see Takenaga (49) and Kraemer(25)].

Other work has been directed at providing a more tractable method of calculating confidence bounds than that of El Mawaziny's generalization to k subsystems, $k \geq 2$, of the Lentner-Buehler bounds which apply to 2 subsystems only [see El Mawaziny and Buehler (12), Sarkar (41) and Grubbs (21)]. The method suggested by El Mawaziny and Buehler depends upon large-sample theory and the others use the fact that a function of the estimator of subsystem mean-time-to-failure has a chi-square distribution. The method of Sarkar does not require that the subsystems be independent and is exact for equal numbers of failures for all subsystems.

Some rather limited numerical comparisons have been made of some of these non-optimal methods for obtaining confidence bounds by, for example, Sarkar (41) and Grubbs (21). Apparently none of these methods have, until this time, been subjected to a thorough comparison with the Lentner-Buehler-El Mawaziny bounds, which must be calculated iteratively from an expression which demands extremely complicated calculations when the number of subsystems is more than two or three. (Problems involving loss of precision and use of excessive amounts of computer time also arise in calculating the El Mawaziny bounds when the product of the number of independent subsystems and the number of failures for any given subsystem is more than about 50.)

Other work dealing with the derivation of confidence bounds for system reliability under the exponential-failure-time model include a Bayesian approach for a parallel system with a single failure for each subsystem by Springer and Thompson (47) and two reports by Allen, Carlson and Hubach (2) and Saunders (42), which discuss the fixed-test-time model for a series system.

For the case in which only pass-fail data are collected for each subsystem many methods involving large-or small-sample approximations or Bayesian techniques have been derived for obtaining confidence bounds on the probability of successful operation of an independent series system. Among the large-sample methods are those suggested by Madansky (34) (based on the asymptotic chi-square distribution of -2 log likelihood ratio), by Myhre and Saunders (37) (which gives a generalization of Madansky's method) and by Rosenblatt (40), DeCicco (11) and Thomas (50) (all three of which are based on the asymptotic normality of maximum-likelihood estimators). The methods of Rosenblatt and Madansky are discussed and compared by Myhre and Saunders (38), who demonstrate that the likelihood ratio method attains its asymptotic properties for smaller sample sizes than the method suggested by Rosenblatt and in practical situations appears to yield more accurate bounds. Madansky (34), however, points out that the Rosenblatt method has slightly higher asymptotic (Bahadur) efficiency. The methods of DeCicco and Thomas use Taylor-series approximations to the variance of the maximum likelihood estimator of the system reliability R and would be expected to have asymptotic properties like those of the Rosenblatt method.

Small-sample approximate confidence bounds on R for an independent series system and binomial data have been derived by Nishime (39), Garner and Vail (20), Connor and Wells (8), Abraham (1) and Lindstrom and Madden [see Lloyd and Lipow (33)]. The first three of these approaches use various methods of combining confidence bounds on subsystem reliability to obtain the desired bounds on system reliability. The others use binomial or Poisson approximations for certain statistics. Some of these methods are sensitive to inequality of sample sizes for subsystems. Lower confidence bounds obtained by most of these approximate methods have been compared by the use of three sets of data by Schick and Prior (44) with three different sets of "exact" bounds obtained using results of Lipow [see (31) and (32)], based on Buehler's theory (6) and Poisson approximations. The data apply to systems composed of two subsystems, and in each of the three cases the sample sizes are equal. Only the Lindstrom and Madden method compares favorably with what appear to be the best of the Lipow "exact" bounds. Since there is some question about the standard used to judge the quality of the approximate methods, however, and since only three sets of data, two subsystems and equal sample sizes have been used in the comparisons, it is very difficult to make useful general inferences concerning these results.

Another method investigated numerically by Schick and Prior (44) is the Bayesian approach wherein reliability for each subsystem is assumed to have a prior distribution which is uniform over the unit interval. Confidence bounds which are exact in the Bayesian sense (under the assumed prior distribution) are derived, by Zimmer, Briepohl and Prairie (51) and by Springer and Thompson (45). The latter authors use a Mellin transform technique for obtaining in closed form the distribution of system reliability, given all the subsystem data. A Monte Carlo application of this Bayesian model is suggested by Mastran (35) for a system which is logically more complex than a series system. In the numerical comparisons given by Schick and Prior (44), there appears to be no particular agreement between the sets of Bayesian bounds calculated on the basis of the procedure prescribed by Springer and Thompson and by Zimmer, et al. (Which incidentally agree to two or three significant figures, as one might expect) and the three sets of "exact" bounds calculated. In particular, the Bayesian lower confidence intervals on R are all larger than those based on what is for these three sets of data the smallest of the "exact" intervals.

One would expect the Bayesian bounds to be exact in a classical sense if sample sizes for all subsystems were "sufficiently" large. This is so because a prior density of the assumed type will have less effect upon the confidence bound as the sample sizes for all subsystems increase. Whether or not the bound is exact in a classical sense has not been established. Furthermore, the accuracy of this bound (see footnote 1) has not been investigated for small sample sizes. It is interesting to note that an approximate method, described by Dalton (10) and attributed to TRW's Florida Operations, yields bounds which agree to within 3 in the third significant figure with the three examples calculated in (44) by means of this particular Bayesian approach. The TRW method has the distinction of being extremely amenable to hand calculation.

Among very recently derived approximate methods for obtaining confidence bounds on the probability of successful operation of a series system are (1) those derived by Woods and Borsting (51) (discussed by Lieberman (30)), which are shown by Monte Carlo investigations in their paper to be very nearly exact; (2) those derived by J. R. Johnson (23) based on the exact multi-variate binomial distribution of component test data, and (3) those arising from a Bayesian approach which formally uses subjective judgment concerning prior knowledge by J. Bram (5).

<u>The Monte Carlo Confidence Bound Problem</u>. We now examine the problem of obtaining lower confidence bounds on the reliability of a logically complex system when testing will be performed on the k independent subsystems only. We assume that an equation relating true subsystem reliabilities to true system reliability is available, say by means of computer programs which can provide such information [see Levy (28) and McKnight, Modiest and Schmidt (36)]. We now, in lieu of an appropriate analytical method of obtaining such bounds, consider the possibility of the use of Monte Carlo techniques as suggested by Burnett and Wales (7), Bosnikoff and Klion (4), Costello, Meisel and Letow (9), Levy and Moore (29) and Mastran (35).

At first glance the creation of a Monte Carlo computer program for obtaining the bounds seems to be a straightforward problem of simulating the distribution of system reliability for a given set of failure data in an efficient manner. It soon becomes apparent, however, that there are important Bayesian questions implicit in the problem. That is, in order to generate the distribution of system reliability for a given data set, one must generate for each subsystem what is essentially the posterior distribution of subsystem reliability, given the subsystem life-test failure data. Hence, some prior distribution or something equivalent to such a prior distribution for subsystem reliability must be implicitly or explicitly assumed. In other words, in carrying out the Monte Carlo approach outlined by the authors mentioned above, one uses the density of some appropriate function of the data and implicitly or otherwise combines this information with a prior density of subsystem reliability by means of Bayes' Theorem, $P(A_i|B) = P(B|A_i)P(A_i) / \sum_{\text{all } j} P(B|A_j)P(A_j)$, to obtain the posterior density function of subsystem reliability, given the data. In agreement with the classical analytical method derived in (48), the Monte Carlo procedures described in (4), (7), (9), (29), and (35), in some cases directly suggest and in others tacitly imply a prior distribution for subsystem reliability which is the appropriate prior leading to the classical optimum bounds when the system consists of one subsystem only. One may then inquire as to whether such an assumption is appropriate when the system consists of more than one subsystem.

Springer and Thompson (47) analytically derive their exact Bayesian confidence bounds on $R(t_m)$ for an exponential-failure-time model, wherein one failure is allowed for each independent subsystem of a parallel system, using an alternative a priori assumption. They assume a uniform prior distribution on subsystem reliability over the unit interval, which leads to the classical optimum bounds on successful system operation for the pass-fail model when the system consists of a single subsystem. Springer and Thompson reason that a flat prior for subsystem reliability is in keeping with the intent of Bayes' Theorem when no prior information is known. They point out that the prior density $p(R_j)$ for the jth subsystem reliability yielding the classical optimum bounds for a system containing a single subsystem and an exponential fixed-failures model, $p(R_j) = R_j^{-1} [\ln(1/R_j)]^{-1}$ or equivalently, $q(\lambda_j) = \lambda_j^{-1}$, where $R_j \equiv R_j(t_m) = \exp(-\lambda_j t_m)$ and $0 \le R_j \le 1$, $j = 1,2,\ldots,k$, is "improper" in that the area under the frequency curve cannot be made equal to unity. Mastran (35) suggests for pass-fail data that prior densities for subsystems which lead to a uniform prior density for system reliability might be appropriate. In other words all the suggested prior distributions are derived from the concept of optimality for one subsystem for some model, even though the model may have little relationship to the one of interest.

In the following, a description is given of results of a study (by members of a group at North American Rockwell Corporation) to determine optimum prior assumptions to be used in generating Monte Carlo confidence bounds on the reliability of a logically complex system. The investigation was conducted principally by K. W. Fertig of Rocketdyne Division and the present author. Mr. Fertig wrote all computer programs needed for the investigation, except for one routine linking the Monte Carlo program to the reliability equation for the complex system. He also provided [see (17)] the important analytical derivation of the necessary form for a special restricted model of an optimum prior density function independent of the data and proved that no such prior density exists. Jerome Spanier of the North American Rockwell Science Center provided consultation on problems related to the Monte Carlo computer program. Shirley Stoneberger of the Los Angeles Division wrote the subroutine which makes use of the reliability equation generated from engineering flow chart information by the SCOPE (28) or the ARMM (36) program for a logically complex system.

RESULTS OF INVESTIGATION

Computer Programs Written and Utilized. An optimum standard against which to judge suggested prior distributions provides a means of attacking the Monte Carlo problem. Therefore, the model was first restricted to a series system wherein the jth independent subsystem has exponentially distributed failure time T_j with Prob $[T_j > t_m] = R_j = R_j(t_m) = \exp(-\lambda_j t_m)$ and n_j prototypes of the jth subsystem, $j=1;2,\ldots,k$, are tested until r_j failures occur. If one can determine an appropriate prior distribution for this model, then it should also be possible to make useful inferences concerning the fixed-failure-time series-system model and to determine a method of using prior information for more complex systems.

A computer program was coded in Fortran H for the IBM S/360 system for calculating for this restricted model the optimum classical confidence bounds of Lentner, Buehler and El Mawaziny discussed in the introduction of this paper. The bounds are based on the conditional distribution of $W = Z_1$, given $Z_1 - Z_2 = u_2, \ldots, Z_1 - Z_k = u_k$, where $Z_j = \sum_{i=1}^{r_j} T_{i,j} + (n_j - r_j) T_{r_j,j}$, with $T_{i,j}$ an observable failure time of the ith prototype of the jth component, and where the subscript 1 is arbitrarily assigned. Then, when u_j is less than zero for $j=2,3\ldots,k$, the optimum classical $(1-\alpha)$-level lower confidence bound $R_B(\alpha)$ on $R(t_m) = \exp(-\emptyset t_m)$, (where $\emptyset = \sum_{j=1}^{k} \lambda_j$) is obtained by finding the solution $\emptyset_B(\alpha)$ of the following equation and then calculating $R_B = \exp[-\emptyset_B(\alpha)\, t_m]$, with $\emptyset_B > 0$,

$$H(w|\underline{u};\phi) = A^{-1}(\underline{u};\phi) \sum_{i_2} \sum_{i_3} \cdots \sum_{i_k} \prod_{j=2}^{k} \left[\binom{a_j}{i_j} (-u)^{a_j - i_j} \right]$$

$$\phi^{-a_1 - \Sigma_j i_j - 1} \Gamma_{\phi w}(a_1 + \Sigma_j i_j + 1) = 1 - \alpha, \tag{1}$$

where

$$A(\underline{u};\phi) = \phi^{-1-\sum_1^k a_j} \sum_{i_2}^{a_2} \sum_{i_3}^{a_3} \cdots \sum_{i_k}^{a_k} \left(a_1 + \sum_{j=2}^{k} i_j\right)$$

$$\prod_{j=2}^{k} \left[\binom{a_j}{i_j} (-\phi u_j)^{a_j - i_j} \right]$$

and where

$$\Gamma_{\phi w}(a_1 + \Sigma i_j + 1) = \int_0^{\phi w} y^{a_1 + \sum_{j=1}^{k} i_j} e^{-\phi y} dy \quad .$$

A similar expression is used if any u_j, $j=2,3,\ldots,k$, is greater than zero, and the solution is obtained by joint application of Newton-Raphson iterative procedures, the method of false position and bisection techniques. Then a computer program for generating Monte Carlo confidence bounds was coded and combined with that for obtaining the Lentner-Buehler-El Mawaziny confidence bounds. A listing and flow chart of the combined computer program are available [see Fertig (16) and (18)].

The Monte Carlo program calculates the confidence bounds on the basis of a specified prior density for subsystem reliability which is a member of the "conjugate" family of prior densities. That is, the prior density yields a posterior density of subsystem reliability, given the subsystem data, of the same general form (belonging to the same family of density functions) as the prior density. The prior density function $p_j(R_j)$ used for the jth subsystem reliability was, therefore,

$$f(R_j) = \frac{(\beta_{oj}+1)^{r_{oj}+1}}{\Gamma(r_{oj}+1)} R_j^{\beta_{oj}} [\ln(1/R_j)]^{r_{oj}} \qquad (2)$$

$\beta_{oj}, r_{oj} > -1$, $j = 1,2,\ldots,k$, with β_{oj} and r_{oj}

subjectively chosen. This yields a posterior density $f(R_j|\hat{\beta}_j;r_j)$ for R_j of the form,

$$f(R_j|\hat{\beta}_j;r_j) = \frac{(\hat{\beta}_j+\beta_{oj}+1)^{r_j+r_{oj}+1}}{\Gamma(r_j+r_{oj}+1)} R_j^{\hat{\beta}_j+\beta_{oj}} [\ln(1/R_j)]^{r_j+r_{oj}} \qquad (3)$$

where the random variable $Z_j/t_m = (\sum_{i=1}^{r_1} T_{i,j}+(n_j-r_j)T_{r_j,j})/t_m$

is equal to $z_j/t_m=\hat{\beta}_j$, $j=1,2,\ldots,k$, for the observed set of data.

If β_{oj} and r_{oj} each have the value -1, then the prior density for R_j corresponds to the "improper" prior which is used by (48), (14), (7), and (29) and which gives the optimal classical bounds for a system consisting of a single subsystem [see Epstein and Sobel (14)], that is, $p(R_j) = R_j^{-1} [\ln(1/R_j)]^{-1}$, $j=1,2,\ldots,k$. (It is true, therefore, that even though the prior density corresponding to $\beta_{oj} = r_{oj} = -1$ is "improper," the corresponding posterior density is proper.) If β_{oj} and r_{oj} are both equal to zero, $j=1,2,\ldots,k$, then each subsystem prior density function for subsystem reliability is uniform over the interval from 0 to 1, as suggested by Springer and Thompson (47) for their special case of a parallel-system model mentioned earlier.

For generating the posterior distribution of R_j using the expression (3), a given set of data and specified values for β_{oj} and r_{oj}, a random number ρ_j is generated for the value of the integral $\gamma_j = \gamma(R_j|\hat{\beta}_j;r_j)$ given by the expression (3) from $R_{\gamma,j}$ to 1. The integration is performed by an evaluation of the incomplete gamma function and the value of $R_{\gamma,j}$ determined iteratively. The Newton-Raphson method of iteration in conjunction with the method of false position is used. Because this procedure is quite expensive in terms of computer time, the computer program was written to calculate for a given set of subsystem data a table of 100 values of $R_{\gamma,j}$ corresponding to equally spaced values of $\gamma(R_j|\beta_j;r_j)$. The computer then samples from and interpolates cubically in this table for $a_1<\beta_j<a_2$ where a_1 and a_2 are functions of the data. For $\beta_j<a_1$ and $\beta_j>a_2$, a different table is sampled. In generating values for this table, $\gamma(R_j|\hat{\beta}_j;r_j)$ is calculated from a specified value of $R_{\gamma,j}$, so that no iteration is necessary, but the values of γ_j are not equally spaced (making interpolation more difficult). The second table, which contains values of γ_j much closer together than the one used for non-extreme values of $R_{\gamma,j}$, is necessary because of the steepness of the curve relating γ_j and $R_{\gamma,j}$ for values of $R_{\gamma,j}$ close to 0 or 1.

The first investigation made by means of the computer was of the two familiar prior distributions corresponding to β_{oj} and r_{oj} both equal to zero and both equal to -1, $j=1,2,\ldots,k$. The Bayesian approach corresponding to $\beta_{oj} = r_{oj} = -1$, incidentally, is sometimes called the fiducial model since the posterior distribution of R_j, $j=1,2,\ldots,k$, can be thought of as obtainable from the distribution of a function of the data for the jth subsystem, as detailed in (25). The preliminary phases of this investigation made use of the Monte Carlo program, but the results given below were obtained using instead a computer program which utilizes a Mellin transform technique [see Springer and Thompson (46)] to calculate the posterior distribution of $R(t_m)$ from the posterior distributions of the R_j's. This Mellin transform program was originally written to calculate the variance of the Monte Carlo confidence bound and is applicable to a series system when the posterior density of R_j has the form given by the expression (3) with r_{oj} an <u>integer</u>. The Mellin transform computer program is faster than the Monte Carlo program and gives better precision, but in its present form cannot be used if r_{oj} is other than an integer.

<u>Study of Suggested Prior Densities</u>. For each combination of input, involving from three to twenty-five components having λ's in various proportions, numbers of failures ranging from 1 to 10 and three or four different values of α ranging from .05 to .50, data were generated and a comparison was made of the two Bayesian bounds with the optimum classical bound. In <u>each</u> case (of a total of 156 cases), the Bayesian bound based on $\beta_{oj} = r_{oj} = -1$ is smaller than the corresponding classical bound obtained. It, therefore, appears that though exact in a Bayesian sense (under the assumed prior distribution for $R_j, j=1,2,\ldots k$), the bounds based on such a prior assumption are conservative in the classical sense.

When the optimum bound is standardized at .800 by adjusting the mission time t_m, the fiducial bound ranges from .538 to .793. When the optimum bound is equal to .368, the fiducial bound ranges from .062 to .354.

El Mawaziny and Buehler (13) show that their large-sample approximation of the optimal bound, a bound obtained by the Rosenblatt method (40) and the fiducial bound will approach the optimal bound as numbers of failures for all subsystems become large. For three samples having ten failures for each of three identical components, the fiducial bounds were of the order .787 and .341 for optimal bounds of .800 and .378, respectively, with deviation between any two corresponding fiducial bounds less than three in the third decimal place.

Analytical results described later indicate that the fiducial bounds for a fixed number of failures per subsystem will agree less well as the number of subsystems increases and the subsystems become more variable with respect to failure rate. Unfortunately, because of the computer-time factor and considerations of precision, it is impossible at present to compare bounds for systems containing as many as ten subsystems when as many as ten failures occur for more than one or two of these subsystems. In any case, the large-sample methods cannot be expected to give bounds agreeing well with the optimal bounds when some of the subsystems have been subjected to few tests. Furthermore, it is impossible on the basis of these results to say whether bounds based on this specified prior might be conservative, liberal, or exact for a particular logically complex system.

The uniform prior distribution for subsystem reliability gives bounds even lower than those based on the fiducial method except in 24 cases (out of 150) in which all three bounds have values fairly close to zero. In these 24 cases they are higher than the optimum classical bounds. It appears that the distribution of the bounds based on the uniform prior may be less disperse than those of the optimal bounds, but these bounds seem to be even more conservative than the "fiducial" bounds given by $\beta_{oj} = r_{oj} = -1$ for true reliabilities of a reasonable size and α's of interest. For systems with low reliabilities, bounds obtained using a uniform prior density for subsystem reliability should be liberal rather than conservative when the confidence level is sufficiently low, but not exact in general. This inconsistent behavior may be due to the fact that a uniform prior density for R_j implies a prior density for λ_j (the failure rate for the jth subsystem) of the form $q(\lambda_j) = t_m \exp(-\lambda_j t_m)$, $j=1,2,\ldots,k$, or, strangely, one which is a function of t_m, the specified mission time.

The result for $\beta_{oj} = r_{oj} = -1$ is keeping with the analysis and numerical results of Saunders (42), who studies a fixed-test-time exponential series-system model and the Bayesian approach suggested in (2). This Bayesian approach uses a prior density for the fixed-test-time model equivalent to the so-called fiducial method. Saunders (42) points out that in using such a Bayesian model for an exponential series system (and his argument applies to any true Bayesian model, that is, one based on a prior assumption which does not involve information concerning the number of components in the system), one can obtain different confidence bounds depending upon what one chooses to call a subsystem. Saunders points out, too, that such inferences apply to more logically complex systems which are highly reliable, since one can approximate an extremely reliable coherent system [see Birnbaum, Esary and Saunders (3) for a definition of a coherent system] by a series-system model, as indicated by Esary, Proschan and Walkup (15).

The Search for Optimum Prior Assumptions. Initially, it has been planned that a trial and error procedure would be used in attempting to determine appropriate prior assumptions for our series-system model. Saunders' argument might lead one to consider trying prior assumptions which are not truly Bayesian in that they are dependent upon the configuration (or number of components in the system). At this point in the study, however, an analytical result was derived, modifying the subsequent approach. The details of the analysis are given by Fertig (17) and are summarized below.

First, the form of a prior density function, or generalization of such a function, for R_j, $j=1,2,\ldots,k$, corresponding to the optimum classical bounds for our system model was determined. This was accomplished by setting the Laplace transform of $H_{\phi,k}(w|\underline{u};\phi)$ equal to the Laplace transform of the posterior density of $\phi = \sum_{j=1}^{k} \lambda_j$, obtained under a general prior assumption (not restricted to conjugate priors) for the special case $u_2 = u_3 = \ldots = u_k = 0$. If the prior assumptions which yield the optimum bounds are independent of the data, true Bayesian priors, for example, then an assumption concerning the value of the u's will have no effect on the result. The fact that the optimum classical bounds are invariant under permutations of the subsystems was used to obtain the improper "prior density" for the jth subsystem yielding these bounds. It is

$$p(R_j) = R_j^{-1}\,[\ln(1/R_j)]^{-(2-1/k)} \quad , \; j=1,2,\ldots,k.$$

We note that this improper density depends upon k, the number of subsystems in the series system and for k=1 does yield the optimum classical bound.

The Monte Carlo program was then used to test whether this prior assumption (which we may think of as a weighting function since it does not correspond to a strict Bayesian prior density) would yield the optimum bounds for variations in the data. For the case where all the u's equal zero ($z_1 = z_2 = \ldots = z_k$), eight values of the Monte Carlo bounds based on 5,000 replications agree with the optimum bounds to within three in the third decimal place. Data were randomly generated for five subsystems using the fact that $2Z_j/\lambda_j$ is distributed as chi-square with 2_{r_1} degrees of freedom [see Epstein and Sobel (14)], where $r_1 = 3$, $r_2 = 4$, $r_3 = 4$, $r_4 = 2$, $r_5 = 2$, $\lambda_1 = 1/12$, $\lambda_2 = 1/13$, $\lambda_3 = 1/15$, $\lambda_4 = 1/10$, $\lambda_4 = 1/11$. The mission time was taken as 1.0. For ten such data sets, the Monte Carlo bounds are uniformly larger than the classical confidence bounds with deviations ranging from 1 to 6 in the second decimal place.

The confidence bound obtained from any set of data by El Mawaziny's formula given by Eq. (1) is the unique optimum (uniformly most accurate unbiased) confidence bound for this exponential-failure-number series-system model. This is proved in the Appendix of Fertig's paper. Thus, any optimum bound is equivalent to the bound defined by Eq. (1) and must give the same result for any given set of data. Since the error in the Monte Carlo procedure is very small compared with the deviations obtained for the u's not all equal to zero, the empirical evidence indicates that the prior assumptions which yield the optimum classical bounds do depend upon the data.

Fortunately, a means of proving this result analytically then presented itself [see Fertig (17) for details]. The Laplace transform of $H(w|\underline{u};\phi)$ is a horrendous expression which gives no apparent clue as to how it might be factored and assigned to the various subsystems. The problem was made tractable earlier by letting all the z's be equal.

Another method of simplifying the expression was found to be to assume $r_1 = r_2 = \ldots = r_k = 1$. If this is done, then it is possible to demonstrate that the "prior density" for the jth subsystem yielding the optimum bound for $r_1 = r_1 = \ldots = r_k = 1$ cannot have the form $p(R_j) = R_j^{-1}\,[\ln(1/R)]^{-(2-1/k)}$ unless $z_1 = z_2 = \ldots = z_k$, as assumed in the earlier case. Hence, as indicated by the numerical evidence, one must incorporate present data into the "prior assumptions" or more properly the weighting functions, for obtaining optimum confidence bounds for the exponential fixed-failure-number series- system model.

Now that we have established what is not fruitful for obtaining the optimal bounds, one may properly inquire as to the next step in the investigation with respect to confidence for a complex system. Two approaches present themselves. The first is to consider each series system which is a part of the complex system as a single subsystem of the total system. If the fiducial approach were to be used, one could obtain an estimate $R_{\rho,\ell}$ of true reliability, given the failure data, for the ℓth subsystem consisting of the k_ℓ independent subsystems making up this series-system subsystem by substituting a random number ρ_ℓ for 1 . in Eq. (1). Then the reliability equation could be used to obtain an estimate of total system reliability given the failure data. This method will probably not yield confidence bounds that are exact in the classical sense, though of course they are exact in a Bayesian sense. In lieu of the fiducial approach one could somehow modify the value obtained for $R_{o,\ell}$ in an attempt to obtain bounds exact in the classical sense. Clues as to how this might be accomplished may possibly be obtained by investigating a simple parallel system (again an exponential-failure-time model with fixed number of failures) and methods of obtaining confidence bounds exact in the classical sense for the simpler model.

A considerable amount of computer time will be required with this approach when the product of the number of subsystems in any series system and the number of failures for any subsystem in that series system becomes large. Hence, one might in such cases, abandon the idea of considering the series system as a single subsystem. Instead one might consider a method of approximating the optimum confidence bound for a series system by using series system data in the prior assumptions (or weighting functions) for the subsystems of the series system that yield the posterior distribution of series system reliability. For the case $r_1 = r_2 = \ldots r_k = 1$, one possible factoring of the Laplace transform gives

$$p_\ell(R_j) = R_j^{-1-\hat\beta_j+\hat\beta_{(1)}} \quad [\ln(1/R_j)]^{-(2-1/k_\ell)} \quad ,j=1,2,\ldots,k_\ell, \quad (4)$$

where $\hat\beta_{(1)}$ is the smallest of z_1/t_m, $z_2/t_m,\ldots,3_{k_\ell}/t_m$. This function or some modification of this function involving the true values for r_1, r_2, ..., r_k could be tried. If the expression (4) is used without modification, then the posterior distribution of series system reliability, given the failure data, depends only upon $\hat\beta_{(1)}$. It can be shown that, in fact, the fiducial distribution of series system reliability for this model depends only upon $\hat\beta_{(1)}$ if and only if $r_1 = r_2 = \ldots = r_{k_\ell} = 1$.

The method of Kraemer (24) depends solely upon $\hat\beta_{(1)}$ and hence would be expected to give poor results for large numbers of failures per component. This is born out by the comparisons made by Sarkar (42) and Grubbs (21).

The bounds derived by Grubbs (21) are approximations to the fiducial bounds, and for the bounds compared during this study, the approximation appears to be excellent. The Grubbs method is based on the fact that $2r_j \cdot \lambda_j/\hat\lambda_j$ is distributed as chi-square with $2r_j$ degrees of freedom,

$j=1,2,\ldots,k$, so that $\phi = \sum_{i=1}^{k} \lambda_j$ can be thought of as distributed as a weighted sum of chi-square variates. The weights used by Grubbs, namely, $\hat{\lambda}_j/2r_j$, $j=1,2,\ldots,k$, are appropriate for obtaining the fiducial bounds. One could obtain, instead, an approximation to the optimal bounds by adjusting the weights appropriately, obtaining clues from the expression (4) above. In this way, one can test approximations to the optimum prior assumptions and avoid the time-consuming Monte Carlo calculations. If successful in closely approximating the optimum bounds by the proper modification of the Grubbs method, one can use this approximation in place of the Lentner-Buehler-El Mawaziny bounds in considering series systems within a complex system as single subsystems.

The investigation is being continued along these lines.

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COMPARATIVE ANALYSIS OF THE LCSS-ETG-3 PERFORMANCE CAPABILITY USING STATISTICAL PROBABILITIES

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ABSTRACT

A test plan is formulated and executed to obtain a random sample of measurements on the U.S. Army Missile Command's Land Combat Support System Electronic Test Group equipment.

Analyses of the data establish sample estimates of bias, accuracy, and stimulus setting errors and the standard deviation of measurement on 14 combinations of parameters and scales (e.g., dc voltage, 10-volt scale). The analyses pose hypotheses about the statistics and test these hypotheses against appropriate frequency distributions. They include the principle of analysis of variance, which makes use of bias error, accuracy error, stimulus setting error, and sample variance. These four parameters are used as response variables to establish the effects of the main factors of test durations, time delays, and machines and combinations of the main factors (i.e., interactions) on the computed response statistics for each of the 14 parameters and scales considered.

The overall estimates of the precision (standard error of measurement) for each parameter and scale are related to actual weapon system tolerances to obtain probability estimates of the risk of passing a bad unit ("undetected defect") or holding a good unit ("false alarm") in a single test in a checkout procedure. Single checkout probabilities are related to multiple sequential checkout probabilities.

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Section I. INTRODUCTION

The purpose of the Land Combat Support System (LCSS) electronic test group (ETG) equipment is to provide maintenance support and check operational readiness of major modules, assemblies, and subassemblies of the Shillelagh, Lance, and TOW missile systems. The primary requirement is for direct and general support missions. A detailed description of the LCSS-ETG can be found in a previous report [1]. The ETG is designed to automate the testing of critical components of the missile systems to achieve:

a) Rapid evaluation of the operational status of the unit under test (UUT).

b) Rapid fault isolation of a defective UUT.

c) Automated decision making as to operational status by comparison of measured values with prescribed standards.

d) A standardized automated test capability for several weapon systems.

Automation of the ETG equipment requires the preparation of a programmed test sequence. The test program instructs the operator on the required manual operations for the checkout such as external connections to "make" and "break." The program includes all necessary tests for functional checkout of the UUT's as prescribed by the weapon system design engineers. Typical tests are to measure stimuli and responses of such parameters as ac and dc voltage, resistance, optical alignment, frequency, phase, and time, and to compare these measurements with prescribed values. The required values of such parameters along with acceptable tolerances (deviations) are prescribed in the test program. The test equipment makes the measurement and compares it with the specified value and decides on a "go/no-go" basis as to a fault determination.

In the testing of missile components on a go/no-go basis there are a combination of conditions which may exist. A unit may be good and check good resulting in a go decision. The unit may be good and check bad resulting in a no-go decision. On the other hand, a unit may be bad and check good resulting in a go decision, or it may be bad and check bad resulting in a no-go decision. There are certain probabilities associated with these combinations of actual component condition and checkout results. These are shown in Table I. The $p(\alpha)$ is the probability that a unit checks good when in fact it is bad. The $p(\beta)$ is the probability that a unit checks bad when in fact it is good. Some authors refer to these probabilities as an "undetected defect" and a "false alarm,"

TABLE I. CHECKOUT PROBABILITY VERSUS UNIT CONDITIONS AND TEST DECISIONS

Unit Condition and Checkout	Decision	
	Go	No-Go
Bad		
Checks good (undetected defect)	$p(\alpha)$	NA
Checks bad	NA	$[1 - p(\beta)]$
Good		
Checks good	$[1 - p(\alpha)]$	NA
Checks bad (false alarm)	NA	$p(\beta)$

respectively. It can be seen that the $p(\alpha)$ is related to a go decision and the $p(\beta)$ is related to a no-go decision based on the test results. A unit that is good and checks good will not result in a no-go decision. Similarly, a unit that is bad and checks bad will not result in a go decision. These combinations are not applicable and are shown as NA in Table I. In a go decision situation there is a probability that a bad unit has checked good; i.e., there is a $p(\alpha)$ chance that a defect exists and it is undetected by the test equipment which is an "undetected defect." In a no-go decision situation there is a probability that a good unit has checked bad; i.e., there is a $p(\beta)$ chance that the test equipment has falsely indicated a defect that does not exist which is a "false alarm." Therefore, it is seen that the $p(\alpha)$ is the probability of simultaneously getting a measured value within the specification limits (or the decision limits) and an actual value outside the specification limits. The $p(\beta)$ is the probability of simultaneously getting a measured value outside the specification limits (or decision limits) and an actual value inside the specification limits.

In other words, given a go decision, the probability that it is wrong (bad checks good) is $p(\alpha)$ and the probability that it is right (good checks good) is $[1 - p(\alpha)]$. Given a no-go decision, the probability that it is wrong is $p(\beta)$ and the probability that it is right is $[1 - p(\beta)]$.

The $p(\alpha)$ and $p(\beta)$ set for the test equipment should be realistically determined in light of the weapon mission and test equipment environment. If the probability $p(\alpha)$ is set too high, an excessive number of bad units going to the troops will result.

On the other hand, if $p(\beta)$ is too high, an excessive amount of time is spent in checking for a defect that does not exist. Therefore experience, knowledge of military tactics and good judgment should govern the compromises between logistics, field troop effectiveness, troop operational conditions, military objectives, etc., to determine the levels of $P(\alpha)$ and $P(\beta)$. (Note: $p(\alpha)$ and $p(\beta)$ refer to single test probability and $P(\alpha)$ and $P(\beta)$ refer to multiple test probability.) The determination of $P(\alpha)$ and $P(\beta)$ considering the above military factors is not germane to this effort. This effort is concerned with the analysis of probabilistic relations between error probabilities $p(\alpha)$ and $p(\beta)$, and standard deviation error of measurement instrument $\left(\sigma_m\right)$, standard deviation of test parameter $\left(\sigma_p\right)$, parameter tolerance for a given $y\sigma$ confidence level (θ), and decision limits (γ). Also included are the relations between single test probabilities $p(\alpha)$, $p(\beta)$ and multiple test probabilities $P(\alpha)$, and $P(\beta)$.

Section II. MATHEMATICAL MODELS

1. General

In the measurement of any one individual parameter by the ETG, there are three things to be considered. The first is the value specified by the weapon system for the UUT. This is called the nominal value of the parameter (N). The second is the actual value of the parameter (X). The third is the measured value of the parameter (M). It is assumed that the actual value (X) of the parameter is related to the measured value (M), according to the normal probability density function. This assumption is based on the fact that there is no inherent bias error as would be caused by coupling, feedback loops, and switching in the instrument and that all errors in measurement are completely random and normally distributed. It is also assumed that the actual value X is distributed normally about the nominal value N, according to a normal probability density function.

Since the go/no-go decision is made on the measured value of the parameter, the normal probability density distribution for the random measurement error is considered in the following way. The density function is considered to be centered at the measured value M of the parameter. The density function with standard deviation σ_m describes the distribution of the possible actual value X_i's that could have resulted in a given measured value M.

2. Single Check Probability

The normal probability density distribution for the measurement error, for a given value M, has the form

$$f(X) = \frac{1}{\sigma_m \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X - M}{\sigma_m}\right)^2} \tag{1}$$

The actual parameter value, X, is also a random variable with a probability density function f(X). It is reasonable to assume that the actual value X is normally distributed about the nominal parameter value N with a standard deviation for the nominal parameter value of σ_p. With no bias error in the measurement device, the measured value M will also be normally distributed about the nominal value N with a density function f(M). If the measurement standard deviation σ_m is an order of magnitude less than the parameter standard deviation σ_p for the nominal value N then

$$f(M) \approx f(X) \ . \tag{2}$$

Therefore

$$f(M) = \frac{1}{\sigma_p\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{M-N}{\sigma_p}\right)^2} \tag{3}$$

The nominal value, N, as specified by the weapon system, also has prescribed tolerances. These tolerances are usually assumed to be $\pm n\sigma_p$ where σ_p is the standard deviation from the acceptable mean value of the parameter. The $+ n\sigma_p$ value is the upper specification limit and the $- n\sigma_p$ value is the lower specification limit, S_u and S_l, respectively. The tolerances may be specified as the $n\sigma_p$ level, where n = 1, 2, 3, 5, etc. Whatever the specified $n\sigma$ level, it represents the allowable limits for the parameter values by the weapon system for proper operation of the unit [2].

In order to assure that the probability of an undetected defect does not exceed a specified maximum, the measured value, M, must fall between even tighter test limits, defined as upper and lower decision limits D_u and D_l, respectively. A go/no-go decision is then based on whether the measured value falls inside or outside the decision limits and not the weapon system specification limits. The decision limits may be set at the specification limits or some fraction of the specification limits. That is:

$$\left(D_u, D_l\right) = a\left(S_u, S_l\right) \tag{4}$$

where $0 < a \leq 1$.

This is shown graphically in Figure 1 [3].

In Figure 1, it can be seen that $\theta = \pm n\sigma_p$ and $\gamma = \pm a\left(n\sigma_p\right)$ represents the upper and lower specification limits and the upper and lower decision (test)

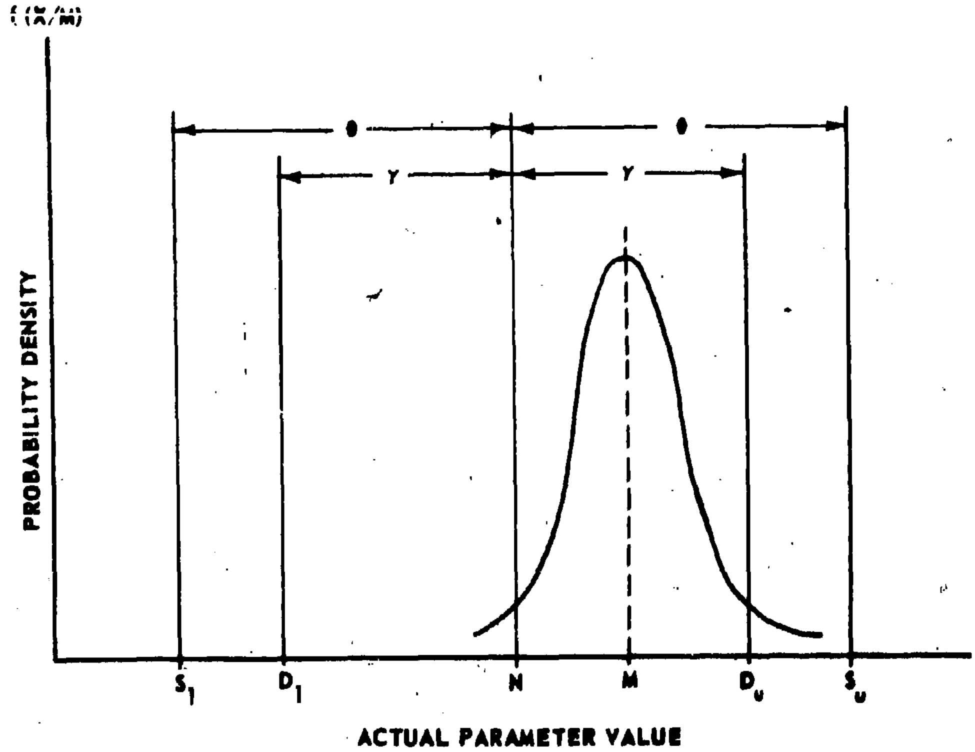

FIGURE 1. MEASUREMENT ERROR PROBABILITY DENSITY DISTRIBUTION

limits. The normal distribution curve around M is also shown to clarify the proposition of setting the decision limits less than the specification limits.

The probability of an undetected defect $p(\alpha)$ is the probability of simultaneously getting a measured value, M, within the decision limits and an actual value X outside the specification limits. The probability of getting a measured value, M, within the decision limits is $f(M)dM$. The probability that the measured value resulted from an actual value, X, outside the specification limits is

$$1 - \int f(X/M)dX \; . \tag{5}$$

The simultaneous probability is the product of the two individual probabilities:

$$\left[f(M)\,dM\right]\left[1-\int_{S_l}^{S_u} f(X/M)\,dX\right]. \qquad (6)$$

The probability of an undetected defect is the summation of the above probability over all possible M's between the decision limits:

$$p(\alpha)=\int_{D_l}^{D_u} f(M)\left[1-\int_{S_l}^{S_u} f(X/M)\,dX\right]dM\,. \qquad (7)$$

The probability of a false alarm $p(\beta)$ is the probability of simultaneously getting a measured value, M, outside of the decision limits and an actual value, X, inside the specification limits. Similarly,

$$p(\beta)=\int_{-\infty}^{D_l} f(M)\left[\int_{S_l}^{S_u} f(X)\,dX\right]dM+\int_{D_u}^{+\infty} f(M)\left[\int_{S_l}^{S_u} f(X)\,dX\right]dM\,. \qquad (8)$$

In equations (7) and (8) the limits are expressed as follows:

$$D_l = N-\gamma$$

$$D_u = N+\gamma$$

$$S_l = N-\theta$$

$$S_u = N+\theta \qquad (9)$$

and $f(X)$ and $f(M)$ are as defined in equations (1) and (3) above.

Substituting $f(X)$ and $f(M)$ into equations (7) and (8) gives an integral equation which is, according to Duncan [4], in the noncumulative form and cannot be integrated in closed form. Numerical approximations have been obtained and set up in tabular form. However, in order to obtain reasonably close engineering estimates of $p(\alpha)$ and $p(\beta)$, an exponential of the form e^{X} and e^{-X}

is used and the integration performed with the limits of equation (9) substituted. In this manner, the $p(\alpha)$ and $p(\beta)$ equations are obtained in terms of σ_p, σ_m, θ, and γ (the desired parameters) and reduce to the following equations:

$$
\begin{aligned}
p(\alpha) = {} & \left[\frac{\sigma_m}{2\,\sigma_p - \sigma_m}\right] e^{-1.15\left(\frac{\theta\sigma_p - \gamma\sigma_p + \gamma\sigma_m}{\sigma_p\sigma_m}\right)} \\
& - \left[\frac{\sigma_m}{2\left(\sigma_p + \sigma_m\right)}\right] e^{-1.15\left(\frac{\theta\sigma_p + \gamma\sigma_p + \gamma\sigma_m}{\sigma_p\sigma_m}\right)} \\
& - \left[\frac{\sigma_m^{2}}{\sigma_p^{2} - \sigma_m^{2}}\right] e^{-1.15\left(\frac{\theta}{\sigma_m}\right)} \, .
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
p(\beta) = {} & e^{-1.15\left(\frac{\gamma}{\sigma_p}\right)} - \left[\frac{\sigma_p^{2} - 2\sigma_p\sigma_m - \sigma_m^{2}}{\sigma_p^{2} - \sigma_m^{2}}\right] e^{-1.15\left(\frac{\theta}{\sigma_p}\right)} \\
& + \left[\frac{\sigma_m}{2\left(\sigma_p - \sigma_m\right)}\right] e^{-1.15\left(\frac{\theta\sigma_p - \gamma\sigma_p + \gamma\sigma_m}{\sigma_p\sigma_m}\right)} \\
& - \left[\frac{\sigma_m}{2\left(\sigma_p + \sigma_m\right)}\right] e^{-1.15\left(\frac{\theta\sigma_p + \gamma\sigma_p + \gamma\sigma_m}{\sigma_p\sigma_m}\right)} \, .
\end{aligned}
\tag{11}
$$

Equations (10) and (11) were solved parametrically assuming that the weapon systems specification limits fall at the 3-sigma points $\left(\text{i.e.}, \ \theta = 3\sigma_p\right)$ for the actual parameter value distribution for a series of values of the following ratios:

$$\frac{\sigma_m}{\sigma_p} \equiv \frac{\text{measurement deviation}}{\text{parameter deviation}} = \text{accuracy ratio}$$

$$\frac{\gamma}{\theta} \equiv \frac{\text{measurement limit}}{\text{specification limit}} = \text{decision ratio.}$$

The computed values were plotted as functions of $p(\alpha)$, $p(\beta)$, σ_m/σ_p, and γ/θ. The plots are shown to different scales in Figures 2, 3, and 4.

3. Multiple Check Probability

The discussion up to this point has been concerned with individual measurement error probability. It is often necessary, in the checkout of a UUT, to make two or more sequential tests on the same unit, Under such conditions the overall error probability becomes a function of the number of sequential tests, m, and the individual test probabilities, $p(\alpha)$ and $p(\beta)$. The multiple check probabilities $P(\alpha)$ and $P(\beta)$ may be computed from the following equations:

$$P(\alpha) = 1 - \prod_{i=1}^{m} \left[1 - p(\alpha)_i\right] \qquad (12)$$

$$P(\beta) = 1 - \prod_{i=1}^{m} \left[1 - p(\beta)_i\right], \qquad (13)$$

where

m = number of tests
$p(\alpha)$ ≡ individual test probability of undetected defect
$p(\beta)$ ≡ individual test probability of false alarm.

Assuming that $p(\alpha)$ and $p(\beta)$ are the same for all m tests, then equations (12) and (13) reduce to

$$P(\alpha) = 1 - [1 - p(\alpha)]^m \qquad (14)$$

$$P(\beta) = 1 - [1 - p(\beta)]^m \quad . \qquad (15)$$

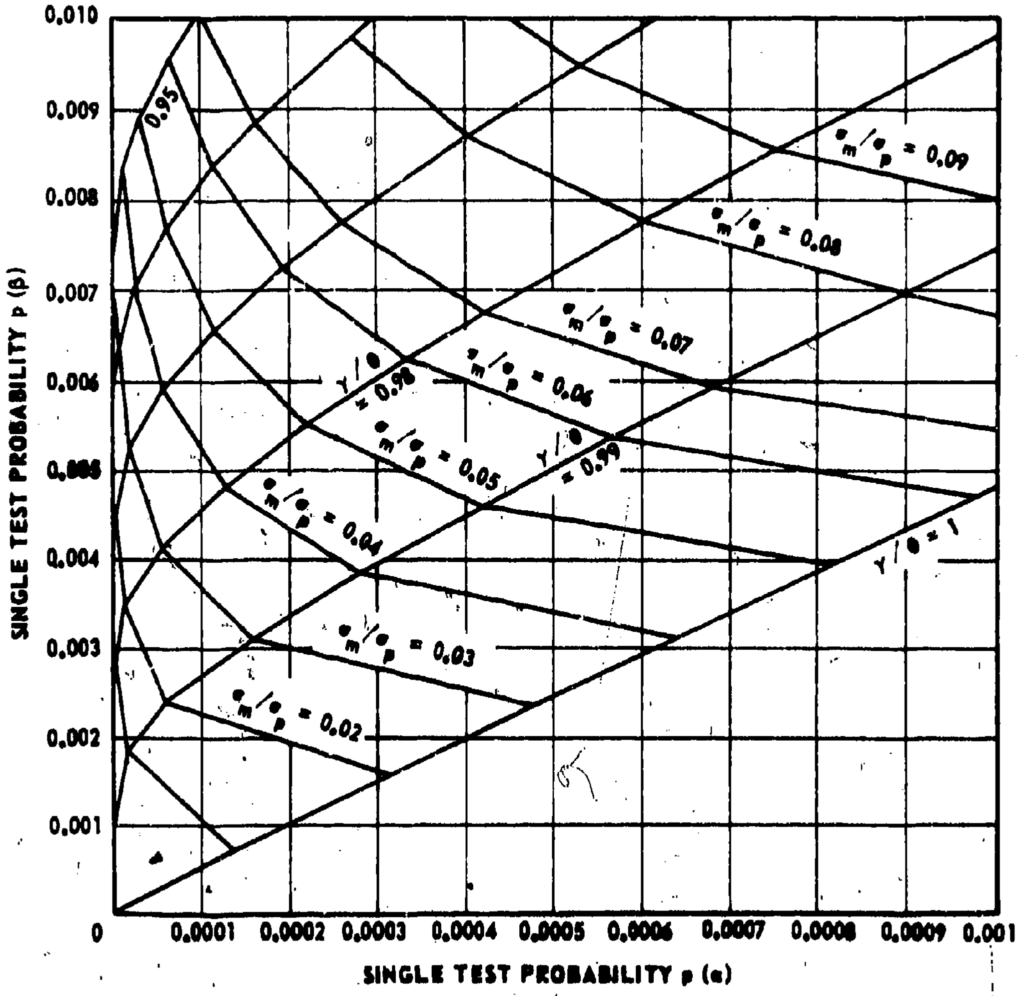

FIGURE 2. SINGLE TEST PROBABILITY VERSUS ACCURACY RATIOS AND DECISION LIMITS

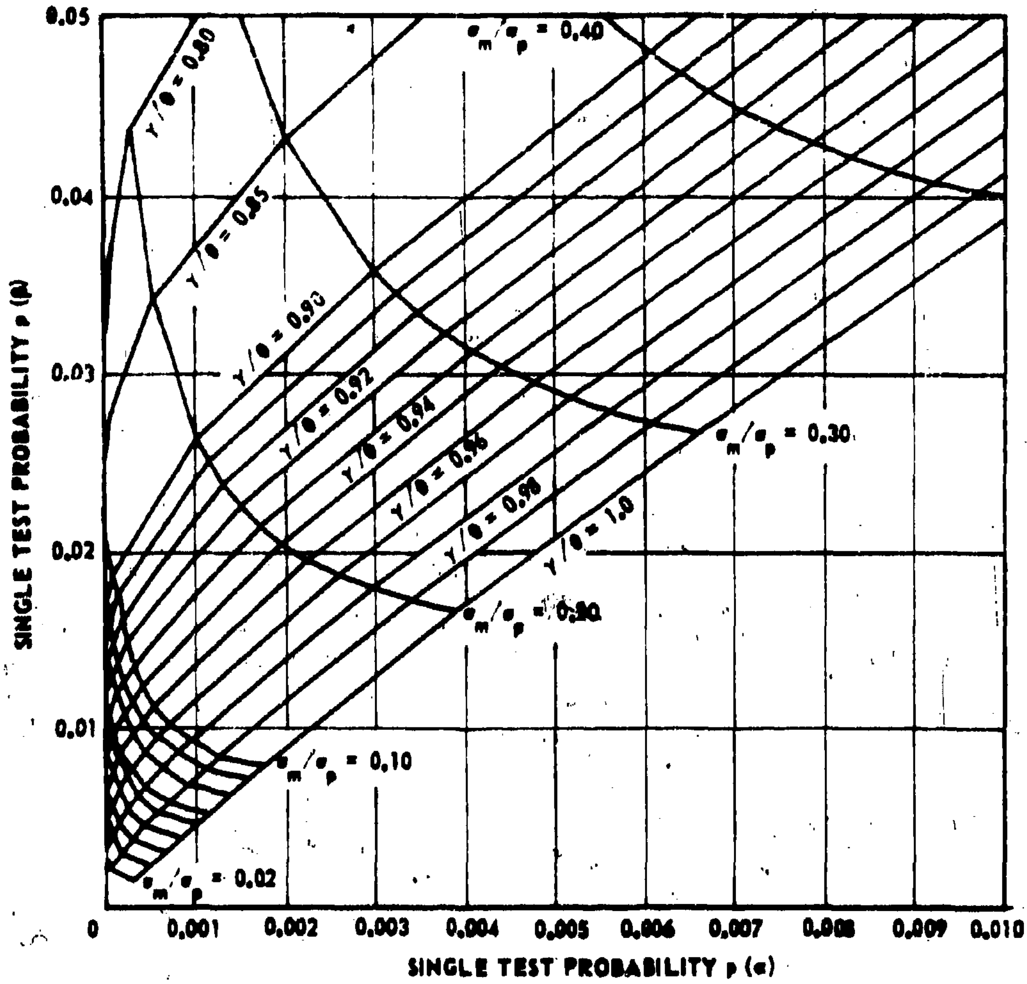

FIGURE 3. SINGLE TEST PROBABILITY VERSUS ACCURACY RATIOS AND DECISION LIMITS

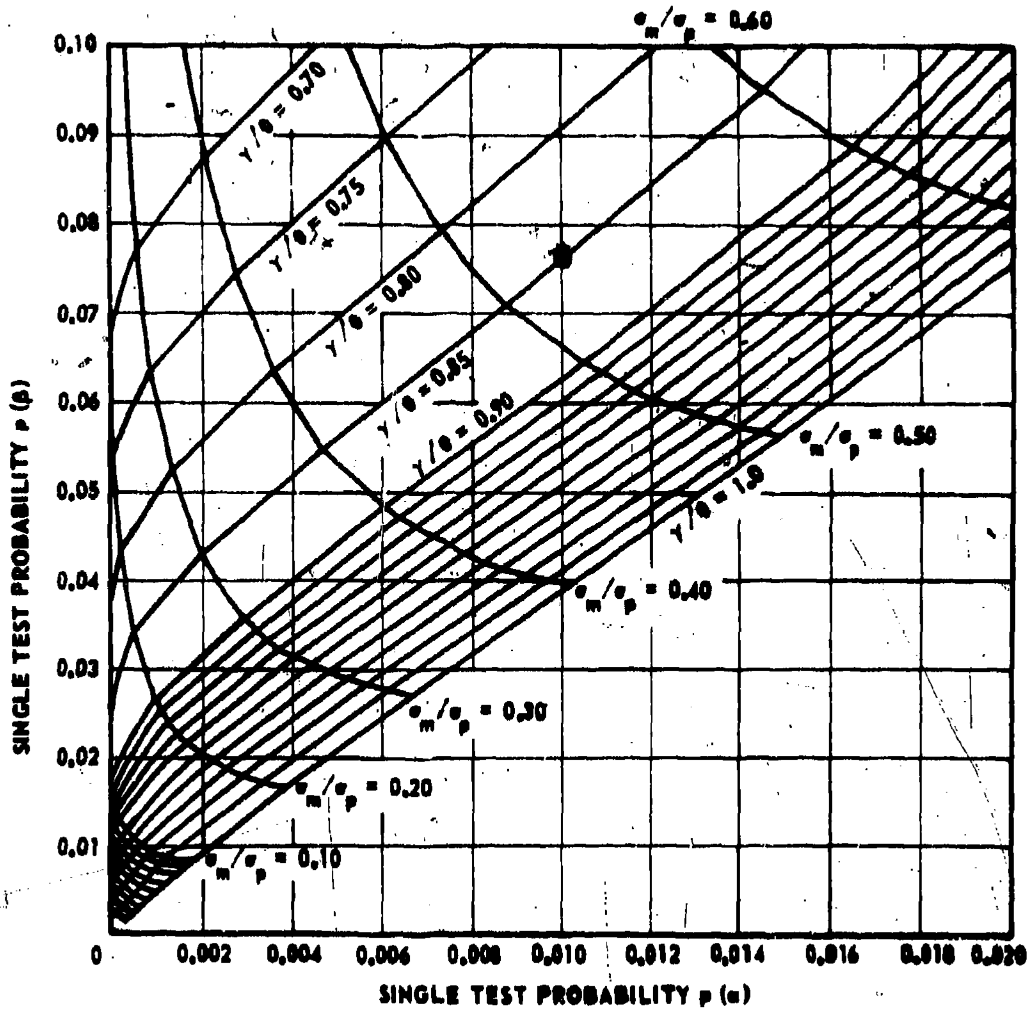

FIGURE 4. SINGLE TEST PROBABILITY VERSUS ACCURACY RATIOS AND DECISION LIMITS

Equations ([illegible]) and ([illegible]) were solved for a series of values of $p(m)$, $p(\sigma)$ [illegible] and m and are shown in plots of two different scales in Figures 5 and 6.

The more common way of expressing measurement accuracy is as a plus or minus percentage of full scale reading with a certain confidence. The standard deviations σ_m or σ_p can be expressed in percentage when the measurements are made at nearly full scale. The relationship between σ and percent is as follows:

$$\pm X\% = \left(\frac{\pm n\sigma}{F.S.} \right) 100 , \qquad (16)$$

where

$\pm X\%$ ≡ accuracy in percent full scale
σ ≡ standard deviation
n ≡ desired confidence level (i.e., 1, 2, 3, ... etc.)
F. S. ≡ full scale deflection of instrument.

A more complete and detailed discussion on the above derivation of error probability density functions and their relationship to test equipment may be obtained from Moon [5, 6]. The objective of this effort is to apply the mathematical models as shown to the design criteria of the LCSS-ETG equipment.

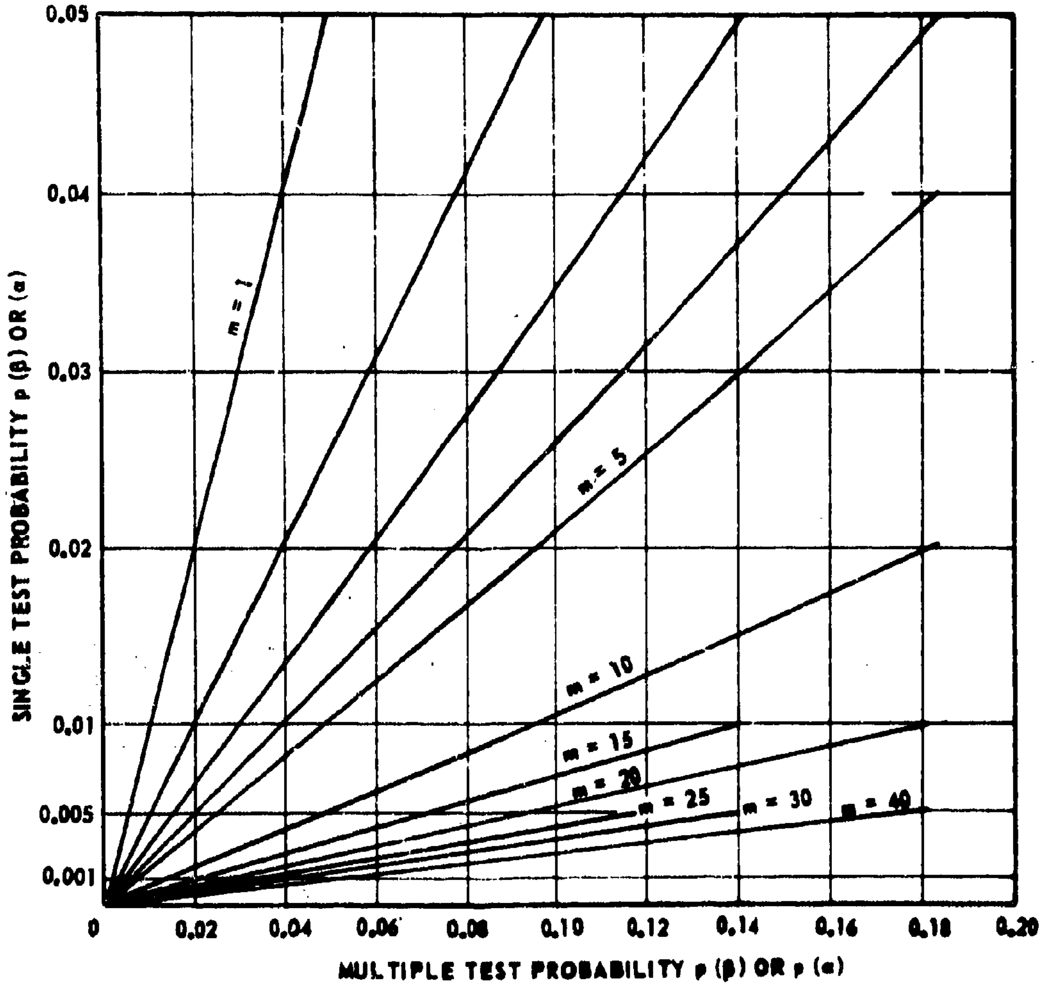

FIGURE 5. SINGLE TEST PROBABILITY [p(α), p(β)] VERSUS MULTIPLE TEST PROBABILITY [P(α), P(β)]

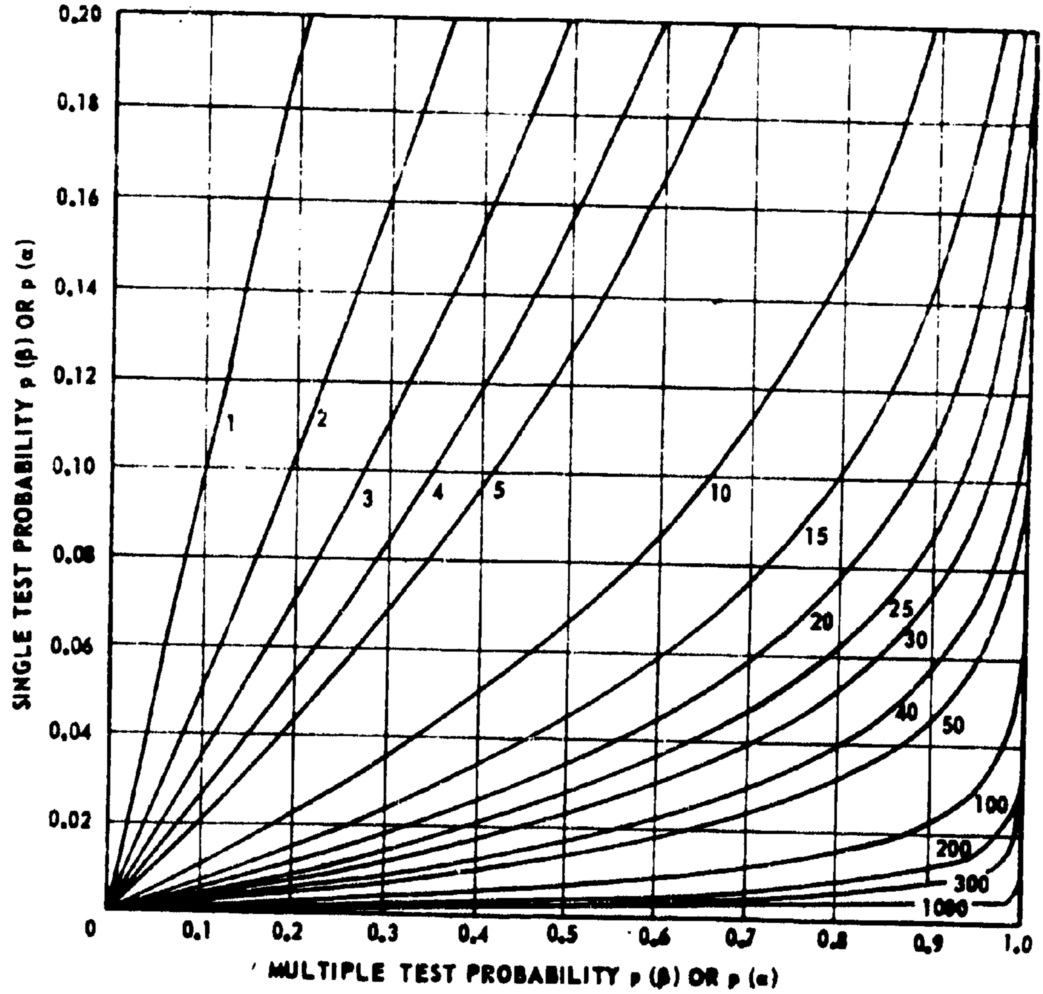

FIGURE 6. SINGLE TEST PROBABILITY [p(α), p(β)] VERSUS MULTIPLE TEST PROBABILITY [P(α), P(β)]

Section III. TEST PLAN

The purpose of the test program is to obtain a random sample of measurements to establish estimates of the accuracy, precision, and stability of the different measuring and stimulating systems of the LCSS-ETG over the full-scale ranges of the equipment. All values are, of course, only sample estimates based on limited samples of values from only two machines. Care should be taken when assuming that these two machines are representative of the population of machines.

1. Test Design

Two ETG sets were available for use. In view of the complexity of the ETG, it was decided to include in the test design the effects of time delay between the measurement command and the actual measurement. ETG specifications called for various time delays. In order to check for the effects of transients and drift upon short delays and long delays, respectively, it was decided to use the specified delay, and 3 times and 5 times the specified delay. The effects of repetitive measurements were also considered by including test time durations. The durations were established as four 1-hour tests and one 4-hour test periods per machine, and delay times. The test design is shown in Table II, from which it can be seen that each combination of machine, delay, and duration is tested for 4 hours for a total test time of 48 hours for all combinations.

In order to filter out as much test environment bias as possible, the test sequence was randomized. The randomization would tend to filter out

TABLE II. TEST DESIGN

Machine			2			5			Σ hr by duration
Tape delay			A	B	C	A	B	C	
Test run duration (Nom. hr)	hr	freq							
	1	4	4	4	4	4	4	4	24
	4	1	4	4	4	4	4	4	24
Σ hr by tape			8	8	8	8	8	8	48
Σ hr by machine			24			24			48

[illegible] variations due to periodic work variations, operators, time-of-day effects, etc. The objective of randomization is to increase the probability that the sample is random and increase the probability that inherent environmental biases are eliminated. The randomization was done using a random number generator coded for days, hours, machines, and delays. The randomized sequence as determined by the random number generator is shown in Table III. The machines are designated as 2 and 5 corresponding to their RCA serial numbers, ETG 3-002 and ETG 3-005, respectively. The delay times are designated as A, B, and C for the 1, 2, and 5 multiplier of specified delays, respectively. The numbers shown in parentheses in Table III refer to the test numbers used for identification in the computer programming. The numbers run sequentially from 1 through 48 corresponding to the total 48 hours of testing. With Tables II and III, all the data can be identified, classified, and grouped in any combination of test conditions for analysis.

2. Test Method

The test method was designed to simulate actual operational tests as performed on a weapon UUT by the English Language Program tapes. It was considered that this method would provide maximum data in a minimum amount of test time. Also, it was desired to leave the ETG sets available for other tests and uses as much as possible during their limited availability at Redstone Arsenal. It was designed to minimize operator error, biases, and interpretation simulating actual UUT test conditions.

A programmed semi-automatic test tape was compiled to make measurements and observations rapidly and automatically. The tapes were identical except for the programmed time delays between stimulus command and measurement execution. General instructions were provided the operator for running the tapes and all measurements were performed and printed automatically by the ETG. In programming the tapes, interrupts were inserted with the proper instructions to the operator where manual switching is required. It was specified that the operator, prior to running the test tape, would:

a) visually inspect the ETG machines for defective or missing components

b) check out the hook-up and main functional arrangement

c) perform all preliminary operational checks

d) make at least one successful run on the ETG with the RCA calibration and maintenance (C&M) tape

e) log all unusual conditions prior to and during the test tape run.

TABLE III. RANDOMIZATION OF LCSS TEST SCHEDULE

Date of Test	Hour of Test							Test Set 002			Test Set 005		
	0900	1000	1100	1200	1300	1400	1500	ΣA	ΣB	ΣC	ΣA	ΣB	ΣC
Feb.													
Tues 13		2C (1)								1			
Wed 14						2A (2)		1					
Thur 15			2B (3)						1				
Fri 16	2B (4)	2B (5)	2B (6)	2B (7)					4				
Mon 19							5A (8)				1		
Tues 20	5C (9)												1
Wed 21						2B (10)			1				
Tues 27	2C (11)		2A (12)					1		1			
Thur 29			5A (13)	5C (14)							1		1
March													
Fri 1	2C (15)	2C (16)	2C (17)	2C (18)		5B (19)				4		1	
Mon 4				5B (20)								1	
Fri 8		2A (21)	2A (22)	2A (23)	2A (24)			4					
Mon 11		2C (25)								1			
Tues 12					5C (26)								
Wed 13				5A (27)							1		
Thur 14		5A (28)	5A (29)	5A (30)	5A (31)						4		
Fri 15	5A (32)										1		
Mon 18	5B (33)	5C (34)	5C (35)	5C (36)	5C (37)							1	4
Tues 19					5C (38)								1
Wed 20	5B (39)	5B (40)	5B (41)	5B (42)								4	
Thur 21		5B (43)										1	
Tues 26					2B (44)				1				
Wed 27						2A (45)		1					

TABLE III. RANDOMIZATION OF LCSS TEST SCHEDULE (Concluded)

Date of Test	Hour of Test							Test Set 002			Test Set 003		
	0900	1000	1100	1200	1300	1400	1500	ΣA	ΣB	ΣC	ΣA	ΣB	ΣC
March													
Thur 28				2C (46)									
Fri 29	2A (47)			2B (48)									
Σ Hours and Tapes	8	9	9	11	6	4	1	8	8	8	8	8	[illegible]
$\Sigma\Sigma$	48							24			24		

The standards used as references during the test program were located inside or adjacent to the FTG. The standards were provided and calibrated by the U. S. Army Metrology and Calibration Center. The following equipment was used as standards:

a)	D-C standard/D-C differential voltmeter	HP 740B
b)	Kelvin Varley divider	ESI 722
c)	Electronic counter	HP 5245L
d)	Audio voltage standard	Holt Avs 323
e)	Ratio transformer	Gertsch PT-2
f)	Oscillator	HP 241 A
g)	RMS A-C differential voltmeter	John Fluke 931 A

The semi-automatic test tape has a general test procedure to program the appropriate ETG system to measure and print the corresponding values in the following sequence:

shorted input
external reference
self test reference
stimuli
external reference.

This sequence provides data to compute estimates:

measurement bias error $\left(\epsilon_b\right)$
measurement accuracy error $\left(\epsilon_a\right)$
stimulus setting error $\left(\epsilon_s\right)$

all relative to a known external standard reference. The following FTG measurement parameters were written into the test program tapes as functional groups:

Test Operation Series Number	Parameter Measured
100	dc voltage
200	Resistance
300	Frequency
400	ac voltage (400 Hz)
500	ac voltage (50 Hz)

Test Operation Series Number	Parameter Measured
600	ac voltage (10 kHz)
700	ac voltage (1 MHz)
800	Pulse train

Each line on the programmed printout is referred to as a single observation. All observations as printed out are identified by their operation number and parameter. On the complete tape going through all series (100 through 800) there are 101 observations (i.e., measurements). During each hour of test, all measurements were replicated five times. Table IV is a compilation of the complete test tape sequence of measurements showing the parameters, test operations numbers, type test, ETG component, ETG full-scale range, test conditions and the expected measured value or standard value.

3. Test Measurements

A complete set of the test tape results are shown in Volume II, Appendix A. The data shown therein have been corrected for obviously bad data and printout errors. In the great majority the data are duplicates of the original test tapes as printed out by the ETG.

TABLE IV. ETG TEST OPERATION IDENTIFICATION CHART

Parameter	Test Operation	Type Test	ETG Component	ETG Full-scale Range	Test Conditions	Expected Measured Value (or STD)	Computations
dc Voltage	100	SI	DMM	10 Vdc		0	
	101	ER	DMM	10 Vdc	7.800 Vdc	7.800	
	102	STR	DMM	10 Vdc	-7.500 Vdc	-7.500	
	103	DC 1	DMM	10 Vdc	9.000 Vdc	9.000	
	104	DA 1	DMM	10 Vdc	9.000 Vdc	9.000	
	105	FR	DMM	10 Vdc	7.800 Vdc	7.800	
	106	SI	DMM	250 Vdc		0	
	107	ER	DMM	250 Vdc	223.00 Vdc	223.00	
	108	PSI - 6	DMM	250 Vdc	230.00 Vdc	230.00	
	109	ER	DMM	250 Vdc	223.00 Vdc	223.00	
	110	SI	DMM	1000 mVdc		0	
	111	ER	DMM	1000 mVdc	503.21 mVdc	503.21	
	112	DA2	DMM	1000 mVdc	900.00 mVdc	900.00	
	113	FR	DMM	1000 mVdc	503.21 mVdc	503.21	
Resistance	200	SI	DMM	10 kΩ		0	
	201	ER	DMM	10 kΩ	8.8198 kΩ	8.8198 kΩ	
	202	STR	DMM	10 kΩ	9.0 kΩ	9.000 kΩ	
	203	ER	DMM	10 kΩ	8.8198 kΩ	8.8198 kΩ	
	204	SI	DMM	100 Ω		0	
	205	ER	DMM	100 Ω	80.05 Ω	80.05	
	206	SI	DMM	1000 kΩ		0	
	207	ER	DMM	1000 kΩ	819.9 kΩ	819.9 kΩ	
Frequency	300	SI	W.C.	1000 kHz		0	
	301	FR	W.C.	1000 kHz	999.996 kHz	999.996 kHz	
	302	STR	W.C.	1000 kHz	900.000 kHz	900.000 kHz	
	303	FR	W.C.	1000 kHz	999.996 kHz	999.996 kHz	
	400	SI	DMM	10 VRMS		0	
	401	FR	DMM	10 VRMS	5.628 VRMS, 100 Hz	5.628 VRMS	
	402	STR	DMM	10 VRMS	7.0 VRMS, 100 Hz	7.0 VRMS	
	403	ER	DMM	10 VRMS	5.628 VRMS, 100 Hz	5.628 VRMS	

TABLE IV. ETC TEST OPERATION IDENTIFICATION CHART (Continued)

Parameter	Test Operation	Type Test	ETG Component	ETG Full-scale Range	Test Conditions	Expected Measured Value (or STD)	Computations
ac Voltage, 400 Hz	404	SI	DMM	10 VRMS		0	
	405	ER	DMM	10 VRMS	5.628 VRMS, 400 Hz	0.317 VTRMS	$(5.628)^2/100 = 0.317$
	406	SI	DMM	10 VRMS		0	
	407	ER	DMM	10 VRMS	5.628 VRMS, 400 Hz	7.958 Vp	$(5.628)\sqrt{2} = 7.958$ Vp
	408	SI	DMM	10 VRMS		0	
	409	ER	DMM	10 VRMS	5.628 VRMS, 400 Hz	15.92 Vpp	$(5.628)\ 2\sqrt{2} = 15.916$
	410	SG 1	DMM/WC	10 VRMS	9.000 VRMS, 400 Hz	9.000 VRMS	
	411	SG 1	DMM/WC	10 VRMS	9.000 VRMS, 400 Hz	0.081 VTRMS	$(9)^2/10^3 = 0.081$
	412	SG 1	DMM/WC	10 VRMS	9.000 VRMS, 400 Hz	0.9000 VRMS	SG 1 on 10:1 attenuator
	413	SG 1	DMM/WC	10 VRMS	9.000 VRMS, 400 Hz	0.400 kHz	
	414	SG 2	DMM/WC	10 VRMS	9.000 VRMS, 400 Hz	9.000 VRMS	
	415	SG P	DMM/WC	100 VRMS	90.00 VRMS, 400 Hz	90.00 VRMS	
	416	SG Q	DMM/WC	10 VRMS	9.00 VRMS, 400 Hz	9.00 VRMS	(90-deg delay)
	417	SG Q	DMM/WC	10 VRMS	9.00 VRMS, 400 Hz	9.00 VRMS	(180-deg delay)
	418	ER	DMM	10 VRMS	5.628 VRMS, 400 Hz	5.628 VRMS	
ac Voltage, 50 Hz	500	SI	DMM	10 VRMS		0	
	501	ER	DMM	10 VRMS	0.977 VRMS, 50 Hz	0.977 VRMS	
	502	SG 1		10 VRMS	1.1 VRMS, 20 Hz	1.1 VRMS	
	503	SG 1		10 VRMS	1.1 VRMS, 20 Hz	0.0200 kHz	
	504	ER	DMM	10 VRMS	0.977 VRMS, 50 Hz	0.955 VTRMS	$(0.977)^2 = 0.955$ VTRMS
	505	ER	DMM	10 VRMS	0.977 VRMS, 50 Hz	1.381 Vp	$(0.977)\sqrt{2} = 1.381$ Vp
	506	ER	DMM	10 VRMS	0.977 VRMS, 50 Hz	2.762 Vpp	$(0.977)\ 2\sqrt{2} = 2.762$ Vpp
	507	SG 1		10 VRMS	1.1 VRMS, 20 Hz	1.21 VTRMS	$(1.1)^2/10^0 = 1.21$
	508	SG 1		10 VRMS	1.1 VRMS, 20 Hz	1.5565 Vp	$(1.1)\sqrt{2} = 1.5565$ Vp
	509	SG 1		10 VRMS	1.1 VRMS, 20 Hz	3.11 Vpp	$(1.1)\ 2\sqrt{2} = 3.11$ Vpp
	510	ER	DMM	10 VRMS	0.977 VRMS, 50 Hz	0.955 VTRMS	$(0.977)^2/10^0 = 0.955$ VTRMS
	511	ER	DMM	10 VRMS	0.977 VRMS, 50 Hz	1.381 Vp	$(0.977)\sqrt{2} = 1.381$ Vp
	512	ER	DMM	10 VRMS	0.977 VRMS, 50 Hz	2.762 Vpp	$(0.977)\ 2\sqrt{2} = 2.762$ Vpp

TABLE IV. ETC TEST OPERATION IDENTIFICATION CHART (Continued)

Parameter	Test Operation	Type Test	ETC Component	ETG Full-scale Range	Test Conditions	Expected Measured Value (or STD)	Computations
ac Voltage, 50 Hz	513	SI	DMM	250 VRMS		0	
	514	ER	DMM	250 VRMS	110.0 VRMS, 50 Hz	110.0 VRMS	
	515	SI	DMM	250 VRMS		0	
	516	ER	DMM	250 VRMS	110.0 VRMS, 50 Hz	121.0 VTRMS	$(110)^2/10^2 = 121.0$ VTRMS
	517	SI	DMM	250 VRMS		0	
	518	ER	DMM	250 VRMS	110.0 VRMS, 50 Hz	155.5 Vp	$(110)\sqrt{2} = 155.5$ Vp
	519	SI	DMM	250 VRMS		0	
	520	ER	DMM	250 VRMS	110.0 VRMS, 50 Hz	311.1 Vpp	$(110)\ 2\sqrt{2} = 311.1$ Vpp
	521	SG P	DMM/WC	250 VRMS	110.0 VRMS, 20 Hz	110.0 VRMS	
	522	SG P	DMM/WC	250 VRMS	110.0 VRMS, 20 Hz	121.0 VTRMS	$(110)^2/10^2 = 121$ VTRMS
	523	SG P	DMM/WC	250 VRMS	110.0 VRMS, 20 Hz	155.5 Vp	$(110)\sqrt{2} = 155.5$
	524	SG P	DMM/WC	250 VRMS	110.0 VRMS, 20 Hz	311.1 Vpp	$(110)\ 2\sqrt{2} = 311.1$
	525	SG P	DMM/WC	1000 kHz	110.0 VRMS, 20 Hz	0.020 kHz	
	526	ER		250 VRMS	110.0 VRMS, 50 Hz	110.0 VRMS	
	527	ER	DMM	250 VRMS	110.0	121.0 VTRMS	$(110)^2/10^2 = 121.0$
	528	ER	DMM	250 VRMS	110.0	155.5 Vp	$(110)\sqrt{2} = 155.5$
	529	ER	DMM	250 VRMS	110.0	311.1 Vpp	$(110)\ 2\sqrt{2} = 311.1$
ac Voltage, 10 kHz	601	ER	DMM	10 VRMS	0.614	0.614 VRMS	
	602	SG I		10 VRMS	1.1	1.1 VRMS	
	603	SG I		10 kHz	1.1	9.0 kHz	
	604	ER	DMM	10 VRMS	0.614 VRMS, 10 kHz	0.614 VRMS	
	605	ER	DMM	10 VRMS	0.614 VRMS, 10 kHz	0.8682 Vp	$(0.614)\sqrt{2} = 0.8682$ Vp
	606	ER	DMM	10 VRMS	0.614 VRMS, 10 kHz	1.736 Vpp	$(0.614)\ 2\sqrt{2} = 1.736$ Vpp
	607	ER	DMM	250 VRMS	110 VRMS, 1 kHz	110 VRMS	
	608	ER	DMM	250 VRMS	110 VRMS, 1 kHz	121 VTRMS	$(110)^2/10^2 = 121$ VTRMS
	609	ER	DMM	250 VRMS	110 VRMS, 1 kHz	155.5 Vp	$(110)\sqrt{2} = 155.5$
	610	ER	DMM	250 VRMS	110 VRMS, 1 kHz	311.1 Vpp	$(110)\ 2\sqrt{2} = 311.1$ Vpp
	611	SG P		250 VRMS	110 VRMS, 1 kHz	110 VRMS	
	612	SG P		250 VRMS	110 VRMS, 1 kHz	121.0 VTRMS	$(110)^2/10^2 = 121.0$ VTRMS
	613	SG P		10 kHz	110 VRMS, 1 kHz	1.0 kHz	
	614	ER	DMM	250 VRMS	110 VRMS, 1 kHz	110 VRMS	
	615	ER	DMM	250 VRMS	110 VRMS, 1 kHz	121.0 VTRMS	$(110)^2/10^2 = 121.0$ VTRMS
	616	ER	DMM	250 VRMS	110 VRMS, 1 kHz	155.5 Vp	$(110)\sqrt{2} = 155.5$ Vp
	617	ER	DMM	250 VRMS	110 VRMS, 1 kHz	311.1 Vpp	$(110)\ 2\sqrt{2} = 311.1$ Vpp

TABLE IV. ETG TEST OPERATION IDENTIFICATION CHART (Concluded)

Parameter	Test Operation	Type Test	ETG Component	ETG Full-scale Range	Test Conditions	Expected Measured Value (or STD)	Computations
ac 1000 kHz	701	ER	DMM	10 VRMS	1.00 Vp, 100 kHz	1.00 Vp	
	702	ER	DMM	10 VRMS	1.00 Vp, 100 kHz	2.00 Vpp	
Pulse Train	801	ER	W.C.	1000 kHz	999.996 kHz	999.996 kHz	
	803	PGN	W.C.	1000 kHz	100 kHz (30 Vp)	99.9999 MS	
	804	PGN	DMM/W.C.	100 Vp	100 kHz (30 Vp)	30 Vp	Volts p in time period
	805	PGN	W.C.	1000 kHz	100 kHz (30 Vp)	99.9999 MS	Pulse rise time
	806	PGN	W.C.	1000 kHz	100 kHz (30 Vp)	0.005 MS	Pulse width
	807	PGN	W.C.	1000 kHz	100 kHz (30 Vp)	10 MS	Pulse delay
	809	ER	W.C.	1000 kHz	999.996 kHz	999.996 kHz	

Section IV. DATA ANALYSIS

1. General

The semiautomatic test tapes were run on the ETG-3 machines in accordance with the randomized sequence shown in Table III. On the tapes are 101 measurement commands. Each tape was replicated five times in each hour. Each machine was tested for 24 hours for a total of 48 hours. Therefore, the total number of observations is computed as follows:

$$N_T = 101 \text{ obs/rep.} \times 5 \text{ rep/hr} \times 48 \text{ hr} = 24,240$$

for all combinations of machines, time delays, durations and parameters. The 101 observations are broken down into a certain number for each parameter. A breakdown of the total number of observations for any combination is shown in Table V.

The measurements data were printed out by the ETG-3 printer on paper tapes. The data were coded for computer use, punched on input cards, and identified by test set numbers 1 through 48. The test set numbers included identification by machine, time delay, duration, date and hour of the day. The data used are subject to at least two sources of error. The first is machine printout errors. If a measurement was different from the last measurement of the same operation by less than a factor of two, it was included in the analysis as recorded. If the difference was greater than a factor of two, the measurement was excluded from the analysis and recorded as a machine "fault" or "dropout." All data replacing dropouts were estimated in accordance with established missing value procedures. Sample computations were made, including dropout values, to assess their effect on a computed statistic. The factor of two may not be the optimum factor but was selected to avoid any arbitrariness on determination of which values to exclude. The second source of error is the transfer of data from machine printout tapes to input cards. Obviously all errors were not eliminated and the results include the errors which could not be identified or which were overlooked. The data used for the computations are shown in Appendix B of Volume II. Lines are drawn under values estimated for dropouts. The total number of dropouts was 54 out of 24,240 observations. A list of dropouts by operation number is shown in Table VI, which also tabulates the dropouts by machine, delay time, and duration on an hourly basis. Shown for comparison are the operational faults on the C&M tapes which occurred at the time the test tapes were run. A description of the faults is listed below the table. Also shown is the ETG-3 parameter being tested by each operation. The 800 series of test operations (pulse train) had the largest

TABLE V. SCHEDULE OF OBSERVATIONS BY MACHINE, TIME DELAYS, DURATION AND PARAMETER

	Machines	ETG3-0002						ETG3-0005						Σ by Parameter and Scale
	Time Delays	A		B		C		A		B		C		
	Durations	1	4	1	4	1	4	1	4	1	4	1	4	
Parameters and Scales	10 Vdc	120	=	=	=	=	=	=	=	=	=	=	=	1440
	250 Vdc	80	=	=	=	=	=	=	=	=	=	=	=	960
	1000 mVdc	80	=	=	=	=	=	=	=	=	=	=	=	960
	10 k ohm resistance	80	=	=	=	=	=	=	=	=	=	=	=	960
	100 ohm resistance	40												480
	1000 k ohm resistance	40												480
	1000 kHz frequency	80												960
	10 Vac 400 Hz	380												4560
	10 Vac 50 Hz	260												3120
	250 Vac 50 Hz	340												4080
	10 Vac 10 kHz	120												1440
	250 Vac 10 kHz	220												2640
	10 Vac 1000 kHz	40												480
	Pulse Train	140												1680
Σ	MTD	2020	=	=	=	=	=	=	=	=	=	=	=	
Σ	TD	4040		=		=		=		=		=		24,240
Σ	M	12,120						=						

TABLE VI. COMPILATION OF PROGRAM DROPOUTS AND C&M FAULTS

Machine Tapes Faults	FIG5-0002 A Oper.	FIG5-0002 A C&M	FIG5-0002 B Oper.	FIG5-0002 B C&M	FIG5-0002 C Oper.	FIG5-0002 C C&M	FIG3-0005 A Oper.	FIG3-0005 A C&M	FIG3-0005 B Oper.	FIG3-0005 B C&M	FIG3-0005 C Oper.	FIG3-0005 C C&M
	500	2 202 318 323		3	804	1 367	806	8 611 806 183 225 287	806	19 318 183	112 801 806	9 124 153 287 306
		12		10		11		13		20	804	14 183 201
		[illegible]		[illegible]		25	806	27 519 201 369 360	806	33	804 806	26
	510 801	17 117 185 176 263 290	102,104 105,205 302,303 401-450 501,511 601,605 801,803 809	18 117 185 176 36 290		16		32	806	15 129 224	804	38 137 224
		21	804 805	4 169 265 360		15		28		29		34
	804	22 117 367 371	803 805	5 169 265 360		16		29		10	108 804	35
	809	23 117 367 371	807 804 805	6 169 265 360		14		30	806	11 187 182		36
		24	805	7 169 265 360		18		31	205 207 608 611 613 615 806	12 187 182	801 804 809	37

C&M Test Function Checks:

112 - test adapter (1A2) routing switches A1 and B1 and controller (1A1) switching data character SD5-2 and SD7-1 drivers.

124 - 1-megohm input impedance of digital multimeter (1A5) dc converter with dc standard output.

129 - digital multimeter (1A5) 0-9.999 K resistance measurement scale.

137 - resistance across capacitor C7. Test continues by checking ±0.5 V dc limit on digital multimeter resistance converter output on 0-99.99 ohm and 0-999.9 ohm measurement scales.

153 - waveform converter (1A6) 0-99.99 V peak voltage measurement scale using probe and 7 V RMS output of self-test reference (1A7).

169 - test points connected to fixed resistors mounted on confidence and maintenance patchboard.

176-188 - check operation of 0-10 V dc power supply (1A8).

201 - 2 bits in pulse generator results counter 3 most significant decades. Also checks waveform converter results counter 3 least significant decades.

202 - changes oscillatory status level pulse repetition frequency to check waveform converter (1A6) results counter 2 bit in second decade.

224 - signal conditioner chopper drive frequency and digital multimeter 10' mVdc measurement range by measuring 500 mVdc output from 0-10 Vdc power supply (1A8).

265 - output of 0-10 Vdc power supply 3 (7A7) on MBA terminals before receipt of apply command and resistance at 0-10 Vdc power supply 3 (7A7-A) output for a 21 ohm load.

287 - Measures 0-10 Vdc power supply 5 (1A18-A) MBA output with supply programmed to 0 Vdc.

306 - 5-50 Vdc power supply (7A12) MBA output.

318 - frequency of signal generator (1A19) oscillator (SG1) output on MBA terminals. Then checks signal generator (SG1) and waveform converter (1A6).

323 - SG1 output frequency on MBA terminals.

360 - period of pulse generator (1A17) (PGN) main output. Test also checks unit apply function of PGN main output.

363 - pulse width of pulse generator (1A17) main output on MBA terminals.

367 - pulse width of pulse generator (1A17) main output.

369 - amplitude of pulse generator (1A17) main output.

371 - amplitude of pulse generator (1A17) main output. Test continues by checking amplitude of pulse generator main output.

398 - TPA11 by measuring continuity from RSA001 normally open contact to arm. RSA521 energized.

519 - (undetermined); 611 - (undetermined); 806 - (undetermined).

Operations: 100-105 10 Vdc; 106-109 250 Vdc; 110-113 100.0 mVdc; 200-203 10 K resistance; 204-205 0.1K resistance; 206-207 1000K resistance; 300-302 1000 kHz frequency; 400-418 10 VRMS 300 Hz; 500-512 10 VRMS 50 Hz; 512-529 250 VRMS 50 Hz; 601-606 10 VRMS 10 kHz; 607-617 250 VRMS 10 kHz; 701-702 10 VRMS 1000 kHz; 801-809 pulse train.

number of dropouts with 32. Test hour 48 had the largest number of dropouts by the hour with 15. A correlation of dropouts with C&M faults is evident. In the table the hourly test set numbers are in the upper right square. In test set 2, operation 400 was a dropout. Operation 400 measures 10 VRMS, 400 Hz ac voltage. The corresponding C&M faults, obtained from the machine daily log, which could be the cause, is 318 and 323 which checks the signal generator frequency. A complete analysis is beyond the scope of this effort, but would give additional insight into test set problems.

2. Statistical Computations

The main objective of this program was to determine an estimate of the standard error of measurement (measurement standard deviation) for the different parameters and scales of the machine. This is referred to earlier as σ_m. This statistic would be used in conjunction with the weapon system parameter standard deviation σ_p to obtain an estimate of the accuracy ratio. This ratio is needed to determine realistically the statistical probability of undetected defects and false alarms in the go/no-go chain. However, an attempt has been made to estimate other characteristics of the ETG-3.

When a measurement is made by the ETG-3, there are three primary sources of error. These are:

a) inherent machine bias error, ϵ_b

b) machine measurement accuracy error, ϵ_a

c) machine stimulus setting error, ϵ_s.

To illustrate, suppose the English Language Test Program Tape commands a certain voltage be applied to the UUT with a specified response to that stimulus to be measured. The circuits providing that stimulus may be effected by a previous operation (bias), the measuring device may sense the bias and read erroneously (accuracy) or the bias and accuracy may be acceptable but the voltage applied was not the value specified (stimulus setting). Naturally, combinations of all three may exist on any given measurement.

Estimates of bias error, ϵ_b, were obtained by programming a shorted input (SI) operation as shown in Table IV and measuring for a response. Estimates of measurement error, ϵ_a, were obtained by measuring an accurately known value from one of the external reference (ER) standards mentioned previously and subtracting the bias error. Estimates of the stimulus setting error were obtained by programming a stimulus from one of the ETG-3 components and subtracting the bias and measurement errors.

The error estimates were computed with the following equations:

$$\epsilon_b = \frac{\sum_{i=1}^{N}\left(SI_i - EV\right)}{N} , \qquad (17)$$

where

SI ≡ shorted input value
EV ≡ expected value
N ≡ number of responses.

$$\epsilon_a = \frac{\sum_{i=1}^{N}\left(ER_i - EV\right)}{N} - \epsilon_b , \qquad (18)$$

where ER_i ≡ external reference measured value.

$$\epsilon_s = \frac{\sum_{i=1}^{N}\left(MV_i - EV\right)}{N} - \epsilon_a , \qquad (19)$$

where MV ≡ measured value of ETG-3 component (e.g., DA-1, DC-1, STR). The above computed errors are the means of the differences between the actual (or expected) value and the measured (or unexpected) value.

The standard deviation of measurement $\left(\sigma_m\right)$ is computed by the following equation:

$$S_M{}^2 = \frac{\sum_{i=1}^{N}\left[(MV - EV)_i - \overline{(MV - EV)}\right]^2}{N - 1} , \qquad (20)$$

which is the measurement variance. The standard deviation is simply the square root $\left(S_M \right)$ of the equation and is designated S rather than σ since it is computed from a limited sample of data.

The statistics as computed with the above equations were computed in two ways. The first computations were made for each hour (test set) of testing. The purpose of this was to use the estimates of these statistics as response variables for a subsequent analysis of variance. The analysis of variance enables one to test for significant differences between factors. In addition to testing for significant differences between the main factors (i.e., machines, time delays and durations), tests can also be made for differences between second order or interaction effects (e.g., machines × time delay effects) as well as third order interaction effects. The second set of calculations was made using all observations for each parameter. These are referred to as the overall values of ϵ_b, ϵ_a, ϵ_s, and S_M. These calculations are more sensitive to subsequent tests due to the much larger sample size and greater degree of freedom.

The hourly computations are shown in Volume II, Appendix C. These values were computed with small sample size and degrees of freedom and are used only in the analysis of variance computations. The overall statistics are shown in Table VII. While the statistics ϵ_b, ϵ_a, and ϵ_s give some indication of the machine error by parameter and scale, the measurement standard deviation S_M is the important statistic from the standpoint of machine variability and determination of statistical control of desired probability levels of "undetected defects" and "false alarms," $p(\alpha)$ and $p(\beta)$.

3. Analysis of Variance (ANOVA)

The hourly computations of ϵ_b, ϵ_a, ϵ_s, and $S_M{}^2$ were used as response variables for analyses of variance of all parameters and scales. Hence, there were 4 × 14 = 56 analyses performed. The statistical models are

$$\epsilon_b(i, j, k, l) = \mu + A_i + B_j + C_k + AB_{ij} + AC_{ik} + BC_{jk} + ABC_{ijk} + e_{l(ijk)} \qquad (21)$$

and similarly for ϵ_a, ϵ_s, and $S_M{}^2$. This model assumes that each response of errors ϵ_b, ϵ_a, ϵ_s and variance $S_M{}^2$ is the algebraic sum of:

TABLE VII. COMPILATION OF OVERALL COMPUTED ERRORS, STANDARD DEVIATION AND PERCENTS BY PARAMETER AND SCALE

No.	Parameter and Scale	Computed Values					Specified		* Comp. % Value > Specified Value
		ϵ_b	ϵ_A	ϵ_S	S_M	±% $3 S_M$	S	±% (3 S)	
1	10 Vdc	0.000946	-0.00197	0.00106	0.02515	0.8	0.0013	0.04	
2	250 Vdc	0.147	-0.0798	1.77	2.975	3.6	0.0666	0.08	
3	1000 mVdc	0.349	-3.46	0.476	2.40	0.7	3.333	1.2	*
4	10 kΩ resistance	0.00327	-0.00171	0.00396	0.0098	0.3	0.0333	1.03	*
5	100 Ω resistance	0.895	-0.843	1.32	0.658	2.0	0.387	1.73	
6	1000 kΩ resistance	0.193	-1.13	0.758	0.747	0.2	3.33 k	1.00	*
7	1000 kHz frequency	0.00100	-0.0953	0.0485	1.52	0.5	0.0037 k	0.0011	
8	10 VRMS ac 400 Hz	0.370	-0.745	0.747	0.636	19.1	0.01	0.3	
9	10 VRMS ac 50 Hz	0.00453	0.421	0.00304	0.0867	2.6	0.01	0.3	
10	250 VRMS ac 50 Hz	2.93	-2.21	3.58	2.62	3.1	0.0666	0.08	
11	10 VRMS ac 10 kHz	0.00453	0.212	-0.0663	0.2088	6.3	0.01	0.3	
12	250 VRMS ac 10 kHz	2.93	-2.09	3.48	2.46	3.0	0.0666	0.08	
13	10 VPP ac 1.0 mHz	0.00453	0.01520	0.00453	0.0364	1.1	0.00108	3.25	*
14	Pulse train	0.925	-1.67	1.52	1.96	0.6	NA	NA	

a) a universal mean of the error μ (i.e., the true error or variance)

b) a machine effect on the error or the variance, C_k

c) a time delay effect on the error or the variance, B_j

d) a duration effect on the error or the variance, A_i

e) an interaction effect on the error or the variance, AB_{ij}, AC_{ik}, BC_{jk}, ABC_{ijk}

f) a random residual experimental error, $e_{l(ijk)}$.

Since the model is fixed, none of the effects can be determined absolutely. They can be measured only as differential deviations, i.e.:

a) the A_i effects as deviations from μ

b) the B_j effects as deviations from μ

c) the C_k effects as deviations from μ

d) the AB_{ij}, AC_{ik}, BC_{jk} as deviations from the $A_i + B_j$, $A_i + C_k$, and $B_j + C_k$ respectively

e) the ABC_{ijk} as deviations from $A_i + B_j + C_k$.

Because of the large volume of data and the number (56) of analyses required to cover all combinations of parameters and statistics, the analyses were computer programmed. The program used was obtained from Edwin Bartee and was compiled by J. A. Svestka. A printout of the computer program is shown in Volume II, Appendix D, including dimensions and correspondence statements and subroutines. Each combination of machines, time delays, and duration, of which there are 12 ($2 \times 3 \times 2$), had four responses in each cell. Since each hour of test was replicated five times, four responses, which represents 4 hours of testing, was the average value for that hour. Table VIII is a completely coded layout of the data input to the analysis of variance. The table identifies the factors in the mathematical model [Equation (21)], the test set numbers, the date, and the hour. To illustrate, the four response values for the cell representing a 1-hour test duration $\left(A_i\right)$, with the specified delay time $\left(B_j\right)$, on the ETG-3-0002 machine $\left(C_k\right)$ are data test sets 2, 12, 45, and 47 which are coded as 1, 2, 3, and 4 for input to the computerized analysis of variance. A computer printout of the input data to each analysis of variance and the ANOVA results are shown in Volume II, Appendix E.

TABLE VIII. COMPILATION OF HOURLY TESTS BY MACHINE, TAPE, DURATION, DAY AND HOUR

Machines (C_k)		ETG3-0002			ETG3-0005			Σ Duration
Tapes (B_j)		1	3	5	1	3	5	
Test Duration (A_i)	1 hr	1 2 2/14-14	9 3 2/15-11	17 1 2/13-10	25 8 2/19-15	33 19 3/01-14	41 9 2/20-9	24
		2 12 2/23-11	10 10 2/21-14	18 11 2/23-9	26 13 2/29-11	34 20 3/04-12	42 14 2/29-12	
		3 45 3/27-14	11 44 3/26-13	19 25 3/11-10	27 27 3/13-12	35 33 3/18-9	43 26 3/12-13	
		4 47 3/29-9	12 48 3/29-12	20 46 3/28-12	28 32 3/15-9	36 43 3/21-10	44 38 3/19-13	
	4 hrs	5 21 3/08-10	13 4 2/16-8	21 15 3/01-8	29 28 3/14-10	37 39 3/20-8	45 34 3/18-10	24
		6 22 3/08-11	14 5 2/16-9	22 16 3/01-9	30 29 3/14-11	38 40 3/20-9	46 35 3/18-11	
		7 23 3/08-12	15 6 2/16-10	23 17 3/01-10	31 30 3/14-12	39 41 3/20-10	47 36 3/18-12	
		8 24 3/08-13	16 7 2/16-11	24 18 3/01-11	32 31 3/14-13	40 42 3/20-11	48 37 3/18-13	
Σ Tapes		8	8	8	8	8	8	ΣΣ = 48
Σ Machine		24			24			

Code:

1 ← Code number for analysis of variance data input
2 ← Test set identification number
← Date of test in 1968
2/14-14 ← Hour of test (e.g. 1400 or 2 pm)

4. Test of Hypotheses

A statistical hypothesis is an assumption about the population being sampled. It usually consists of assigning a value to one or more parameters of the population. A test of a hypothesis is simply a rule by which a hypothesis is either accepted or rejected. The rule is usually based on sample or test statistics used to test the hypotheses. The critical region of a test statistic consists of all values of the test statistic where the decision is made to reject or accept the hypothesis. Since hypothesis testing is based on observed sample statistics computed on N observations, the decision is always subject to errors. If the hypothesis is really true and is rejected by the sample statistic, a Type I error is committed. The probability of a Type I error is designated as α. If the hypothesis is accepted when it is not true, i.e., if some alternate hypothesis is true, then a Type II error has been made. The probability of a Type II error is designated β.

The overall values of ϵ_b, ϵ_a, and ϵ_s for all parameters were tested with the following hypotheses for the sample error statistics:

$$Ho_1 \quad \epsilon_b = 0 \quad \alpha = 0.05$$

$$Ho_2 \quad \epsilon_a = 0 \quad \alpha = 0.05$$

$$Ho_3 \quad \epsilon_s = 0 \quad \alpha = 0.05 \, .$$

The alternate hypotheses are:

$$H1_1: \quad \epsilon_b \neq 0$$

$$H1_2: \quad \epsilon_a \neq 0$$

$$H1_3: \quad \epsilon_s \neq 0 \, .$$

The computed estimates of the errors are the differences in means. The hypothesis (Ho) is that the errors are not significantly different from zero. This is based on the assumption that the universe mean of the errors is zero. If the basic hypothesis is rejected [at the $(1 - \alpha)$ level of confidence] the alternate (H_1) is accepted. However, if the basic hypothesis is accepted at a specified level of confidence there is still a chance that an error of the first kind has been made. In testing hypotheses pertaining to the universe mean the procedure is simplest if the standard deviation of the universe is known. In this case the sample errors are treated as having a normal distribution with a mean equal to the universe mean; but, the universe standard deviation is unknown. Hence, the correct test statistic is the 't' statistic. The 't' statistic is used when the standard deviation must be estimated from the sample data. The 't' statistic is computed as follows:

$$'t'_{(\alpha)} = \frac{\overline{X} - \overline{X}'}{S/\sqrt{N}} \, , \tag{22}$$

where

$\overline{X}$ = calculated mean
$\overline{X}'$ = universe mean
N = sample size
α = risk = 0.05

and

$$S = \left[\frac{\sum_{i=1}^{N} \left(X_i - \overline{X} \right)^2}{N - 1} \right]^{1/2} \tag{23}$$

If the computed value of 't' is less than the table value, for N - 1 degrees of freedom and $\alpha = 0.05$, then the basic hypothesis is accepted and the alternate is rejected. A computer printout of the results is shown in Volume II, Appendix F. A summary is shown in Table IX in the row labeled "'t' Test of Overall Values."

In the ANOVA the basic hypotheses are as follows:

$$Ho{:}\ A_i = 0 \qquad \alpha = 0.10$$

$$Ho{:}\ B_j = 0 \qquad \alpha = 0.10$$

$$Ho{:}\ C_k = 0 \qquad \alpha = 0.10$$

and similarly for the second order (ABij, etc.) effects and the third order (ABCijk) effects. The alternate hypotheses are:

TABLE IX. COMPILATION OF RESULTS OF TESTS OF HYPOTHESES ('t' TEST AND 'F' TESTS)

	[illegible]
Test [illegible] A_i	
Delay [illegible] $B_j = 0$	
Duration X Delay $AB_{ij} = 0$	
Machine $C_k = 0$	
Duration X Machine $AC_{ik} = 0$	
Delay X X Machine $BC_{jk} = 0$	
Duration X Delay X Machine $ABC_{ijk} = 0$	

Indicates in 't' test that hypotheses: [illegible] 0 is rejected and alternate hypotheses: [illegible] $\neq 0$ is accepted.

Indicates in ANOVA 'F' test that hypotheses: $A_i = 0$; $B_j = 0$; $C_k = 0$; etc. are rejected and alternate hypotheses: $A_i \neq 0$; $B_j \neq 0$; $C_k \neq 0$; etc. are accepted.

$$H_1: A_i \neq 0$$

$$H_1: B_j \neq 0$$

$$H_1: C_k \neq 0$$

and similarly for the second and third order effects.

The objectives of basic hypothesis tests are to determine that the main factors of duration $\left(A_i\right)$, time delay $\left(B_j\right)$ and machines $\left(C_k\right)$ as well as interactions between main factors $\left(e.g., AB_{ij}\right)$ do not significantly effect the response variables, ϵ_b, ϵ_a, ϵ_s, and S^2 and they are essentially zero, i.e., there is no treatment effect. The test statistic is the F distribution which is the ratio of two independent chi-square distributions. This means that the F distribution is the ratio of the mean squares between treatments to the mean squares (MS) within treatments or mean square for error.

$$F_{(\alpha)} = \frac{\text{MS (treatments)}}{\text{MS (error)}} , \tag{24}$$

where

$$\text{MS (treatments)} = \frac{\text{sum of squares}}{\text{degrees of freedom}} \tag{25}$$

$$\text{MS (error)} = \frac{\text{sum of squares}}{\text{degrees of freedom}} . \tag{26}$$

If the computed value of F is equal to or greater than the table value of F for the set level of confidence (α) and the proper degrees of freedom then Ho: is rejected and Hl: is accepted. A summary of the results of the ANOVA is shown in Table IX. The ANOVA printouts for each statistic and parameter are shown in Appendix G of Volume II.

Section V. DISCUSSION

This analysis has been primarily concerned with the estimation of the measurement variance and secondarily with the bias error, measurement error and stimulus setting error. The estimation of these statistics has been done with a sample of data taken from two machines. The analyses have also estimated the effects of time delays and duration of tests on these statistics. The statistics were computed for fourteen different measurement parameters and scales performed by the ETG-3 test sets. These parameters are not all of the functions of the machines and represent, at best, only a partial test program. The results represent estimates for a small sample of machines, factors, and measurements, and, should be accepted as such. The results shown in Tables VII and IX will be discussed briefly. The discussion will be categorized by machine parameters which are probably of greater interest to machine users than the sample statistics.

1. 10-Volt dc Scale

In Table VII, the estimated value of the bias error is 0.000946, the accuracy error is -0.00197 and the stimulus setting error is 0.00106. The standard deviation is 0.02515. Table IX shows that the hypothesis that the bias error is not significant Ho: $\epsilon_b = 0$ is rejected. The hypotheses that $\epsilon_a = 0$ and $\epsilon_s = 0$ were accepted in the 't' tests. In the ANOVA portion of Table IX, it is shown that machine effects were significant; i.e., Ho: $C_k = 0$ was rejected for both statistics ϵ_b and ϵ_s. In Table VII, the 3S level of standard deviation expressed as percent full scale is 0.8 as compared with the specified (assumed 3S) value of 0.04 percent. This is a factor of 20 higher than specified assuming the specified values are correct.

2. 250-Volt dc Scale

For this parameter, the hypotheses that ϵ_b and $\epsilon_s = 0$ were rejected indicating that there are significant bias and stimulus errors on this parameter. The ANOVA reveals that the contributing factors to ϵ_b and ϵ_s are main effects of delay time and machines and interactions between delay times and machines. The estimated values as shown in Table VII are $\epsilon_b = 0.147$, $\epsilon_a = -0.0798$, $\epsilon_s = 1.77$ and S = 2.975. The 3S level expressed as a percent = 3.6 as compared with the specified value of 0.08 percent, a factor of about 45 greater.

3. 1000-Millivolt dc Scale

The sample of data reveals that all errors were significantly different from zero. The ANOVA indicates that all factors significantly contribute to bias error. Delay times, machines and delay time-machine interactions contribute to ϵ_a. All factors except durations contribute to ϵ_s and delay times and machines contribute to the variance, S^2. The estimated values in Table VII are $\epsilon_b = 0.349$, $\epsilon_a = -3.46$, $\epsilon_s = 0.476$, and $S = 2.40$. The 3S level is 0.7 percent as compared to the specified value of 1.2 percent.

The rest of the parameters are left to the reader. An overall look at Table VII shows that the machines have met the specified standard error of measurement on the 1000-millivolt dc scale, the 10-kilohm resistance scale, the 1000-kilohm resistance scale and the 10-volt PP ac voltage scale at 1-megahertz frequency. The 100-ohm resistance scale is close with 2 percent as compared with 1.73 percent specified. The 10-volt ac 400-hertz scale is not good with a 19 percent as compared with 0.3 percent specified. The worst seems to be the 1.0 megahertz frequency parameter with 0.5 percent as compared with 0.0011 percent specified a factor of about 500 greater.

Table IX shows the bias error was significant for all parameters and scales. Accuracy error was significant for all parameters except 10 volts dc, 250 volts dc, and 1000-kilohertz frequency, i.e., 11 out of 14. Stimulus setting error was significant on all parameters except 10 volts dc, 1000-kilohertz frequency and 10 volts ac, 50 hertz, i.e., 11 out of 14.

The ANOVA portion of Table IX shows that test duration $\left(A_i\right)$ was detected as a significant effect on the responses of ϵ_b, ϵ_a, ϵ_s, and S^2 in only 4 analyses out of 56. The main effect, delay time $\left(B_j\right)$ was found to be significant 21 times out of 56. The main effect of machines $\left(C_k\right)$ was found to be significant 46 times out of 56. The second order effect, AB_{ij}, 6 times; AC_{ik}, 13 times; BC_{jk}, 11 times; and ABC_{ijk}, 8 times out of 56. Machine effects far outweigh the other main and interaction effects.

The estimates of the standard deviation of measurement $\left(S_M\right)$ as obtained from this sample will be used as an estimate of σ_m. Previously, the use of σ_m in the determination of an accuracy ratio was discussed. Also, the use of the accuracy ratio, decision limits and $p(\alpha)$ and $p(\beta)$ in the determination of the statistical capability of the test sets for single tests. With the values of σ_m as obtained a few examples will be made with actual values of the weapon system components tolerances (σ_p). Also, decision limits will be assumed. The parameters 1000-millivolt dc scale, 10-kilohm resistance scale, 1000-kilohm resistance scale met the MIS-6000 specifications. These parameters and their values are used in comparison with missile specification values for components which would require testing on these scales. Also, one of the parameters which did not meet MIS-6000 specs, the 10-volt dc scale, is included for comparison purposes. The sample estimated values of the standard deviation (σ_m) for each of these parameters is divided by the component standard deviation (σ_p) to obtain an accuracy ratio (σ_m/σ_p). Decision limit ratios (δ/θ) of 1.00, 0.95, and 0.90 were assumed for this example.

The values for each weapon system are actual values obtained from the system specifications. The TOW system tolerances are based on 2σ levels and the SHILLELAGH system tolerances are based on 3σ levels. The values of σ for both systems were obtained from the system tolerances accordingly. The values of $p(\alpha)$ and $p(\beta)$ were obtained from the computer printouts used to plot Figures 2, 3, and 4. The results are shown in Table X.

The system components are as shown in Table XI. Similar analyses can be made for other parameters and scales and weapon systems.

Table X demonstrates the relationships between accuracy ratios, decision limits, undetected defects and false alarms for single tests for actual system components. It also demonstrates the trade-offs and compromises available to hold certain $p(\alpha)$ and $p(\beta)$. A component of Shillelagh has a 50-percent tolerance of measurement on the 1000 millivolt dc scale which gives an accuracy ratio of 0.014 or about 70 to 1 when compared with sample σ_m found for the ETG on that scale. Therefore, the probability is that practically no undetected defects will get by the tests; however, some false alarms varying from 1 out of 1000 to 1 out of 100 depending on the decision limits will probably occur. An accuracy ratio of 0.504 (about 2 to 1) shown for TOW component on 10-volt dc scale will result in about 1.5 units out of 100 at decision limit of 1.00 to 1.1 units out of 100 at decision limit of 0.90 passing with "undetected defects." Similarly, 5.6 and 6.6 "false alarms" will occur out of 100 tests.

TABLE X. SINGLE TEST $p(\alpha)$ AND $p(\beta)$ FOR RATIOS OF CALCULATED ETG σ_m AND SYSTEM SPECIFIED σ_p

	Parameter and Scale	10 Vdc			1000 MVdc			10 kΩ Resistance			1000 kΩ Resistance		
	ETG-3 STD Dev. σ_m	0.0252 V*			2.40 mV			0.0098 kΩ			0.747 kΩ		
TOW (2σ)	Sys. STD Dev. σ_p	0.050 V			5.00 mV			0.050 kΩ			5.[illegible] kΩ		
	Accuracy Ratio σ_m/σ_p	0.504 (1.97 to 1)			0.48 (2.08 to 1)			0.196 (5.62 to 1)			0.149 (6.72 to 1)		
	Decision Limit (γ/θ)	1.00	0.95	0.90	1.00	0.95	0.90	1.00	0.95	0.90	1.00	0.95	0.90
	$p(\alpha)\approx$**	0.015	0.013	0.011	0.014	0.011	0.008	0.003	0.0018	0.0009	0.0026	0.0015	0.0005
	$p(\beta)\approx$**	0.056	0.059	0.066	0.054	0.056	0.060	0.0166	0.019	0.026	0.011	0.016	0.021
Shillelagh (3σ)	Sys. STD Dev. σ_p	0.233			167.0			0.0334			3.334		
	Accuracy Ratio σ_m/σ_p	0.108 (9.26 to 1)			0.014 (71.4 to 1)			0.29 (3.45 to 1)			0.224 (4.46 to 1)		
	Decision Limit (γ/θ)	1.00	0.95	0.90	1.00	0.95	0.90	1.00	0.95	0.90	1.00	0.95	0.90
	$p(\alpha)$**	0.0017	0.0003	0.00007	0.0	0.0	0.0	0.006	0.004	0.0029	0.0047	0.0027	0.0030
	$p(\beta)$**	0.0078	0.012	0.019	0.001	0.007	0.01	0.026	0.03	0.35	0.018	0.022	0.02

* This parameter and scale does not meet requirements of ETG MIS-6000 specifications based on the sample of data obtained in this effort.

** Multiply $p(\alpha)$ and $p(\beta)$ values in table by 100 to obtain units per 100 of undetected defects and false alarms.

TABLE XI. IDENTIFICATION OF SYSTEM COMPONENTS USED IN SAMPLE COMPUTATIONS OF TABLE X

Parameter	10 Vdc	1000 mVdc	10-kΩ Resistance	1000-kΩ Resistance
TOW	Yaw CSG Card, Part No. 10191893	Excitation Generator and Self Test Card, Part No. 10191908	Pitch Command Signal Generator, Part No. 10225421	Pitch Command Signal Generator, Part No. 10225421
Shillelagh	SDC Program, Test Step 304, Part No. 1500	Transmitter Signal Analyzer, Test No. 247	19 Vdc Regulator Card, G.M. System Test Set, Test No. 41	Lamp Driver Card, G.M. System, Test No. 74, Part No. 1618

Section VI. SUMMARY

This analysis effort has resulted in estimates of bias error, accuracy error, stimuli setting error and overall standard deviation of measurement (ETG precision) for 13 different combinations of parameters and scales that the test set is capable of measuring. The sample statistics are, of course, subject to sampling errors, hidden or undetected effects and to human errors in the test program and the computations. The test program could be improved and the statistical computations expanded for a more complete and detailed analysis which may reduce some of the inherent errors in the test program and computations. However, the results found from this sample of data give good indications of the following conditions of the ETG-3 test sets.

a) There is significant bias error (ϵ_b) on all parameters and scales tested.

b) There is significant accuracy error (ϵ_a) on 11 out of 13 parameters and scales. (10 Vdc, 250 Vdc, and 1 MHz frequency accepted).

c) There is significant stimulus setting error (ϵ_s) on 10 out of 13 parameters and scales. (10 Vdc, 1 MHz frequency, 10 VRMS, 50 Hz ac excepted).

d) A significant effect of the main factor test duration (A_i) was detected on ϵ_b two out of 13 parameters and on s^2 two out of 13 parameters.

e) A significant effect of the main factor delay time (B_j) was detected on ϵ_b nine out of 13 parameters, on ϵ_a three out of 13, on ϵ_s seven out of 13, and on s^2 two out of 13.

f) A significant effect of the main factor machines (C_k) was detected on ϵ_b 12 out of 13, on ϵ_a ten out of 13, on ϵ_s 13 out of 13, and on s^2 11 out of 13.

g) The ETG is meeting the MIS 6000 specification for precision on only four parameters out of 13 considered or about 30 percent.

h) On the basis of the weapon system tolerances used in the sample comparisons, the ETG will not be able to hold the specified "across the board" 1 out of 100 probabilities for single checkouts except where the system tolerances are broad resulting in a high accuracy ratio. Multiple sequential test probabilities will be worse.

i) The present situation of incompatibility is between

   1) across-the-board 1 out of 100 probabilities

   2) MIS 6000 specified LCSS precision

   3) unreasonably close tolerances for weapon system UUT's

   4) English Language Test Program requirements which must be resolved in a comprehensive manner.

The pulse train is not included in the parameters as no suitable specified capability for the pulse train was found and also due to the excessive percentage of the total dropouts (60 percent) found on the pulse train portion of the test program.

The test set functional dependence cannot be commented on even though it was considered to have had only 56 command dropouts out of 24,240 commands; future tests could possibly obtain an estimate of functional dependence in light of the presently obtained estimates of precision. That is, the factor of 2 variation from an expected value used in this analysis could now be tightened up to obtain a better estimate of functional dependence. This would probably lower the variability and functional dependence.

A more complete test program is considered highly desirable. It should be performed on a continuing basis by the LCSS Project Office or the prime contractor. Tests should be conducted in the field to obtain estimates of the effects of other factors such as temperature, humidity, pressure, dust, etc., on the ETG-3 performance. Such a program to be properly executed would require a considerable level of effort; but, it would be a notable achievement and contribution in the area of evaluation of the test capability of complex test equipment.

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ESTIMATION OF LAUNCH

VEHICLE PARAMETERS FOR THE GIVEN MODEL

$$y = \theta_1 e^{\theta_2 t} \sin(\theta_3 t)$$

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SUMMARY

This paper presents analysis techniques of a launch vehicle of an antitank missile system. Since the development of a mathematical model from an analysis of components subsystems' responses must be checked against overall performance, it was assumed that the vehicle responded as a second order differential equation. The solution of this equation is fitted to the experimental data.

The parameters θ_1, θ_2, θ_3 are estimated for the model $y = \theta_1 e^{\theta_2} \sin(\theta_3 t)$ and given data points $\left(T_h, Y_h\right)$, h = 1,2 ... N. Several techniques of estimation are used. The following methods are included:

(1) Prony's Exponential Approximation

(2) Least Squares Polynomial — Taylor Series

(3) Differential Correction

(3) Gradient-Descent

(5) Modified Newton-Gauss.

A comparison of the techniques is presented and a "best" method of estimation is selected.

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BACKGROUND

During the investigation of a particular antitank missile system, it appeared that launcher motion may have an effect on the missile trajectory. This phenomenon results from the fact that the missile is command guided and the tracker is mounted rigidly on the launcher. The program requires that a method be developed for estimating the inertial restoring, and damping parameters of a vehicle from measured motion of the vehicle during the launch phase of the flight. It will then be possible to determine more readily if the vehicle meets the desired performance requirements and to trace any degradation which may occur after a number of hours of use in the field.

The mathematical model utilized for this investigation is a damped sine wave which comes from the solution of a second order differential equation. For the purpose of this investigation the mathematical model has been assumed to be correct and is of the form: $y = \theta_1 e^{\theta_2 t} \sin(\theta_3 t)$.

I. INTRODUCTION

If a scatter diagram in the x, y plane indicates that a straight line will not fit a set of points satisfactorily because of the nonlinearity of the relationship, it may be feasible to fit a simple curve that will yield a satisfactory fit. Since an investigator always strives to explain relationships as simply as possible, with the restriction that his explanation be consistent with previous knowledge, he will prefer to use a simple type of curve. It follows, therefore, that the type of curve to use will depend largely on the amount of theoretical information one has concerning the relationship and, also, convenience.

In the problem under study, it was assumed that a second order differential equation described the motion of the vehicle. The data recorded $D = \left\{ \left(t_h, y_h\right) | h = 1, 2, \ldots N \right\}$, which was the displacement, y_h, from equilibrium position at time, t_h. Thus, the solution to the differential equation,

$$Y = \theta_1 e^{\theta_2 t} \sin \theta_3 t \tag{1}$$

is to be fitted to D by determining

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$$

so as to minimize the residuals.

An attempt is made to solve directly the least squares problem resulting from (1) and D. Let

$$Q(\theta) = \sum_{h=1}^{N} \left(y_h - \theta_1 e^{\theta_2 t_h} \sin \theta_3 t_h \right)^2 . \tag{2}$$

By taking partials of $Q(\theta)$ with respect to each of the parameters and setting the results equal to zero, the following nonlinear system results:

$$\frac{\partial Q(\theta)}{\partial \theta_1} = \theta_1 \sum_{h=1}^{N} e^{2\theta_2 t_h} \sin^2\left(\theta_3 t_h\right) - \sum_{h=1}^{N} y_h e^{\theta_2 t_h} \sin\left(\theta_3 t_h\right) = 0$$

$$\frac{\partial Q(\theta)}{\partial \theta_2} = -\theta_1 \sum_{h=1}^{N} t_h e^{2\theta_2 t_h} \sin^2\left(\theta_3 t_h\right) + \sum_{h=1}^{N} y_h t_h e^{\theta_2 t_h} \sin\left(\theta_3 t_h\right) = 0$$

$$\frac{\partial Q(\theta)}{\partial \theta_3} = \theta_1 \sum_{h=1}^{N} t_h e^{2\theta_2 t_h} \sin\left(\theta_3 t_h\right) \cos\left(\theta_3 t_h\right) - \sum_{h=1}^{N} t_h y_h e^{\theta_2 t_h} \cos\left(\theta_3 t_h\right) = 0. \tag{3}$$

Although the above system can be reduced to a two-dimensional system, the two-dimensional system indicated that the direct approach is not feasible.

To take a slightly different approach, the Laplace transform of $Q(\theta)$ was taken. The partial derivatives of the resulting expression, $L\{Q[(\theta), t]\}$, with respect to each of the three parameters were taken and the results set equal to zero. The exponential and the sine function are suppressed in the resulting system, but the rational functions involved proved unmanageable and this approach was also abandoned.

Also attempts at simplification by taking the logarithm of (1) and (2) proved futile.

Since the direct approach to the problem would not yield a solution, several indirect methods were applied. These methods were:

(1) Proney's Exponential Approximation

(2) Taylor Series — Least Squares Polynomial

(3) Modified Newton-Gauss

(4) Steepest Descent — Method of Optimum Gradients

(5) Differential Correction.

Based on machine time, number of iterations and minimum of residuals, some methods gave better results than others. In Section III, advantages and disadvantages of each method are given and a "best" method is chosen.

II. DISCUSSION OF METHODS

In this section a detailed discussion of the four indirect methods is given.

A. Prony's Exponential Approximation

From the set D, four equally spaced (with respect to time) points, (t_1, y_1), (t_2, y_2), (t_3, y_3), (t_4, y_4) are chosen. Let

$$A_1 = \frac{\theta_1}{2i} \qquad A_2 = \frac{-\theta_1}{2i}$$

$$a_1 = \theta_2 + \theta_3 i \qquad a_2 = \theta_2 - \theta_3 i.$$

Then $\theta_1 e^{\theta_2 t} \sin \theta_3 t = A_1 e^{a_1 t} + A_2 e^{a_2 t}$. Prony's theory states that e^{a_1} and e^{a_2} satisfy the equation,

$$r^2 + C_1 r + C_2 = 0 \tag{4}$$

when (4) is the characteristic equation of the assumed difference equations

$$C_2y_1 + C_1y_2 + y_3 = 0$$

$$C_2y_2 + C_1y_3 + y_4 = 0 \quad . \tag{5}$$

Thus, from (4) and (5), a_1 and a_2 are determined. Next, the system

$$y_1 = A_1e^{a_1t_1} + A_2e^{a_2t_1}$$

$$y_2 = A_1e^{a_1t_2} + A_2e^{a_2t_2} \tag{6}$$

is solved to find A_1 and A_2. Since θ_1, θ_2, and θ_3 are given in terms of a_1, a_2, A_1, and A_2, θ_1, θ_2, and θ_3 can be found.

B. Taylor Series — Least Squares Polynomial

Motivated by the analysis of the scatter diagram of the data, a third degree polynomial $P(t) = a_0 + a_1t + a_2t^2 + a_3t^3$ was least squares fitted to the data. Thus, a_0, a_1, a_2, and a_3 are determined. Next, $y(t) = \theta_1e^{\theta_2t}\sin\theta_3t$ is expanded in a Taylor series about $t = 0$ and the series is truncated after the first four terms. From the least squares polynomial and the truncated Taylor series, coefficients of equal powers of t are equated yielding:

$$a_0 = 0$$

$$a_1 = \theta_3\theta_1$$

$$a_2 = \theta_3\theta_2\theta_1$$

$$a_3 = \frac{1}{6}(\theta_3\theta_2^2\theta_1 + 2\theta_2^2\theta_3\theta_1 - \theta_3^3\theta_1) \quad . \tag{7}$$

Since a_1, a_2, and a_3 are known, (7) can be solved to obtain θ_1, θ_2, and θ_3:

$$\theta_2 = \frac{-a_2}{a_1}$$

$$\theta_3 = \sqrt{\frac{3\theta_2^2 - 6a_3}{a_1}}$$

$$\theta_1 = \frac{a_1}{\theta_3} \quad . \tag{8}$$

Unfortunately, considerable error may be introduced by the least squares fit of the polynomial and by the truncation of the Taylor series after four terms. To improve our estimate of θ, the following technique is applied. From the set D, the set D^*CD is selected such that $(t^*, y^*) \epsilon D^*$ and $(t, y) \epsilon D$ and $t = t^* \rightarrow y^* \geq y$. The Taylor series — least squares polynomial method is applied to the set D^* and we obtain

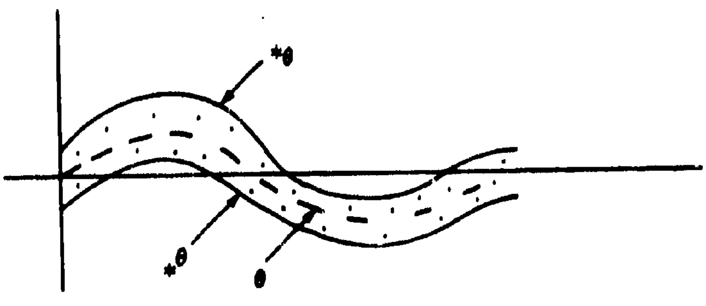
$$^*\theta = \begin{pmatrix} ^*\theta_1 \\ ^*\theta_2 \\ ^*\theta_3 \end{pmatrix}.$$

Now select $D_* CD$ such that $(t_*, t_*) \epsilon D_*$ and $(t, y) \epsilon D$ and $t = t_* \rightarrow y_* \geq y$. Again apply the Taylor series — least squares polynomial method and obtain

$$_*\theta = \begin{pmatrix} _*\theta_1 \\ _*\theta_2 \\ _*\theta_3 \end{pmatrix}.$$

Actually, what we want to do now is to "scan the interval between $^*\theta$ and $_*\theta$" to get the best estimate for θ.

Ideal Picture



Let

$$ {}_0\theta_i = \min \left\{ {}_*\theta_i, {}^*\theta_i \right\} \quad i = 1, 2 $$

$$ {}_0\theta_3 = \max \left\{ {}_*\theta_3, {}^*\theta_3 \right\} . $$

Note: These choices are made to help insure that we start with "lower" curve

Let

$$ \Delta_i = \frac{\left| {}_0\theta_i - {}^*\theta_i \right|}{K} . $$

K is some preassigned constant that determines step length.

Let

$$ \Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ -\Delta_3 \end{pmatrix} . $$

For notational purpose, let

$$ \theta_i = {}_j\theta_i , \qquad i = 1, 2, 3 , \qquad j = 0, 1, 2, \ldots K $$

$$ {}_j\theta = \begin{pmatrix} {}_j\theta_1 \\ {}_j\theta_2 \\ {}_j\theta_3 \end{pmatrix} \quad j = 0, 1, 2, \ldots, K . $$

Note: j is the number of the step in the "scanning" procedure.

Let

$$ Q\left({}_j\theta\right) = \sum_{h=1}^{N} \left[y_h - {}_j\theta_1 e^{{}_j\theta_2 t_h} \sin\left({}_j\theta_3\right) \right]^2 . $$

Now $Q\left({}_j\theta\right)$ is computed, beginning with $j = 0$

Let ${}_{j+1}\theta = {}_j\theta + \Delta$ and $Q\left({}_{j+1}\theta\right)$ is computed. If $Q\left({}_{j+1}\theta\right) > Q\left({}_j\theta\right)$, set

${}_{j+1}\theta = {}_j\theta + \Delta\left(\frac{1}{2}\right)^m$ beginning with $m = 1$.

Again we compute $Q\left({}_{j+1}\theta\right)$. If $Q\left({}_{j+1}\theta\right) > Q\left({}_j\theta\right)$, we put $m = 2$ and repeat the above (if necessary for $m = 3, 4, 5$). If $Q\left({}_{j+1}\theta\right) < Q(\theta)$ for $m = 1, 2, 3, 4$, we set ${}_{j+1}\theta = \theta_j + \Delta$ and repeat the original procedure. Then we use $m = 5$; even though we may not have $Q\left({}_{j+1}\theta\right) < Q\left({}_j\theta\right)$, we put ${}_{j+1}\theta = {}_j\theta + \Delta$ and repeat the original procedure.

It is desirable that θ and the residuals be printed out in each step of the search so as to gain some insight of the relationship between θ and the residuals.

C. Modified Newton-Gauss

In a complete description of the first iteration, we are trying to solve

$$Q(\theta) = \sum_{h=1}^{N} \left(y_h - \theta_1 e^{\theta_2 t} \sin \theta_3 t\right)^2 = 0$$

for

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}.$$

To this end, an initial estimate

$${}_0\theta = \begin{pmatrix} {}_0\theta_1 \\ {}_0\theta_2 \\ {}_0\theta_3 \end{pmatrix}$$

is made. Next $\theta_1 e^{\theta_2 t} \sin(\theta_3 t)$ is expanded in a Taylor series about ${}_0\theta$ and only the first two terms of the series are used. The truncated Taylor series replaces $\theta_1 e^{\theta_2 t} \sin \theta_3 t$ in $Q(\theta)$. Then, the partial derivatives with respect to θ_1, θ_2, and θ_3 of $Q(\theta)$ are computed and set equal to zero. This results in a system of equations that is linear in $\left(\theta_i - {}_0\theta_i\right) i = 1, 2, 3$. Thus, if the coefficient matrix of the system is invertible, we can solve for

$$\Delta\theta = \begin{pmatrix} \theta_1 - {}_0\theta_1 \\ \theta_2 - {}_0\theta_2 \\ \theta_3 - {}_0\theta_3 \end{pmatrix}.$$

There are cases in which the coefficient matrix may not be invertible. This is certainly true if the parameter surface is flat. Thus, a singular coefficient matrix, in this case, is indicative of a non-unique solution. In cases in which a unique solution does exist, the addition of second partials in computing the correction, $\Delta\theta$ helps to insure the non-singularity of the coefficient matrix.

Effectively, we have a new estimate for θ. (This is where the differential correction method, alias Newton's Method, begins a new iteration.)

Next, we optimize the magnitude of the correction $\Delta\theta$. The expression ${}_0\theta + v\Delta\theta \quad 0 \leq v \leq 1$ is considered.

The term Q is evaluated at ${}_0\theta$, ${}_0\theta + \frac{1}{2}\Delta\theta$, and ${}_0\theta + \Delta\theta$. A parabola is then passed through these three points and the minimum of the parabola is calculated in terms of v. Then $Q({}_0\theta + v\Delta\theta)$ is computed. The θ associated with the $\min\left\{Q({}_0\theta),\ Q\left({}_0\theta + \frac{1}{2}\Delta\theta\right),\ Q({}_0\theta + \Delta\theta),\ Q({}_0\theta + v\Delta\theta)\right\}$ is chosen and called ${}_1\theta$. If $Q({}_1\theta) < Q({}_0\theta)$ the <u>initial estimate</u> of θ, ${}_0\theta$ is replaced by ${}_1\theta$ and the entire procedure is repeated. If this is not the case, the domain of v is diminished $\left(\text{usually by } \frac{1}{2}\right)$ and the optimization of the magnitude of the correction $\Delta\theta$ is repeated the required number of times to produce $Q({}_1\theta) < Q({}_0\theta)$, or the domain of v is sufficiently small and we terminate the procedure.

NOTE: The reason that the check of the min $\left\{Q({}_0\theta), Q\left({}_0\theta + \frac{1}{2}\Delta\theta\right), Q({}_0\theta + \Delta\theta), Q({}_0\theta + v\Delta\theta)\right\}$ is considered in that the minimum of the parabola does not necessarily occur at that value of θ that will produce the minimum Q.

D. The Method of Steepest Descent — Optimum Gradient

Again we consider the expression

$$Q(\theta) = \sum_{h=1}^{N} \left[y_h - \theta_1 e^{\theta_2 t_h} \sin\left(\theta_3 t_h\right)\right]^2 . \tag{9}$$

An initial estimate is made; call it ${}_0\theta$. The gradient, ∇Q, of Q is computed at ${}_0\theta$. Since the gradient points in the direction of maximum increase of Q, the negative of the gradient will point in the direction of greatest decrease of the function. Now the gradient is normalized by dividing each component of the gradient by the maximum of the absolute values of the components.

We next optimize the step-length in the direction of steepest descent by considering the function

$$g(\alpha) = Q\left[{}_1\theta - \alpha\nabla Q\left({}_1\theta\right)\right] ; \tag{10}$$

we find that value of α that will make Q a minimum.

Now

$$g'(\alpha) = -\nabla Q\left({}_1\theta\right) \cdot \nabla Q\left[{}_1\theta - \alpha\nabla Q\left({}_1\theta\right)\right] . \tag{11}$$

By setting $g'(\alpha) = 0 \rightarrow \nabla Q\left({}_1\theta\right)$ and $\nabla Q\ {}_1\theta - \alpha\nabla Q\ {}_1\theta$ are orthogonal to each other for the value of α that makes Q a minimum.

We now compute the function values of g beginning at $g(0)$ and continuing [illegible] until the slope of g changes from negative to non-negative or until [illegible] $g(N-1)$, whichever comes first. Suppose we have the change of [illegible] the magnitude of Q) at N. Then define

$$b = N$$

$$a = N - 1.$$

Using [illegible] $(a, g(a))$, $(b, g(b))$, $(a, g'(a))$, $(b, g'(b))$ we will pass a cubic through these four points and determine the value of α that produces a minimum in the cubic. As in the parabolic interpolation in the Newton-Gauss program, a check is made to see if we are decreasing the magnitude of Q.

Once the "best" α is determined, we put

$$_{i+1}\theta = {}_{i}\theta - \alpha \nabla Q\left({}_{i}\theta\right)$$

and compute

$$Q\left({}_{i}\theta\right) \quad \text{and} \quad Q\left({}_{i+1}\theta\right);$$

when the changes in Q are negligible over four iterations, the process ceases.

E. Method of Differential Correction

$$Y = f(t, \theta_1, \theta_2, \theta_3). \tag{12}$$

We want this formula to be a good fit to the data $\left(t_h, y_h\right)$ $(h = 1, \ldots N)$. The residuals are given by

$$\left\{\begin{array}{l} R_1 = f(t_1,\theta_1,\theta_2,\theta_3) - Y_1 \\ R_2 = f(t_2,\theta_1,\theta_2,\theta_3) - Y_2 \\ \cdot \quad \cdot \cdot \cdot \cdot \cdot \quad \cdot \\ R_h = f\left(t_n,\theta_1,\theta_2,\theta_3\right) - Y_h \end{array}\right\} \tag{13}$$

where $y_h (h = 1, 2, \ldots n)$ are the given (observed) values from the original data.

Let ${}_0\theta_1$, ${}_0\theta_2$, ${}_0\theta_3$ be an initial guess. Now we need to correct this guess by some incremental amount, say α, β, γ such that

$$\left\{\begin{array}{l} \theta_1 = {}_0\theta_1 + \alpha \\ \theta_2 = {}_0\theta_2 + \beta \\ \theta_3 = {}_0\theta_3 + \gamma \end{array}\right\} \tag{14}$$

will yield a better fit to our data.

If we substitute the values of (14) into the residuals (13) and transpose the y_h, we have the following:

$$R_h + y_h = f\left(t_1,\ {}_0\theta_1 + \alpha,\ {}_0\theta_2 + \beta,\ {}_0\theta_3 + \gamma\right). \tag{15}$$

Expanding the right-hand side by Taylor's theorem we get

$$R_h + y_h = f\left(t_h, {}_0\theta_1, {}_0\theta_2, {}_0\theta_3\right) + \alpha\left(\frac{\partial f_h}{\partial \theta_1}\right)_0 + \beta\left(\frac{\partial f_h}{\partial \theta_2}\right)_0 + \gamma\left(\frac{\partial f_h}{\partial \theta_3}\right)_0$$

$$+ \text{ higher order terms in } \alpha,\ \beta,\ \gamma \tag{16}$$

where $\left(\frac{\partial f}{\partial y}\right)_0$ = the value of the partial derivative $\frac{\partial f}{\partial y}$ at

$$t = t_h, \quad \theta_1 = {}_0\theta_1, \quad \theta_2 = {}_0\theta_2, \quad \text{and} \quad \theta_3 = {}_0\theta_3 . \tag{17}$$

Our first approximation is obtained from

$$Y' = f(t, {}_0\theta_1, {}_0\theta_2, {}_0\theta_3)$$

so that we have

$$f\left(t_h, \theta_1, \theta_2, \theta_3\right) = Y'_h \ . \tag{18}$$

This can also be put into equation (16). Now let

$$r_h = Y'_h - Y_h \ . \tag{19}$$

Ignoring the higher order terms the residuals now have the form

$$R_h = \alpha\left(\frac{\partial f_h}{\partial \theta_1}\right)_0 + \beta\left(\frac{\partial f_h}{\partial \theta_2}\right)_0 + \gamma\left(\frac{\partial f_h}{\partial \theta_3}\right)_0 + r_h \tag{20}$$

which are linear in α, β, γ. Therefore, we may determine the corrections by the method of least squares.

III. RESULTS AND CONCLUSIONS

These five methods of estimating θ were used on data taken from actual test firings in which we had four test modes (a standard and three modifications). Plots of typical data are shown in Figures 1 through 4. The results of all five methods are shown in Tables I through IV.

Some advantages and disadvantages of each method are given in Table V. It turns out that the "best" method to use depends on the tools that one has on hand. For example, if one has to estimate the parameters by use of only paper and pencil, he naturally would chose Prony's method. If one had access to a small computer, he might chose the Taylor-least squares approach. Of the five methods discussed in this paper, the method of steepest descent, and the method of modified Newton-Gauss are the most accurate in terms of the smallest residuals.

TABLE I. STANDARD DATA

	θ_1	θ_2	θ_3	Residual
Prony's	6.9	-1.14	7.3	374
Taylor's modified	6.4	-2.8	7.6	94
Differential correction	Did not converge			
Steepest descent	6.5	-2.5	7.9	91
Modified Newton-Gauss	6.5	-2.5	7.9	91

TABLE II. MOD 1 DATA

	θ_1	θ_2	θ_3	Residual
Prony's	2.64	-1.24	10.3	171
Taylor's modified	2.4	-3.7	10.0	88
Differential correction	2.07	-1.85	12.3	68
Steepest descent	2.07	-1.85	12.3	68
Modified Newton-Gauss	2.09	-1.84	12.3	68

TABLE III. MOD 2 DATA

	θ_1	θ_2	θ_3	Residual
Prony's	5.3	-0.7	10.46	407
Taylor's modified	4.14	-2.5	8.92	177
Differential correction	Did not converge			
Steepest descent	4.7	-1.5	9.4	102
Modified Newton-Gauss	4.8	-1.5	9.4	101

TABLE IV. MOD 3 DATA

	θ_1	θ_2	θ_3	Residual
Prony's	8.6	-0.6	7.2	820
Taylor's modified	8.0	-2.1	7.7	328
Differential correction	Did not converge			
Steepest descent	8.5	-1.4	7.3	153
Modified Newton-Gauss	Relative min due to guess			1609

TABLE V. ADVANTAGES AND DISADVANTAGES

	Easily Hand-fit	Easily Adapted to Small Computer	Converges to Relative Minimum	Relative Accuracy	Sensitive to Guess
Prony's	yes	yes	N/A	4	N/A
Taylor's - least squares	no	yes	yes	3	N/A
Differential-correction	no	no	no	2	yes
Steepest descent	no	no	yes	1	no
Modified Newton-Gauss	no	no	yes	1	no

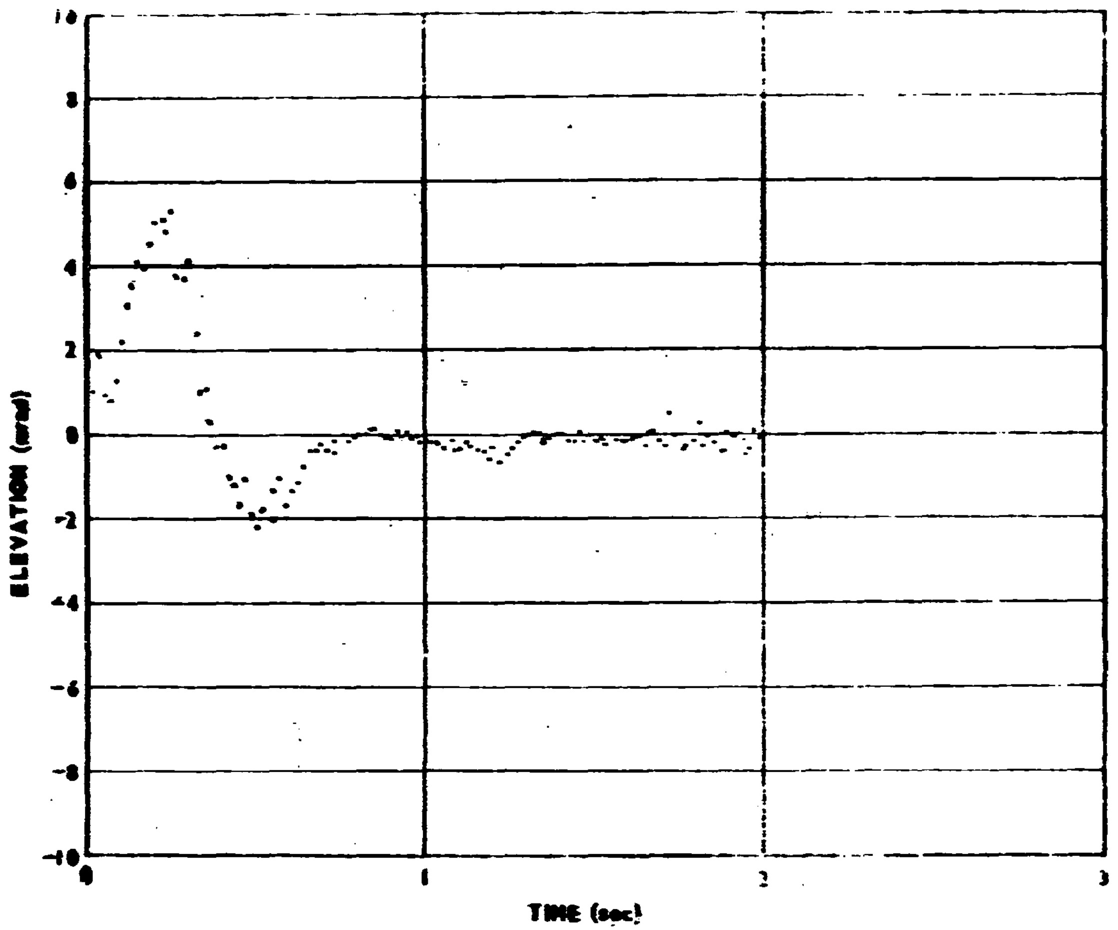


FIGURE 1. STANDARD VEHICLE

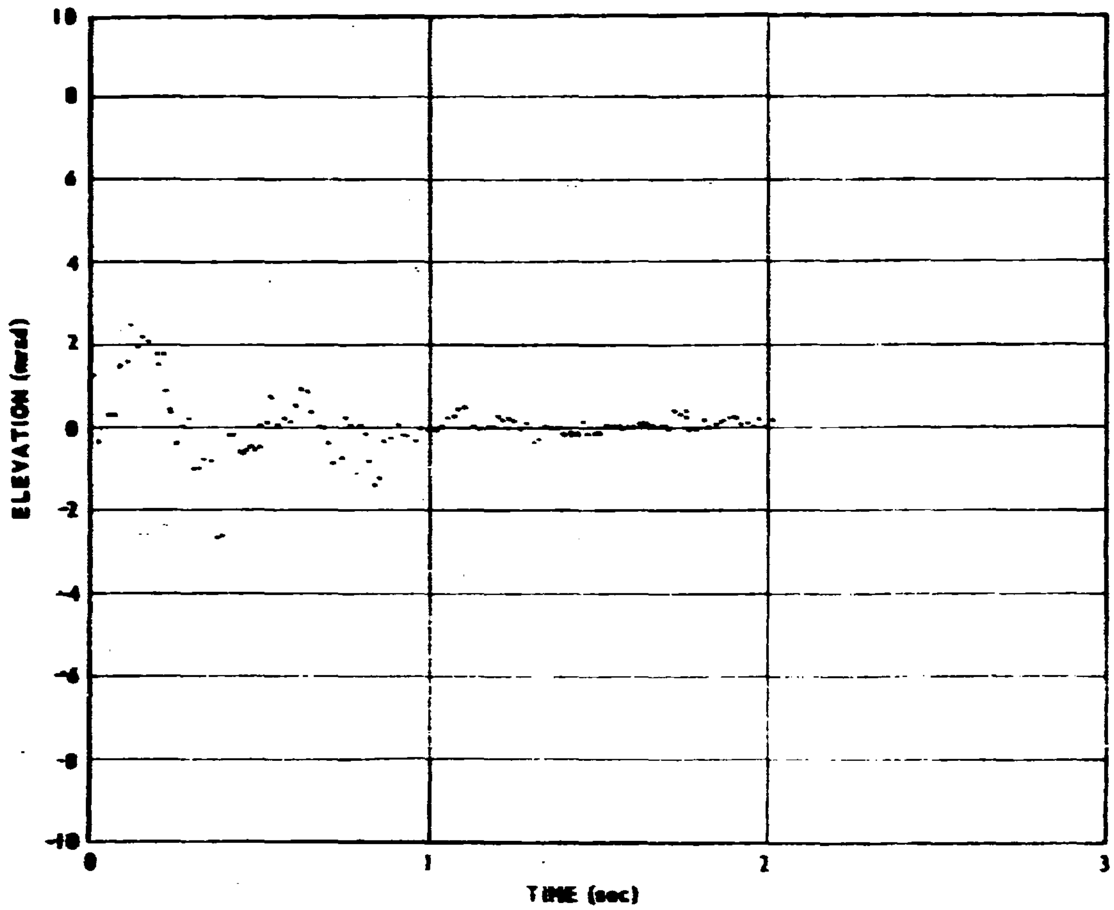


FIGURE 2. MODIFICATION 1

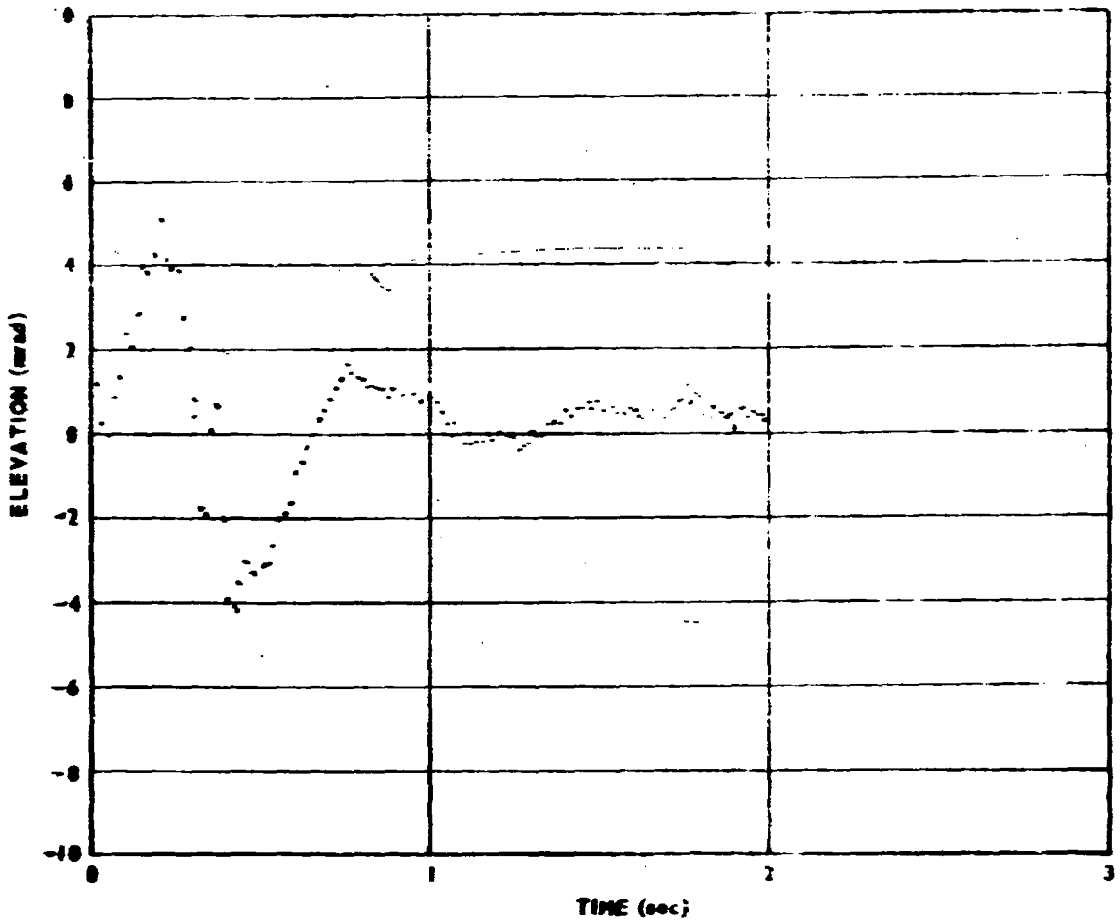


FIGURE 3. MODIFICATION 2

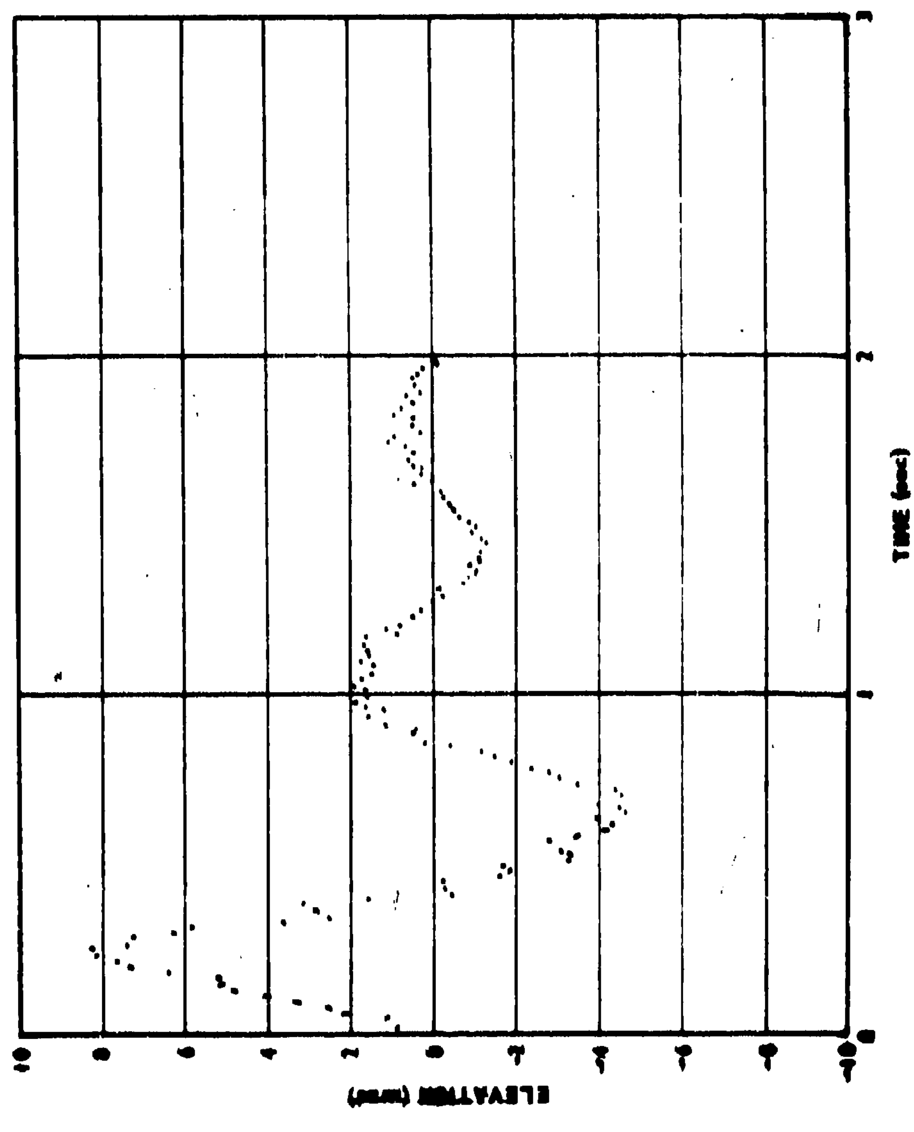


FIGURE 4. MODIFICATION 3

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A METHOD OF IMPROVING THE ESTIMATION OF VARIANCE

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1: Introduction.

Consider a situation where an experimenter has an unbiased estimator s_1^2 of the population variance σ_1^2 from a normal distribution. Let us further suppose that another independent unbiased estimator s_2^2 of the population variance is also available. The two estimators may have been based on samples taken at different times and places. Because of the circumstances it may be known that s_1^2 estimates σ_1^2 while s_2^2 may be estimating σ_2^2 ($\gtrless \sigma_1^2$). This shift in the variance, if any, may have taken place because of the time lapse between obtaining independent samples or it could be due to the shift in places. It is obvious that if $\sigma_1^2=\sigma_2^2$ then the two estimators can be pooled to obtain a "better estimator" of σ_1^2. On the other hand, if $\sigma_2^2\neq\sigma_1^2$ then one may estimate σ_1^2 by s_1^2 alone. A preliminary test of the hypothesis H_0: $\sigma_2^2=\sigma_1^2$ can be carried out by utilizing the $F(=s_2^2/s_1^2)$ statistic and if the hypothesis is rejected then one uses s_1^2 to estimate σ_1^2 otherwise a pooled estimator of σ_1^2 is obtained by pooling s_1^2 and s_2^2 appropriately. The estimator designated here as U has been obtained by following this approach, which is similar to the approach used by Bancroft [1].

Another method of estimating σ_1^2 is to use weights which are continuous functions of F. The estimator S described below is constructed in this manner and turns out to be more effective than the estimator U.

Let $x_{11}, x_{12}, \ldots, x_{1N}$ and $x_{21}, x_{22}, \ldots, x_{2N}$ be independent samples from normal populations with unknown variances σ_1^2 and σ_2^2 respectively, and let $k = \sigma_2^2/\sigma_1^2$. It is required to estimate σ_1^2. Define as usual

$$s_1^2 = \frac{1}{N-1}\sum_{j=1}^{N}(x_{1j}-\bar{x}_1)^2 \quad ; \quad s_2^2 = \frac{1}{N-1}\sum_{j=1}^{N}(x_{2j}-\bar{x}_2)^2$$

We consider an estimator T of weighted sums of s_1^2 and $\frac{s_1^2 + s_2^2}{2}$

of the form

$$T = \psi s_1^2 + (1-\psi) \frac{s_1^2 + s_2^2}{2} \tag{1}$$

where ψ is some function of F. The reason for considering T as an estimator of σ_1^2 is that s_1^2 is an estimator based only on the first sample and

$$\frac{s_1^2 + s_2^2}{2}$$

is an estimator based on combining the two samples when $\sigma_1^2 = \sigma_2^2$. As an estimator of σ_1^2 we consider a subclass S of T which reduces to the form

$$S = s_1^2 \left[\frac{\alpha_1 - \alpha_2 F + \alpha_3 F^2}{\beta_1 - \beta_2 F + \beta_3 F^2} \right] \tag{2}$$

where the constants of α_i and β_i are given below. This estimator has also been considered elsewhere (Mehta and Gurland [6] but we include it here for comparison.

The estimator U mentioned above is also a subclass of T with the weight function ψ given by

$$\psi = \begin{cases} 0 & \text{for } 1/F_o < F < F_o \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

The constant k_0 is determined by the level of significance of the preliminary test for equality of variances.

The behaviour of S and U will be investigated in regard to expected mean square and relative bias.

2. The estimator U

The estimator U is of the form T with the weight function ψ defined by (3) above. It is based on the random outcome of a preliminary test of whether $\sigma_1^2 = \sigma_2^2$. If the preliminary test rejects equality of variances then s_1^2 is employed as the estimator; but if it does not reject equality of variances then the average of s_1^2 and s_2^2 is used as the estimator of σ_1^2. This estimator is similar to one used by Bancroft [1] except that we use a two-sided test for equality of variances whereas he uses a one-sided test.

First we consider the expected mean square of U and compare it with that of s_1^2 which utilizes only the first sample. Since s_1^2 is unbiased we define the efficiency of U as

$$\text{Eff } U = \frac{\text{Var } s_1^2}{E(U-\sigma_1^2)^2} \tag{4}$$

Subsequently we shall also consider the relative bias of U given by

$$\frac{E(U) - \sigma_1^2}{\sigma_1^2} \tag{5}$$

In order to obtain expressions for the efficiency and bias of U we need to evaluate the first two moments of U.

It can be shown by straight-forward integration that

$$E(U) = \sigma_1^2 \left[1+\frac{1}{2}\left(I_{y_0}\left(\frac{N+1}{2},\frac{N-1}{2}\right)-I_{x_0}\left(\frac{N+1}{2},\frac{N-1}{2}\right)\right)+\frac{k}{2}\left(I_{x_0}\left(\frac{N-1}{2},\frac{N+1}{2}\right)-I_{y_0}\left(\frac{N-1}{2},\frac{N+1}{2}\right)\right)\right] \quad (21)$$

$$x_0 = \frac{kF_0}{1+kF_0} \quad ; \quad y_0 = \frac{F_0}{1+F_0}$$

$$B(a,b)\, I_{t_0}(a,b) = \int_0^{t_0} t^{a-1}(1-t)^{b-1}\,dt \quad (22)$$

[illegible]ilarly the [illegible] moment of U is given by

$$E(U^2) = \frac{\sigma_1^4}{4(N-1)^2}\left[(N^2-1)\left\{I_{x_0}\left(\frac{N+3}{2},\frac{N-1}{2}\right)-I_{y_0}\left(\frac{N+3}{2},\frac{N-1}{2}\right)\right\} + 2(N-1)^2 k\left\{I_{x_0}\left(\frac{N+1}{2},\frac{N+1}{2}\right)-I_{y_0}\left(\frac{N+1}{2},\frac{N+1}{2}\right)\right\} + (N^2-1)k^2\left\{I_{x_0}\left(\frac{N-1}{2},\frac{N+3}{2}\right)-I_{y_0}\left(\frac{N-1}{2},\frac{N+3}{2}\right)\right\}\right] + \sigma_1^4\left[\frac{N+1}{N-1}\left\{I_{y_0}\left(\frac{N+3}{2},\frac{N-1}{2}\right)-I_{x_0}\left(\frac{N-1}{2},\frac{N+1}{2}\right)+1\right\}\right]$$

where x_0, y_0 and $I_{t_0}(a,b)$ have been defined in (22). Thus the efficiency of U can be written as follows

$$\text{Eff } U = \frac{\text{Var}(s_1^2)}{E(U^2)-[E(U)]^2+[\text{Bias } U]^2}$$

where

$$\text{Bias } U = E(U) - \sigma_1^2$$

and $E(U)$, $E(U^2)$ are given above.

3. Some computed values of Efficiency and Relative Bias of U.

In Tables 1-5 are given calculated values of the efficiency of U_2 for sample sizes 3,5,7,9,11, for the range $0.1 \leq k \leq 10.0$, and for values of the constant F_0 which correspond to 1%,5%,10%,20% and 50% levels of significance. Examination of these tables reveals that there is a gain of efficiency for some values of k and a loss for other values of k. Furthermore the extent of these gains or losses depends on sample size. For $k \leq 1$ there is generally a gain in efficiency but the magnitude of this gain decreases with increase in sample size. As a matter of fact for small values of k, e.g. around 0.1, there is even a loss of efficiency as N becomes larger.

As far as values of $k > 1$ are concerned there is, without exception, a general loss of efficiency as manifested for all the sample sizes considered. The relationship of this loss with the values of k and N is more complicated than for the case $k \leq 1$ considered above. For some values of $k > 1$, e.g. $k = 2$, the loss in efficiency becomes more pronounced as N increases while for other values of $k > 1$, e.g. $k = 10$, it becomes less pronounced.

For all values of k and N considered the relation of efficiency to the value of the constant F_0 follows a definite pattern. Whatever be the sample size, the efficiency for values of $k > 1$ increases with decrease of F_0 (or equivalently with increase of level of significance of the preliminary test). On the other hand for $k \leq 1$ the trend is reversed, that is to say, the efficiency decreases with increase of the level of significance of the preliminary test.

On the basis of efficiency only, one would be tempted to select an estimator of this class with a low level of significance of the preliminary test involved. However, when this level is low the relative bias is high. In Table 10 is presented the relative bias of U for sample sizes 3,5,7,9 and 11. The value of F_0, for each sample size, corresponds to a 20% level of significance of the preliminary test of the hypothesis H_0: $k = 1$. Other levels of significance for the preliminary test are possible, of course, and as a matter of fact in Tables 1-5 we have already discussed the efficiency of U for values of F_0 corresponding to levels 1%,5%,10%,20% and 50%. Now for levels greater than 20% the relative bias will be smaller but at the same time the gain in efficiency will also be smaller. On the other hand for levels less than 20% the efficiency will be higher but the relative bias will also be higher. At this particular level, namely, 20%, the relative bias is reasonably well controlled for $0.1 \leq k \leq 1.0$ and at the same time there are gains of efficiency, at least in a subset of this range. On refering to Table 10 we note that the maximum relative bias in this range of k for sample size $N = 11$ is 6% and for $N = 3$ it is 14%.

4. The estimator S

In the general estimator T defined by (2) let us regard ψ for the moment as a constant, and minimize the expected mean square error of T with respect to ψ. The minimum is reached when

$$\psi = \frac{N-3-2k(N-1)+k^2(N+1)}{(N+1)-2k(N-1)+k^2(N+1)}$$

If we substitute this value for ψ in the estimator T and replace k, which is, of course, unknown, by $c+dF$, where c and d are arbitrary constants, we obtain the estimator S given by (2), where

$$\alpha_1 = (N-1)(1-2c) + (N+1)c^2 \qquad \beta_1 = (N+1)(1+c^2)-2(N-1)c$$

$$\alpha_2 = 2\{(N-1)d-(N+1)cd-1\} \qquad \beta_2 = 2d\{(N-1) + (N+1)c\}$$

$$\alpha_3 = (N+1)d^2 \qquad \beta_3 = (N+1)d^2$$

The relevant underlying details involved in obtaining the above estimator S are outlined in the paper by Mehta and Gurland [4].

Estimation of k by a simple function such as $c + dF$ has been applied similarly in other contexts (cf.[2],[3]). The constants c and d must be appropriately chosen, and the results of certain choices will appear in Tables 6, 7, 8, 9, 11, 12 considered below.

As in the case of U we require the first two moments of S in oder to evaluate its efficiency and relative bias. For odd values of the sample size N these moments can be expressed as a finite series of integrals which can be evaluated by reduction. The precise form of these moments appears in the work by Mehta and Gurland [6] cited above.

5. Some computed values of Efficiency and Relative Bias of S

In examining the behaviour of the estimator S we employ the same criteria of efficiency and relative bias defined above as in (4), (5), for the estimator U. The behaviour of the estimator S has been considered previously in [6], but for convenience of making comparisons with the estimator U we sketch these results here briefly. In table 6 the efficiency is shown when values of the constants c and d have been selected to emphasize the range [illegible] $\leq k \leq$ [illegible]. Except for very small sample sizes, such as $N = 3$, the gain in efficiency is not substantial. In fact, a loss in efficency begins to occur for sample size 7 and for larger values of k.

In Table 7 the efficiency is shown corresponding to constants c and d which emphasize the range $0.1 \leq k \leq 1.0$. There is a considerable gain in efficiency for this range, especially for small sample size. This gain, however, is accompanied by a large relative bias.

In Table 8 the efficiency is given for estimator S where the constants c and d are chosen so that the relative bias remains numerically below 10% for the range $0.1 \leq k \leq 10.0$. Table 9 indicates the efficiency that results when constants c and d are chose to hold the relative bias numerically below 5%. In this case the gain in efficiency is slight, especially for larger sample sizes.

In Tables 11 and 12 are presented the relative bias of the estimators in the class S for which the efficiency has been discussed in Tables 8 and 9. From Table 12 it is evident that the relative bias of the estimators considered there is much smaller. In fact for $N = 11$ the maximum relative bias [illegible] and for $N = 3$ it is 5.8%. The gain in efficiency, however, as indicated in Table 9 is only very slight for the range $0.1 \leq k \leq 1.0$; [illegible] slight loss of efficiency for the range [illegible] $\leq$ [illegible]. From the point of view of efficiency this estimator is not so attractive; however, it is of interest if one is mainly interested in controlling the bias.

It is evident from Table 11 that the maximum relative bias for the whole range $0.1 \le k \le 10$ and for all the sample sizes considered is 16%. It is also evident that for many of these values of k for the sample sizes considered this relative bias is very small. This control of relative bias together with the gain in efficiency discussed previously would indicate that this member of the class S merits consideration as an estimator of σ_1^2.

5. Comparison of the behaviour of estimators U and S

In comparing the behaviour of S and U it is necessary to keep in mind that the parameter k can assume values greater than or less than one. It is evident that we can find members of the class S which in most of the range $0.1 \le k \le 10.0$ are more efficient than members of U. For example on comparing the estimator S in Table 8 corresponding to N = 9 and the estimator U in Table 4 corresponding to a preliminary test at a 20% level we observe that for all values of $k > 1$ the efficiencies of S are very much higher than those of U. For the range $0.1 \le k \le 1.0$ the efficiencies of S exceed those of U except for values of k in the subset $0.7 \le k \le 1.0$ in which subset the efficiencies of U are only slightly greater than those of S. The comparative behaviour of S and U for other sample sizes considered follows a similar pattern. Generally speaking therefore in the whole range $0.1 \le k \le 10.0$ the estimator S appears preferable as far as efficiency is concerned.

Let us now consider the relative bias of these estimators. Values of the relative bias of S and U are given in Tables 11 and 10 respectively. On comparing these biases corresponding to sample size N = 9, for example, the relative bias of S in the range $1.0 < k \le 10.0$ is very much less than that of U, while for k in the range $0.1 \le k \le 1.0$ the relative bias of S remains less than that of U except in the subset $0.1 \leq k \leq 0.3$. The comparative behaviour of the relative bias [illegible] follows a similar pattern, namely, for values [illegible] the relative bias of S is much less than that of U while for values of $k \geq 1$ the relative bias of S remains less than that of U except in the subset [illegible].

[illegible] U [illegible] as in Table [illegible] to keep the relative bias below a maximum of [illegible] and U is chosen with a preliminary test at [illegible] level the results indicate that if we regard efficiency and relative bias over the whole range [illegible] the estimator S is preferable.

Other comparisons between S and U are also possible by selecting different members of these classes of estimators; however, we believe, the basis of comparison considered here is the most generous towards U. If, for example, we consider the class U with a level of 50% for the preliminary test, its relative bias is improved but the efficiencies, with very slight exceptions, are all less than those of S. On the other hand if we consider the class U with a preliminary test at a low level, for example 1% the efficiencies for $k < 1$ generally exceed those of S; however, the disadvantages of such a U would be overwhelming because its relative bias is prohibitively large and its loss of efficiency for $k > 1$ is terrible.

Other estimators S besides those given in Table 6 might also have been considered for the comparison. In Tables 6 and 7, for example, estimators are presented which emphasize the ranges $1.0 < k \leq 10.0$ and $0.1 \leq k \leq 1.0$ respectively. On the other hand the estimators considered in Tables 9 and 12 control the relative bias within a maximum of 5%. For the whole range $0.1 \leq k \leq 10.0$, however, the estimator considered in Table 8 is worth recommending because the relative bias is reasonably well controlled and there are noticeable gains of efficiency.

Examples:

(1) The following example illustrates how to obtain S and U in a practical situation. For this we have drawn a sample of size 7 from N(3,1) and named it as the first sample. The unbiased estimate of $\sigma_1^2 = 1$ as given by s_1^2 from this sample is $s_1^2 = 1.653$. We draw another sample of size 7 from N(5,0.36) and designate it as the second sample. The unbiased estimate of $\sigma_2^2 = 0.36$ as given by s_2^2 is $s_2^2 = 0.448$. Consequently $F = s_2^2/s_1^2 = 0.271$ and the hypothesis that $k = 1$ is rejected at the 20% level of significance. Thus $U = 1.653$. On the other hand we obtain

$S = 1.527$ if we restrict the relative bias to be less than 10%.
$ = 1.623$ if we restrict the relative bias to be less than 5%.
$ = 1.011$ if we use the estimator S which emphasizes the range $0.1 \le k \le 1.0$.
$ = 0.969$ if we use the estimator S which emphasizes the range $1.0 < k \le 10.0$.

In all the cases we note that S is nearer the true value of unity than U.

(2) The example considered here differs from that of (1) in that here the value of the ratio k is greater than 1. Suppose now we have a second sample also of size 7 from N(4,4). The unbiased estimate of $\sigma_2^2 = 4$ is $s_2^2 = 4.305$. The value of $F = s_2^2/s_1^2$ is now 2.603 which is not significant at 20% level of significance and consequently the null hypothesis that $k = 1$ is not rejected. Therefore in this case the estimate $U = \dfrac{1.653 + 4.305}{2} = 2.979$.

On the other hand we obtain

$S = 1.660$ if we restrict the relative bias to be less than 10%.
$ = 1.660$ if we restrict the relative bias to be less than 5%.
$ = 1.881$ if we use the estimator S which emphasizes the range $0.1 \le k \le 1.0$.
$ = 1.663$ if we use the estimator S which emphasizes the range $1.0 < k \le 10.0$.

In all the cases we note that the estimator S is nearer the true value than U.

Table 1: Efficiency of B_2 for N=3

k	F_o=199 1%	F_o=[illegible] 5%	F_o=19 10%	F_o=9.0 20%	F_o=3.0 50%
10	0.025	[illegible]	[illegible]	0.171	[illegible]
9	0.031	[illegible]	[illegible]	0.181	[illegible]
8	0.039	[illegible]	[illegible]	0.194	[illegible]
7	0.051	[illegible]	[illegible]	0.213	[illegible]
6	[illegible]	[illegible]	[illegible]	0.241	[illegible]
5	0.102	[illegible]	[illegible]	[illegible]	[illegible]
4	0.162	[illegible]	[illegible]	[illegible]	[illegible]
3	0.295	[illegible]	[illegible]	[illegible]	[illegible]
2	0.673	[illegible]	[illegible]	[illegible]	[illegible]
1	1.942	[illegible]	[illegible]	1.344	[illegible]
0.9	2.122	[illegible]	[illegible]	1.384	[illegible]
0.8	2.283	[illegible]	[illegible]	1.404	[illegible]
0.7	2.410	[illegible]	[illegible]	1.412	[illegible]
0.6	2.485	[illegible]	[illegible]	1.390	[illegible]
0.5	2.493	2.017	[illegible]	1.343	[illegible]
0.4	2.430	[illegible]	1.576	1.263	[illegible]
0.3	2.294	[illegible]	[illegible]	1.174	[illegible]
0.2	2.086	1.511	[illegible]	1.067	[illegible]
0.1	1.764	[illegible]	[illegible]	0.975	[illegible]

Table 2: Efficiency of [illegible]

k	F_o=23.1[illegible] 1%	F_o=9.60 5%	F_o=6.[illegible] 10%	[illegible]	[illegible]
10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Table 3: Efficiency of B_2 for N=7

k	F_o=11.07 1%	F_o=5.82 5%	F_o=4.28 10%	F_o=3.05 20%	F_o=1.78 50%
10	0.041	0.147	0.298	0.584	0.94[illegible]
9	0.044	0.142	0.281	0.553	0.97[illegible]
8	0.048	0.140	0.267	0.522	0.927
7	0.054	0.141	0.257	0.494	0.913
6	0.065	0.148	0.254	0.471	0.897
5	0.083	0.163	0.261	0.459	0.881
4	0.120	0.199	0.289	0.467	0.869
3	0.210	0.286	0.369	0.525	0.874
2	0.516	0.569	0.627	0.729	0.930
1	1.898	1.645	1.469	1.257	1.046
0.9	2.039	1.724	1.506	1.277	1.048
0.8	2.102	1.736	1.502	1.268	1.046
0.7	2.063	1.670	1.442	1.228	1.037
0.6	1.925	1.534	1.335	1.161	1.024
0.5	1.713	1.455	[illegible]	1.093	1.008
0.4	1.460	1.163	1.062	1.004	0.994
0.3	1.197	0.990	0.948	0.946	0.996
0.2	0.953	0.873	0.990	0.932	0.998
0.1	0.802	0.884	0.934	0.973	0.997

Table 4: Efficiency of [illegible] for N=9

k	F_o=7.6 1%	F_o=4.43 5%	F_o=3.[illegible] 10%	F_o=2.5[illegible] 20%	[illegible]
10	0.066	0.256	0.475	0.762	[illegible]
9	0.064	0.234	0.436	0.722	[illegible]
8	0.065	0.215	0.395	0.677	[illegible]
7	0.067	0.206	0.360	[illegible]	[illegible]
6	0.074	0.192	0.33[illegible]	0.582	[illegible]
5	0.087	0.193	0.315	[illegible]	[illegible]
4	0.116	0.213	0.31[illegible]	0.517	[illegible]
3	0.191	0.260	0.372	0.539	[illegible]
2	0.465	0.579	0.598	0.709	[illegible]
1	1.897	1.626	1.448	1.247	[illegible]
0.9	2.014	1.692	[illegible]	1.262	[illegible]
0.8	2.035	1.674	1.457	1.243	[illegible]
0.7	1.935	1.569	1.37[illegible]	1.190	[illegible]
0.6	1.73[illegible]	1.322	1.242	1.113	[illegible]
0.5	1.48[illegible]	1.204	[illegible]	1.02[illegible]	[illegible]
0.4	1.221	1.022	0.972	0.958	[illegible]
0.3	0.982	0.885	0.888	0.922	[illegible]
0.2	0.804	0.930	0.878	0.935	[illegible]
0.1	0.792	0.917	0.959	0.986	[illegible]

Table 5: Efficiency of [illegible] for N=11

k	$\Gamma_{o_{1\%}}$=[illegible]	$\Gamma_{o_{5\%}}$=3.[illegible]	$\Gamma_{o_{10\%}}$=[illegible]	$\Gamma_{o_{20\%}}$=2.32	$\Gamma_{o_{50\%}}$=1.5[illegible]
10	0.198	[illegible]	[illegible]	0.881	0.991
9	0.099	0.361	0.674	0.947	0.988
8	0.092	0.317	0.547	0.894	0.983
7	0.088	0.279	[illegible]	0.751	0.975
6	[illegible]	0.24[illegible]	[illegible]	[illegible]	0.962
5	[illegible]	0.231	0.386	[illegible]	0.945
4	0.117	0.232	0.356	0.572	0.921
3	0.179	0.279	0.381	[illegible].560	0.900
2	0.426	0.437	0.567	0.691	0.922
1	1.680	1.615	1.433	1.239	1.042
0.9	1.994	1.671	[illegible]	1.250	1.043
0.8	1.976	1.627	1.422	1.222	1.037
0.7	1.826	1.488	1.315	1.158	1.025
0.6	1.585	1.296	1.171	1.074	1.008
0.5	1.313	1.097	1.028	0.990	0.992
0.4	1.056	0.932	0.915	0.930	0.983
0.3	0.852	0.829	0.860	0.913	0.984
0.2	0.737	0.825	0.687	0.946	0.993
0.1	0.831	0.949	0.978	0.993	0.999

Table 6: Efficiency of [illegible]

k	N=3 c=-.2[illegible] d=3.5	[illegible]	[illegible]	[illegible]	[illegible]
10	1.106	[illegible]	[illegible]	[illegible]	[illegible]
9	1.119	[illegible]	[illegible]	[illegible]	[illegible]
8	1.1[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7	1.157	[illegible]	[illegible].603	[illegible]	[illegible]
6	1.1[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5	1.225	[illegible]	[illegible].020	[illegible]	[illegible]
4	1.283	[illegible]	1.029	[illegible]	[illegible]
3	1.377	[illegible]	[illegible].072	[illegible]	[illegible]
2	1.551	[illegible]	[illegible]	[illegible]	[illegible]
1	1.931	[illegible]	[illegible]	[illegible]	[illegible]
0.9	1.988	[illegible]	[illegible]	[illegible]	[illegible]
0.8	2.047	1.721	[illegible]	1.117	[illegible]
0.7	2.106	[illegible]	[illegible]	[illegible]	[illegible]
0.6	2.160	[illegible]	[illegible].671	[illegible]	[illegible]
0.5	2.200	[illegible]	[illegible]	1.173	[illegible]
0.4	2.213	[illegible]	[illegible]	[illegible]	[illegible]
0.3	2.182	[illegible]	[illegible]	[illegible]	[illegible]
0.2	2.087	[illegible]	[illegible]	[illegible]	[illegible]
0.1	1.932	1.37[illegible]	[illegible]	1.26[illegible]	[illegible]

Table 7: Efficiency of S (Emphasizes range -1≤k≤1.0)

k	N=3 c=0.49 d=0.49	N=5 c=0.49 d=0.49	N=7 c=0.49 d=0.49	N=9 c=0.49 d=0.49	N=11 c=0.49 d=0.49
10	0.760	0.654	0.646	0.664	0.683
9	0.777	0.657	0.641	0.652	0.671
8	0.800	0.664	0.637	0.642	0.655
7	0.831	0.677	0.638	0.633	0.643
6	0.875	0.698	0.646	0.630	0.628
5	0.939	0.736	0.666	0.637	0.627
4	1.038	0.804	0.713	0.667	0.642
3	1.206	0.936	[illegible]	[illegible]	0.709
2	1.531	1.234	[illegible]	1.013	0.949
1	2.221	1.948	[illegible]	1.822	1.798
0.9	2.311	2.027	[illegible]	1.892	1.865
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	2.1[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	2.1[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	2.454	1.76[illegible]	[illegible]	[illegible]	1.151
[illegible]	[illegible]	[illegible]	[illegible]	1.111	[illegible]
[illegible]	[illegible]	1.364	[illegible]	[illegible]	[illegible]

Table 8: Effic[illegible] of [illegible] relative [illegible]

k	N=3 c=-2.0 d=3.5	N=5 c=-2.0 d=[illegible]	N=7 c=-2.0 d=[illegible]	N=9 c=[illegible] d=[illegible]	N=[illegible] c=[illegible] d=[illegible]
10	1.052	[illegible]	[illegible]	[illegible]	[illegible]
9	1.059	[illegible]	[illegible]	[illegible]	[illegible]
8	1.067	[illegible]	[illegible]	[illegible]	[illegible]
7	1.078	[illegible]	[illegible]	[illegible]	[illegible]
6	1.092	[illegible]	[illegible]	[illegible]	[illegible]
5	1.109	[illegible]	[illegible]	[illegible]	[illegible]
4	1.133	[illegible]	[illegible]	[illegible]	[illegible]
3	1.166	[illegible]	[illegible]	[illegible]	[illegible]
2	1.20[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1	1.23[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.9	1.238	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Table [illegible] [illegible] Relative Pi[illegible]

k	N=[illegible]	N=[illegible]	N=7	N=[illegible]	N=11
	c=[illegible]	c=[illegible]	c=[illegible]	c=[illegible].0	c=[illegible]
	d=[illegible]	d=[illegible]	d=[illegible]	d=[illegible].0	d=[illegible]
[illegible]	1.[illegible]	[illegible]	[illegible]	0.[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	0.[illegible]	0.[illegible]
[illegible]	1.011	[illegible]	[illegible]	0.7[illegible]	[illegible]
[illegible]	1.[illegible]	0.[illegible]	[illegible]	0.[illegible]	0.[illegible]
[illegible]	1.[illegible]	0.[illegible]	[illegible]	0.[illegible]	[illegible]
[illegible]	1.[illegible]	1.[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	1.[illegible]	1.[illegible]	[illegible]	0.[illegible]	[illegible]
[illegible]	1.[illegible]	1.[illegible]	1.0[illegible]	1.0[illegible]	[illegible]
2	1.[illegible]	1.[illegible]	1.01[illegible]	1.00[illegible]	1.[illegible]
1	1.0[illegible]3	1.0[illegible]	[illegible]	1.0[illegible]	[illegible]
0.9	1.067	1.[illegible]	[illegible]	1.0[illegible]	1.02[illegible]
0.8	1.0[illegible]	[illegible]	[illegible]	1.030	[illegible]
0.7	1.097	[illegible]	[illegible]	1.038	[illegible]
0.6	1.103	1.[illegible]	[illegible]	1.040	1.0[illegible]
0.5	1.110	1.07[illegible]	[illegible]	1.046	1.[illegible]
0.4	1.117	1.[illegible]	[illegible]	1.052	1.0[illegible]
0.3	1.12[illegible]	1.[illegible]	1.07[illegible]	1.058	1.0[illegible]
0.2	1.135	1.0[illegible]	1.070	1.066	1.05[illegible]
0.1	1.1[illegible]	1.1[illegible]	[illegible]	1.073	1.0[illegible]

Table [illegible] Relative Pi[illegible] of [illegible] considered in Table [illegible]9.

k	N=3	N=[illegible]	N=7	N=9	N=11
	c=[illegible]	c=[illegible]	c=[illegible]	c=1.81	c=1.7[illegible]
	d=3.5	d=[illegible]	d=[illegible]	d=1.81	d=1.7[illegible]
10	-.01[illegible]	-.0[illegible]	0.[illegible]	0.006	0.00[illegible]
9	-.015	-.00[illegible]	[illegible]	0.006	0.005
8	-.017	-.0[illegible]	[illegible]	0.006	0.00[illegible]
7	-.02[illegible]	[illegible]	[illegible]	0.007	0.006
6	-.024	[illegible]	[illegible]	0.007	0.00[illegible]
5	-.029	[illegible]	.0[illegible]	0.00[illegible]	0.00[illegible]
4	-.03[illegible]	-.017	[illegible]	0.00[illegible]	0.00[illegible]
3	-.046	-.027	-.0[illegible]	0.005	0.00[illegible]
2	-.061	-.046	-.0[illegible]	0.001	0.00[illegible]
1	-.08[illegible]	-.0[illegible]	[illegible]	-.01[illegible]	-.011
0.[illegible]	-.0[illegible]	-.[illegible]	[illegible]	-.017	-.01[illegible]
0.[illegible]	-.0[illegible]	-.0[illegible]	[illegible]	-.021	-.018
0.7	-.091	-.097	[illegible]	-.026	-.02[illegible]
0.6	[illegible]	-.0[illegible]	[illegible]	-.03[illegible]	-.02[illegible]
0.5	[illegible]	-.1[illegible]	[illegible]	-.04[illegible]	-.0[illegible]
0.4	-.0[illegible]	[illegible]	[illegible]	-.0[illegible]	-.0[illegible]
0.3	-.09[illegible]	-.0[illegible]	[illegible]	-.062	-.0[illegible]
0.2	-.0[illegible]	-.0[illegible]	[illegible]	-.080	-.0[illegible]
0.1	-.0[illegible]3	-.0[illegible]1	-.0[illegible]	-.105	-.1[illegible]

Table [illegible]

k	[illegible]	[illegible]	[illegible]	[illegible]
10	[illegible]	[illegible]	[illegible]	[illegible]
0.1	[illegible]	[illegible]	[illegible]	[illegible]

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A [illegible]

[illegible]

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I <u>INTRODUCTION</u>. Eisenhart (1947) distinguished two uses of analysis of variance which he designated as Type I and Type II. Type I provides a test of significance of the difference between estimates of population means. Type II provides a test for estimates of population variances. Eisenhart's treatment covered the general case of analysis of variance--but involved two important types of restrictions. First the "residual error" was assumed homogeneous with zero expected value. Second, all other parameters in Type I were assumed to have zero variances, and in Type II to have zero means. In the mixed model, parameters could be of either form, but, individually, where the means are not assumed zero the variances are and vice versa.

The present paper is limited to the case of two classes (the bivariate case) but removes both of these restrictions. This leads to a greatly enlarged variety of types and to a close parallelism with bivariate correlation. Two special cases, not previously treated, are discovered and appropriate formulae derived.

II <u>MATHEMATICAL MODEL</u>. Given

$$x = \alpha + \beta + \gamma + \delta + \varepsilon + \dots$$

we have

$$E(x) = \bar{\alpha} + \bar{\beta} + \bar{\gamma} + \bar{\delta} + \bar{\varepsilon} + \dots$$

$$V(x) = V(\alpha) + V(\beta) + V(\gamma) + V(\delta) + V(\varepsilon) + \dots$$

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[illegible]

then,

$$x = \bar{\alpha}' + \bar{\beta}' + \bar{\gamma}' + \ldots + \alpha' + \beta' + \gamma' + \ldots,$$

where as in Eisenhart's model, the parameters either have zero expected values or zero variances, but not both. In an exactly parallel way we may express a second variable y as

$$y = \bar{\alpha}^* + \bar{\beta}^* + \bar{\gamma}^* + \ldots \alpha^* + \beta^* + \gamma^* + \ldots .$$

For special circumstances it will be permissible to group the constant and/or the variable parameters in x and y into some lesser number adequate for the purpose in hand. If in particular, x and y are distributed in a bivariate normal, then the distribution factors into the product of one involving the means only, and the other involving the variances and covariances. For our purposes, it will be enough to ignore the distribution of the means, and to express x and y as the sum of two variables of the form

$$\begin{pmatrix} x = \tau + \zeta \\ y = \tau + \eta. \end{pmatrix} \tag{1}$$

If this is done, the variance-covariance (dispersion) matrix

$$\begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} \tag{2}$$

becomes

$$\begin{pmatrix} \sigma_1^2+\sigma^2 & \sigma^2 \\ \sigma^2 & \sigma_2^2+\sigma^2 \end{pmatrix} \tag{3}$$

where

$$\sigma^2 = V(\tau); \; \sigma_1^2 = V(\zeta); \; \sigma_2^2 = V(\eta).$$

If ρ in (2) is negative,* then (1) is of the form

$$\begin{pmatrix} x = \tau + \zeta \\ y = -\tau + \eta \end{pmatrix} \tag{4}$$

and (3) becomes

$$\begin{pmatrix} \sigma_1^2+\sigma^2 & -\sigma^2 \\ -\sigma^2 & \sigma_2^2+\sigma^2 \end{pmatrix} \tag{5}$$

so that comparison of the two models can be obtained from (2) and (3).

III <u>CORRELATION MODEL</u>. Viewed as a purely mathematical object, the various special estimating and testing problems in the literature and certain simple extensions can be classified on the basis of restrictions on the elements of matrix (2) as follows:

A. Estimate ρ, σ_x^2 and σ_y^2. Test $\rho = 0$. This is the most common situation. The t-test applies; most naturally as a test of ρ.

B. Given σ_x^2 and σ_y^2, test $\rho = 0$. The t-test with infinite degrees of freedom, the normal test, for $\rho = 0$ applies.

C. Given $\rho = 0$, test $\sigma_x^2 = \sigma_y^2$, and estimate the common variance. This is the now classic case of estimating and testing equality of two independent variances.

D. Given $\sigma_x^2 = \sigma_y^2 = \sigma_c^2$, estimate ρ and σ_c^2, test $\rho = 0$. This example was treated by DeLury (1938).

E. Test $\rho = 0$, $\sigma_x^2 = \sigma_y^2$. This is the compound symmetry problem of Mauchly (1940).

* This possibility seems to have previously been overlooked (Anscombe).

[illegible] but known, test $\sigma_x^2 = \sigma_y^2$. This problem was [illegible] Finney (1938). Morgan (1939) showed that it has little greater power than the test in G.

G. Estimate ρ and test $\sigma_X^2 = \sigma_Y^2$ whatever the value of ρ. This test was independently supplied by Pitman (1939) and Morgan (1939). It is described in Snedecor and Cochran (1967), Section 7.12.

Now suppose X is the sum of two independent random variables τ and ζ and similarily Y is the sum of τ and η, as given in equation (1). Then ρ, the correlation coefficient of X and Y is greater than or equal to zero and the matrix V was shown above to be

$$V = \begin{pmatrix} \sigma_1^2 + \sigma^2 & \sigma^2 \\ \sigma^2 & \sigma_2^2 + \sigma^2 \end{pmatrix} \tag{3}$$

The various possible tests in this formulation are:

H. Test $\sigma^2 = 0$. Estimate σ^2, σ_1^2, and σ_2^2. This is a test of the correlation of X and Y, expressed in the language of common and specific variances.

I. Given σ_1^2 and σ_2^2, test $\sigma^2 = 0$. This is the components of variance analogue of B above. The same test applies.

J. Given $\sigma^2 = 0$, estimate and test $\sigma_1^2 = \sigma_2^2$. This is the usual case of repeated measurements on constant but not necessarily equal standards; more generally, the test for equality of two independent variances as in C.

K. Given $\sigma_1^2 = \sigma_2^2 = \sigma_G^2$, estimate σ_G^2 and σ^2. Test $\sigma^2 = 0$. This is DeLury's problem expressed in the language of variance components. But whereas in its correlation formulation the problem appears to arise infrequently, in its components of variance formulation it is Eisenhart's Model II for Analysis of Variance.

L. Test $\sigma^2 = 0$, $\sigma_1^2 = \sigma_2^2$. This is Mauchly's problem expressed in the language of variance components. The components of variation interpretation of this test would be applicable wherever (a) observations occur in pairs, (b) the variance of the first member of each pair is to be compared with the variance of the second member. Thus in a paired comparison experiment, the residuals of the first members of each pair could be compared with the second as a test of the mathematical model underlying the design.

M. Test $\sigma_1^2 = 0$. This tests the legitimacy of treating the precision of one of two instruments, standards, or techniques as subject to no (negligible) error. The test is supplied in Maloney and Rastogi.

N. Given σ^2 known, test $\sigma_1^2 = \sigma_2^2$.

[illegible] and Rastogi, from which the following [illegible]. The likelihood criterion is

$$\lambda^{2/n} = \frac{\Sigma(X_i - \bar{X})^2\Sigma(Y_i - \bar{Y})^2 - n^2\sigma^4}{[\Sigma(X_i - \bar{X})^2/2 + (Y_i - \bar{Y})^2/2]^2 - n^2\sigma^4}. \tag{23}$$

For large n, -2 log λ is a chi-square r.v. with one d.f. (see Wilks, 1963, Chapter 13). Therefore, we reject the hypothesis H_o if -2 log λ > χ^2_α where χ^2_α is upper 100 percentile point of chi-square random variable with one degree of freedom.

Equation (23) can be written in a form which will be useful below and is mnemonic as well. Since $S_x S_y$ is the geometric mean of S_x^2 and S_y^2 and $\frac{1}{2}(S_x^2 + S_y^2)$ is their arithmetic mean, equation (23) becomes:

$$\lambda^{2/n} = \frac{(GM)^2 - e^2\sigma^4}{(AM)^2 - e^2\sigma^4} \tag{24}$$

writing GM for geometric mean, AM for arithmetic mean, and $e = n/(n-1)$.

Returning to equation (23) if, in particular $\sigma^2 = 0$, the likelihood ratio becomes

$$\lambda^{2/n} = \frac{S_x^2\,S_y^2}{\left(\frac{S_x^2 + S_y^2}{2}\right)^2} = \frac{(GM)^2}{(AM)^2} = \frac{4F}{(1 + F)^2} \tag{25}$$

where $F = S_x^2 / S_y$ is distributed as Snedecor's F r.v. with (n-1, n-1) d.f., since, when $\sigma^2 = 0$, X and Y are independent r.v.'s.

The test based on λ is to reject H_0 if $0 < \lambda < k$, which is equivalent to $0 < F < k_1$ or $k_2 < F$, where $k_1 < 1$. As F is always taken to be greater than 1, the rule becomes reject H_0 if $F > k^*$ at the chosen probability level, i.e., our test reduces to the ordinary F test when it is known that the population variance is zero.

Comparison of equations (24) and (25) exhibits the effect on the test of the existence and magnitude of population variance. Equation (24) is

$$\begin{aligned}
\lambda^{2/n} &= \frac{(GM)^2}{(AM)^2} \cdot \frac{(AM)^2}{(GM)^2} \cdot \frac{(GM)^2 - e^2\sigma^4}{(AM)^2 - e^2\sigma^4} \\
&= \frac{(GM)^2}{(AM)^2} \cdot \frac{(AM)^2 \quad (GM)^2 - e^2\sigma^4 \ (AM)^2}{(AM)^2 \quad (GM)^2 - e^2\sigma^4 \ (GM)^2} \\
&\lessgtr \frac{4F}{(1 + F)^2} \qquad \text{(using 25)}
\end{aligned}$$

since AM $\geq$ GM for any set of positive numbers. It follows that, if a standard F test is applied to the variance estimates for the two instruments or procedures as if the effect of population variance were zero (equation (25)) the test will sometimes accept when the correct test (equation (23)) might reject. Conversely, when equation (25) is appropriate, discrimination will be sharper than in a test situation where population variance is present so that equation (23) must be used. In addition equation (23) can be used to gain insight into the benefit to be derived, hence into the care and expense which is justified, when the population variance is reduced; whether by that using test specimens, animals, solutions or other material in an assay, or [illegible] of ambient conditions or test practice.

Case 1: Estimate and test $\sigma_1^2 = \sigma_2^2$ whatever the value of σ^2. This statement of the problem differs from that of Morgan and Pitman in that here $\sigma_1^2 + \sigma^2 > \sigma^2$, whereas in their case the range of the main diagonal elements in (1) is $0 < \sigma_i^2 + \sigma^2 < \infty$. That their test is the likelihood ratio test in case 0 has been shown elsewhere.

The above cases [illegible] be assembled into a table that brings out the symmetry between corresponding correlational and components of variance forms of expression.

Various Tests of Precision

MODEL					
CORRELATION			COMPONENTS OF VARIANCE		
Case	Assumption	Test	Case	Assumption	Test
A		$\rho = 0$	H		$\sigma^2 = 0$
B	σ_x^2, σ_y^2	$\rho = 0$	I	σ_1^2, σ_2^2	$\sigma^2 = 0$
C	$\rho = 0$	$\sigma_x^2 = \sigma_y^2$	J	$\sigma^2 = 0$	$\sigma_1^2 = \sigma_2^2$
D	$\sigma_x^2 = \sigma_y^2$	$\rho = 0$	K	$\sigma_1^2 = \sigma_2^2$	$\sigma^2 = 0$
E		$\rho = 0;\ \sigma_x^2 = \sigma_y^2$	L		$\sigma^2 = 0;\ \sigma_1^2 = \sigma_2^2$
			M		$\sigma_1^2 = 0$
F	$\rho \neq 0$(known)	$\sigma_x^2 = \sigma_y^2$	N	$\sigma^2 \neq 0$(known)	$\sigma_1^2 = \sigma_2^2$
G		$\sigma_x^2 = \sigma_y^2$	O		$\sigma_1^2 = \sigma_2^2$

Test of significance for variance-covariance parameters of a bivariate relation according to all possible non-trivial parameter restrictions. For all correlational models, the corresponding components of variance model yields the same test (not the same estimates) as its correlational analogue. No correlational model exists corresponding to item M.

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COMPUTERIZED QUALITY CONTROL AS APPLIED TO UPPER ATMOSPHERE DATA

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ABSTRACT. Any observational program, even if carried out with the best available instrumentation and carefulness to avoid instrumental error, may contain erroneous data introduced by preparation and transmission of data. Thus a good concept of quality assurance must precede any data analysis to avoid distortion and bias of results by erroneous records.

Three groups of analytical methods of quality assurance are discussed, inconsistencies, interrelationship of data, and frequency distributions. These methods have been developed at the Army Missile Command for screening radiosonde data by high speed computers. The goal is flagging of erroneous or suspicious records that these may be corrected.

Checking procedures include tests for trivial errors such as duplication, wrong sequence, missing data, special checks on identification numbers, etc. Other procedures utilize data interrelationships, in this special case the vertical structure of the atmosphere. Further checks employ screening of maxima and minima by exceedance criteria derived from the frequency distribution. The Weibull distribution has proven especially useful in this last phase of the checking procedure. Some pitfalls and limitations in the utilization of evaluation criteria are discussed.

The remainder of this article has been reproduced photographically from the author's manuscript.

I. INTRODUCTION

It is a generally accepted fact that all raw data from sample surveys and experiments contain errors. Even if an observational program has been carefully prepared and is carried out with the best available instrumentation which the program can reasonably afford, some errors are always present. They may be caused by instrumental deficiencies or inaccuracies or by unqualified observers but can also be introduced by preparation or transmission.

Analysis of observational data can be no better than the quality of the available data. Thus a careful attention to quality assurance of the data must precede any data analysis. This vital part of any investigation should be a major concern to all investigators. Its main purpose is to avoid distortion of the analysis resulting from erroneous observations. This goal will determine the magnitude of the effort to be put into a quality control program and will influence the methods selected for quality assurance. Some results can be evaluated for soundness by qualified professionals and then a quality check could be omitted. The complexity of the atmosphere or the amount of the end product (such as tables of matrices or computer produced maps, etc.) made it virtually impossible in our case to judge correctness of the results afterwards.

Of course, one cannot make good data out of bad records, but a so-called "editing" process can make data more useful for analytical purposes. No editing program can eliminate the small random error. It is the big mistake, such as a 10 or 1[illegible] degree error in temperature, which needs correction. Since the influence of erroneous records on results increases as the length of the observational series of record decreases, the need for quality assurance is the greater the shorter the record.

In some instances censoring of frequency distributions by eliminating extreme values may save elaborate screening procedures. This cannot be applied, however, if one of the analysis goals is the study of extremes. In the case of radiosonde data a second reason against censoring can be pointed out. Because of vertical consistency, data elimination at one level without attempt of correction may lead to discontinue the ascent from the censored level up. This may further reduce the already decreasing number of observations with altitude and may leave very few data reaching a top level of [illegible] km, for example. Thus the cure is worse than the disease.

The availability of high speed computers has opened a new field in applying quality control methods and many methods considered too elaborate and cumbersome without computer use can now be employed without difficulties.

Some of the few basic principles, which reappear and can be commonly applied, may be demonstrated from the Army Missile Command's screening program of radiosonde data.

Since the author's detailed article is already scheduled for publication, (See 1969c) only some basic principles will be presented here.

II. CHECKING PROCEDURES[a]

Any automated screening procedure must be so designed that particular (consistent) errors as well as inconsistent errors can be recognized. This goal is rendered more difficult by the requirement that screening procedures should have a simple logic for computerized treatment.

1. Trivial Errors Check

Under this first category fall all errors which are easily recognized, and in many instances an automatic correction can be made. The errors can be divided largely into three groups: coding errors, data and limit checks.

In the first group one may encounter errors such as wrong location number, incorrect elevation, false identification code, mistakes in coding the type of observation, erroneous time, etc.

A second group comprises checks for completeness (missing data), duplication and sequence of the records. If it is intended to supplement the original data by automatic fill-in procedures, they can be incorporated in this phase of the screening procedure or at a later date.

The last group is the limitation violation, e.g. data are outside established tolerance limits or physical boundaries. For example in our case the dew point temperature cannot be greater than the air temperature and the wind direction cannot exceed 16 compass points or 360 degrees.

The examples given for the above error groups are some guidelines and are not exhaustive. They serve only as a demonstration for the type of errors. Clearly, a procedure for trivial error checks evidently depends largely on the type of available data. The correction of the deficiency may also vary, e.g., if dealing with one station, an automatic correction could be made for wrong station code. In other instances elimination of the data may be necessary. If one had to establish a map by computer, this may be the only way to reduce the effect of large errors, while for other analyses time and personnel may be available to go through flagged observations and to painstakingly check their validity.

2. Error Checking by Adjacent Data

In this group inconsistencies are checked against adjacent data or a field of data in the horizontal (map or equations), vertical (cross-sections or equations), or by time relationship. The checking procedure depends largely on established physical or derived empirical laws. Again, procedures aim at flagging suspicious values by computer methods or correcting them if such procedures can be established. Tolerance limits of differences between two or more observations must be derived first.

a. Horizontal Checks

This type of checking process can be applied if computerized maps are available or become the end product or if records for neighboring stations for the same period of record are given. Under physical laws one may understand conditions like the gradient wind relationship etc. Empirical relationships between neighboring stations or thresholds of tolerance differences between stations could be established. Whether these empirical relationships are derived in tabular form or as analytical expressions is not important, except that it is more convenient to work with mathematical statements for which computer programming is usually very simple. Changes of errors are considerably lower as opposed to table inputs, especially when these have more than one entry.

b. Vertical Relationship

The U. S. Army Missile Command's procedure of screening radiosonde data relies heavily on vertical relationships. Cross-sections could be used, but only if they are readily available or calculation of the cross-section by computer methods is the goal. The author does not know of any program at the present where space cross-sections have been utilized for data control. Time-sections have been employed by Canfield et al. (1966).

Our program checks two groups of elements, thermodynamic quantities and wind. In the thermodynamic portion the lapse rate between two consecutive observations at different altitudes is computed and compared with the dry adiabatic lapse rate. This method has proven quite efficient and satisfactory, as usually any error in pressure or temperature will show up eventually in a superadiabatic lapse rate either at the tested data pair or at the next step. For example, assume a 10° negative error in the temperature. If it is the higher of two altitudes, it creates a superadiabatic lapse rate. If the error is positive, one would have an inversion for this step of the program and the record is not flagged. The next step, however, would give a superadiabatic lapse rate.

The last observation of a radiosonde ascent cannot be checked by this method, as there is no other observation to compute the lapse rate. This last point could be checked by tolerance limits or other tools.

It should be added that superadiabatic lapse rates are not automatically eliminated by our program. The cause can be manifold. There may exist the unusual case of a true superadiabatic lapse rate in nature. One may have a temperature or pressure error or the data can be out of sequence by erroneous pressure. Thus all "suspicious" data are flagged and checked by a qualified meteorologist.

Since this simple tool worked so well for the thermodynamic parameters a similar principle was sought for the wind. In the beginning wind data were checked by the frequency distribution of wind shear with techniques established by Essenwanger et al. (1961). This is usually cumbersome and expensive, as computations of frequency distributions are generally costly. The difficulty in establishing a unique relationship was recognized by Finger et al. (1965) who established vertical shear limits for wind checks in tabular form for a few thresholds of layer thickness. However, their method requires detailed criteria depending on layer thickness, wind speed, and difference of direction or speed of [illegible]. The author (1969) has derived a relationship between the vector shear (Δv) and the shear interval (Δh)

$$\Delta v = a_0 \, (\Delta h)^{a_1} \, , \qquad (1)$$

The exponent for extreme value was found to be 1/3 (see also Essenwanger and Reiter 1969a). For use in our program eqn. (1) had to be modified to accommodate normalized shear interval, thus $\Delta v = V_e \, \Delta h$, resulting in

$$V_e = a \, (\Delta h)^{-\frac{2}{3}} \qquad (2)$$

Where V_e denotes the total vector shear. With $a = 2.5$, a reasonable threshold V_e (m sec^{-1} per interval) is found. All values exceeding V_e are flagged.

Equation (2) expresses a unique relationship similar to the lapse rate as a convenient and simple tolerance criteria.

c. Time Series

All elements showing some form of time relationship could be checked by methods taking advantage of this relationship. It does not matter whether the time relationship is periodic or aperiodic. However, in all time related checking procedures the time relationship must be established first.

In case of periodic variations it is quite convenient to represent records by a Fourier series and check an expected versus an observed value. A tolerance limit for a maximum (absolute) difference from the expected value may be determined by statistical methods of error theories. Sometimes it may be quite sufficient and suitable to use subjective tolerance limits.

If an aperiodic time relationship (e.g. persistence) has been found, tolerance criteria for time differences can be employed. In all cases an expected value is tested against the observation.

A time checking procedure can be applied, even if no functional relationship can be found. Although time differences may be randomly distributed, a tolerance criterion can be developed similar to that described in a later chapter on frequency distributions. If the difference exceeds a certain magnitude, it may indicate an erroneous observation.

7. Frequency Distribution Checks

Although methods described in the previous sections should catch the bulk of errors, some mistakes may slip through. Let us assume that the surface observation of a radiosonde ascent is missing. Vertical consistency could not discover this mistake. Although it could have been flagged in the trivial error check, other examples can be given where vertical consistency existed, but the total ascent was either too warm or cold. These errors can be checked against a frequency distribution.

In the Army Missile Command's earlier screening procedure preliminary frequency distributions were established, with printout of the first five maxima and minima, mean and standard deviation. Visual inspection of the frequency distribution then revealed isolated observations. Vertical profiles for the maxima and minima were drawn and suspicious records could be detected by irregularities in profiles.

This process was time consuming, and not too many erroneous ascents were discovered, since the majority of corrections had been made. Nevertheless, all frequency distributions had to be inspected. This phase of the program was costly, too, as frequency distributions had to be grouped by small class intervals to detect isolated records and class intervals shifted from month to month or by altitude. This phase of the program was modernized by utilizing only mean and standard deviation and selecting suspicious values by a predetermined threshold x_{th} to be exceeded only a certain percentage of the time. This eliminates the establishment of frequency distributions and reduces the printout as only flagged observations appear.

It is evident that correct as well as incorrect observations will be flagged and printed out, as one should expect a number of observations exceeding the threshold x_{th} in agreement with the selected percentage figure. Unfortunately there is no easy way to separate the two groups by computer methods, as large deviation can be caused by extreme weather events. All cases must be judged by their own merits. It is reiterated that acceptance, correction or deletion of an observational record depends largely on the purpose of any analysis and existing possibilities. We have found it quite convenient to make available for any flagged value the threshold for 99% and the frequency of occurrence, which the flagged observation would have in a theoretical distribution law. These values are helpful guidelines for evaluation, but are generally not sufficient by themselves for a decision that the observation is erroneous or not. It should further be pointed out that censoring of the frequency distribution cannot be applied in our particular case. Especially extreme value data analysis is part of the subsequent research topics. Censoring would not solve the quality assurance problem.

a. Gaussian Distribution

The critical problem is the determination of the threshold x_{th}, outside of whose boundary observations should be flagged. In statistical terms, one has to select a certain point of the cumulative distribution on one or both sides of this curve. The computation of the cumulative distribution is cumbersome for most types of distribution laws as it involves integrating frequency density functions. In the Army Missile Command's earlier version empirical cumulative distributions were computed to secure close agreement with the observed frequency. This had the advantage of the frequency curve being independent from the statistical type, or the mean and the standard deviation of the distribution. Later this was replaced by establishment of frequency distributions, which display less complexity in computer programming.

If the element follows an approximate Gaussian normal distribution, one could determine the threshold by

$$x_{th} = \bar{x} \pm a\sigma \tag{3}$$

where $\bar{x}$ is the mean value, σ the standard deviation and the coefficient "a" would be determined by the desired percentage exceedance, e.g. $a = 3.0$ for .135% of the observations beyond that point. All observations above x_{th} would then be flagged and printed out.

Since the relationship between the cumulative distribution and the standard deviation is known for the Gaussian distribution, the establishment of thresholds should not create any problem for meteorological elements following this distribution law. Gaussian laws apply to most thermodynamic quantities.

b. The Weibull Distribution

If Eqn. (3) were applied to meteorological elements not in agreement with the Gaussian law, one would have either too many flagged observations or not enough, depending on the deviation. Since the relationship between standard deviation and cumulative distribution for other types of distributions is complex and generally cannot be found in simple tables, the ideal solution would be a cumulative frequency law versatile enough to adjust to a variety of types with good approximation. Thus we applied the Weibull distribution with considerable success in our screening procedure.

The Weibull distribution is defined as a cumulative type

$$F(x) = 1 - e^{-\left(\frac{x-\gamma}{\theta}\right)^{\beta}} \qquad (4)$$

with γ, θ and β as the reference, scale and shape parameter, respectively. Any percentage $F(x)$ can be related to x_{th} by the modification of equation (4) to

$$x_{th} = \theta \sqrt[\beta]{\ln P} + \gamma \qquad (5)$$

where

$$P = 1 / (1 - F(x)) \qquad (5a)$$

The estimation of the parameters is the only difficulty left. Maximum likelihood estimation for all three parameters cannot be performed analytically and solution is very time consuming. Thus the utilization of the maximum likelihood method for three parameters would have increased costs compared to frequency distributions. Simpler methods exist when $\gamma = 0$ (see Kao, 1958 or Menon, 1963), however, the assumption $\gamma = 0$ would reduce the flexibility of adjustment for the Weibull distribution and would limit the ability to fit the frequency distribution. Since the major goal in the checking procedure is the establishment of a threshold value x_{th}, the reader may find a parameter estimation by moments. developed by the author (1968, 1969b) quite convenient.

$$\gamma_1 = (c - 3ab + 2a^3)/(b - a^2)^{3/2} \tag{6a}$$

γ_1 denotes the skewness, the ratio of the third moment (reference mean) to the cube of the standard deviation, $\gamma_1 = \mu_3/\sigma^3$

Since a, b and c depend on β only, a computer solution of (6a) is relatively easy or tables can be used (see Essenwanger, 1968, 1969b)

$$a = \Gamma(1 + 1/\beta) \tag{7a}$$

$$b = \Gamma(1 + 2/\beta) \tag{7b}$$

$$c = \Gamma(1 + 3/\beta) \tag{7c}$$

With θ known, the other parameters become

$$\theta^2 = \sigma^2/(b - a^2) \tag{6b}$$

and

$$\gamma = \bar{x} - \theta \cdot a \tag{6c}$$

The three moments of the distribution must be known for application of eqns. 5, 6, and 7. In two cases two moments are sufficient.

As shown by the author (1968) the γ_1 can be approximated by

$$\gamma_1 = 1.4047E - .0646\sigma + .0987 \qquad (8a)$$

for wind and by

$$\gamma_1 = 3.1223E - .3680\sigma - .4515 \qquad (8b)$$

for the total vector wind shear

$$E = \epsilon_3/\sigma^3 = \bar{x}\,(1 + 3d + 2d^2)/\sigma^3 \qquad (9a)$$

with
$$d + 1 = \sigma^2/\bar{x} \qquad (9b)$$

The second case employs the Weibull distribution for elements whose distributions follow the Gaussian law.

Thus β can be determined a priori. Eqn. (6a) gives $\beta = 3.60$. If the squared difference of the Gaussian and the Weibull distribution at steps of half a standard deviation σ within $\pm$ 3.5σ is computed and summed up, a minimum is discovered at $\beta = 3.55$. Table 1a exhibits the frequencies for the Gaussian and the Weibull distribution (cumulative at left and density at right) for $\beta = 3.55$. All differences are less than 1%. The last columns in both sections contain the differences for the β, if selection is made for the smallest possible maximum deviation of any frequency within $\pm$ 3.5σ range.

Of more importance may be the agreement between the x-values as these are used to establish the flagging limit of Eqn. (5). The

Table 1. Comparison of Weibull Distribution With Gaussian Distribution

a. Frequency (in %)

Cumulative Frequency

x/σ	Gauss	Weibull ($\beta = 3.55$)	Difference ($\beta = 3.55$)	Difference ($\beta = 3.62$)
-3.5	.023	-	.023	.023
-3.0	.135	.004	.131	.128
-2.5	.621	.313	.308	.273
-2.0	2.275	2.099	.176	.110
-1.5	6.681	7.041	-.360	-.409
-1.0	15.866	16.664	-.798	-.779
- .5	30.854	31.413	-.559	-.462
.0	50.000	49.799	.201	.321
.5	69.146	68.453	.693	.758
1.0	84.134	83.621	.513	.495
1.5	93.319	93.259	.060	-.001
2.0	97.725	97.895	-.170	-.221
2.5	99.379	99.524	-.145	-.169
3.0	99.865	99.926	-.061	-.067
3.5	99.977	99.992	-.015	-.017

Frequency Density

x/σ	Gauss	Weibull ($\beta = 3.55$)	Difference ($\beta = 3.55$)	Difference ($\beta = 3.33$)
< -2.5	.621	.313	.308	.258
-[illegible] to -2.5	1.654	1.786	-.132	-.153
-1.5 to -2.0	4.406	4.942	-.536	-.583
-1.0 to -1.5	9.185	9.623	-.438	-.676
- .5 to -1.0	14.988	14.749	.239	-.034
.0 to - .5	19.146	18.386	.760	.674
.5 to .0	19.146	18.655	.491	.668
1.0 to .5	14.988	15.167	-.180	.096
1.5 to 1.0	9.185	9.638	-.453	-.302
2.0 to 1.5	4.406	4.637	-.231	-.253
2.5 to 2.0	1.654	1.629	.025	-.060
> 2.5	.621	.475	.146	-.173

Table 1. (Continued)

b. x-scale (in σ Units)

	Two-Sided				One-Sided		
	β = 4.36		β = 4.26	β = 5.53	β = 5.54	β = 2.97	β = 2.96
Gauss	Weibull	Difference	Difference	Difference	Difference	Difference	Difference
-4.0	-3.467	.533	.585	.002	-.002	1.365	1.371
-3.5	-3.255	.245	.306	-.147	-.150	.955	.961
-3.0	-2.933	.067	.105	-.217	-.219	.603	.608
-2.5	-2.535	-.035	-.015	-.217	-.219	.326	.329
-2.0	-2.074	-.074	-.063	-.166	-.167	.131	.133
-1.5	-1.562	-.062	-.058	-.089	-.089	.016	.017
-1.0	-1.026	-.026	-.027	-.013	-.013	-.035	-.036
- .5	- .488	.012	.009	.041	.042	-.043	-.044
.0	.036	.036	.033	.064	.064	-.026	-.027
.5	.537	.037	.036	.051	.051	-.001	-.001
1.0	1.013	.013	.014	.001	.001	.025	.02[illegible]
1.5	1.463	-.037	-.033	-.081	-.082	.045	.04[illegible]
2.0	1.888	-.112	-.104	-.194	-.194	.055	.05[illegible]
2.5	2.290	-.210	-.197	-.332	-.333	.054	.057
3.0	2.680	-.320	-.311	-.454	-.495	.043	.046
3.5	3.062	-.438	-.441	-.674	-.676	.022	.038
4.0	3.389	-.611	-.583	-.869	-.871	-.002	.0[illegible]4

minimum sum of the squared differences between the Gaussian and the Weibull distribution for half units of σ within $\pm 4\sigma$ lies at $\beta = 4.36$ (Table 1b). It should be noticed, however, that the differences for the two-sided fit increase towards the marginal classes. This is a handicap, but is acceptable since it is preferable to flag more than the expected number of observations rather than less. The threshold could be adjusted, too. Again, if a minimum of the absolute deviation is desired, one would select $\beta = 4.26$ with deviations smaller than 0.6σ at the ends.

Since it is known whether an observation is below or above the mean value, a one sided fit solves the problem of poor agreement towards the ends. Good approximation for the minimum threshold can be obtained with a β of 5.53 or 5.54, while one may select a β of 2.96 or 2.97 for the maximum end. The differences are displayed in the right portion of Table 1b.

The advantage in using the Weibull distribution for flagging instead of the concept of the normal distribution lies in the easy computation of related frequency values for the flagged observation with Eqn. (5) and (5a). This eliminates any tabular input as necessary for Eqn. (3) and one program can be applied to all types of frequency distributions.

c. Elements With Various Types of Distributions

Thermodynamic quantities and wind can be treated with techniques as outlined previously. The Weibull distribution is very flexible and thus can be utilized for the purpose of flagging for numerous elements. Some distributions may display untolerable discrepancies.

Transformation of scale sometimes helps, such as a logarithmic progression of visibility data. This must be left to the individual analyst. The Weibull distribution is very flexible and transformation can usually be avoided.

6. Some Remarks of Caution

It is reiterated that no quality assurance program can make good data out of bad records. These programs can only contribute to an "editing" of data, after which the larger errors (hopefully) have been eliminated. Since these large errors can bias any statistical or computer result, the correction of these obvious mistakes is necessary. It must be cautioned, however, that correction methods cannot be geared to an expected analysis result, as all observations contradictory to an assumed hypothesis to be tested by these data are then eliminated. Correction methods must be independent of subsequent analysis. One cannot check persistence, for example, if the majority of data have been filled in by methods derived from persistence.

The editing process by "experts" is usually cumbersome, but correction methods by computers must be carefully designed. Where consistency equations can be obtained, methods for random error corrections can be developed. With the complexity of the atmosphere it is difficult, however, to pinpoint unequivocally differences between a rare event and an obvious mistake.

Any correction method should be based upon known or derived principles of error sources. Sometimes data are rectified which later prove correct in the light of expanded knowledge.

Establishment of threshold values is arbitrary. Threshold values must be designed to catch all the large errors without the burden of reviewing too many data by the expert. Any time a large pile of flagged data appears for a particular data sample, a search for a systematic error should precede any detailed correction operation. This systematic error can then be corrected before other computer runs are made. Sometimes a big bulk of printout can be caused by improper selection of the thresholds. Then an adjustment will give reasonable amounts.

It should be further mentioned that selection of thresholds succeeds for unlimited distributions only. It would be absurd, for instance, to flag all calms in surface wind distributions or all dry records for precipitation data. Elements with U-shaped distributions could in general not be checked by frequency methods.

III. CONCLUSIONS

Analytical methods for the editing of observational data have been divided into three major groups, the checking of inconsistencies (trivial errors), the procedures employing a set of data with interrelationship, and utilization of frequency distributions. The methods presented may serve as a guideline and cannot be exhaustive, as the complexity of the atmosphere with its different meteorological parameters necessitates individual techniques depending on the treated element. The three described groups of error checks are common with any quality assurance program.

It is repeated that editing of data cannot replace a carefully carried out observational program with adequate instrumentation. One can assure, however, that large mistakes and systematic errors from various sources are discovered and any bias of the results due to erroneous data is largely reduced. The small random error cannot ordinarily be eliminated.

Although the data may have gone through quality assurance programs several times before they reach the investigator, it is nevertheless advisable to resubmit the data to a screening procedure. Editing of data by other investigators or installations does not automatically guarantee that the received data are free of mistakes.

The goal of the editing process should not be to correct nature and reject data which do not fit into a predetermined model or hypothesis, [illegible] either [illegible] the data should be based upon known or dis-covered sources of error only. If the latter is kept in mind, analytical methods of quality assurance will serve their useful purpose.

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ABSTRACT

A STATISTICAL MODEL FOR THE ANALYSIS OF SIMULTANEOUS TWO-STATION IONOSPHERIC SOUNDINGS

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1. Ionospheric sounder data characteristics change as the distance between two sounder stations is increased from 0-500 Km. It is therefore desirable to know about the degree of correlation one can expect between vertical incidence (single station) data and oblique incidence (two-station) data. It will be shown that a single ionospheric sounder (ionosonde) operating in the vertical incidence mode can provide useful data over an area of 60 Km radius.

2. Experimentation was performed in the 2-16 MHz frequency range using two ionosondes, one as a fixed terminal and the other as a mobile terminal. Each terminal made scheduled soundings every ten minutes from 0530 to 1730 hours for ten days. While the fixed terminal was transmitting and receiving its own signal, the mobile terminal would simultaneously receive the same transmission; likewise for the mobile with respect to the fixed terminal. As each ionosonde transmitted and received in the vertical incidence mode, the other sounder, receiving the same transmission, completed the oblique ionospheric mode. (An oblique mode or path is one between two stations space a distance apart; a vertical mode or path occurs when either station receives its' own transmission.)

3. The experiment was designed primarily for a paired difference model, that is, the pairing of data occurred as planned by the experiment. The data were also analyzed by a paired comparison method to focus on the gain of information achieved with the paired difference or randomized block design, and to show that vertical incidence and oblique incidence ionosonde data are good estimators of each other over short distances.

4. The application of a similar method of analysis will hopefully be used in future experimentation to substantiate a high degree of correlation between vertical and oblique incidence soundings over field army distances (0-300 Km).

The objective of the present data analysis is to show that daily ionospheric soundings taken at vertical incidence (VI) are very nearly the same as oblique incidence (OI) soundings taken over a 60 Km path (see Figure 1). We are interested in formulating hypothesis tests to determine whether or not the Vertical Incidence data (population I) is nearly the same or is, in fact, identical to the Oblique Incidence data (population II). The analysis investigates a total of 85 daily measurements of critical frequencies performed over a nine-day period, taken every ten minutes from 0530 hours to 1930 hours, for a 60 Km path (see Figures 6, 7). This yielded nine observations of critical frequency per time slot. Samples of raw data appear in Figures 2, 3, 4, 5. In order to test whether or not a given hypothesis is supported by a set of data, we devised a rule of procedure dependent upon certain calculations obtained from a sample of the data, and decided to accept or to reject the hypothesis formulated[3]. Two experiments, E_1 and E_2 were used in comparing the means of population I (VI) and those of population II (OI). The homogeneity of variance was tested by the use of the F test, where σ_o^2 was compared to σ_v^2.

To test homogeneity of variance, the variances of the vertical and oblique incidence data were paired. The 85 grouped values were:

$$(1) \qquad T = \left\{ \left(\frac{S_o^2}{S_v^2}\right)_1, \left(\frac{S_o^2}{S_v^2}\right)_2, \ldots\ldots, \left(\frac{S_o^2}{S_v^2}\right)_{85} \right\}$$

Tests of hypotheses for the equality of two variances were formulated as follows:

(2)
$$H_o: \sigma_o^2 = \sigma_v^2 \qquad \text{vs.} \qquad H_1: \sigma_o^2 \neq \sigma_v^2$$

or:
$$H_o: \frac{\sigma_o^2}{\sigma_v^2} = 1 \qquad \text{vs.} \qquad H_1: \frac{\sigma_o^2}{\sigma_v^2} \neq 1$$

The rejection region is: $\frac{\sigma_o^2}{\sigma_v^2} \geq k$, where k is found by specifying the significance level $\alpha = .01$. The following probability function describes the relationship:

(3)
$$\Pr\left[\frac{\sigma_o^2}{\sigma_v^2} \geq k \;\middle|\; \frac{\sigma_o^2}{\sigma_v^2} = 1\right] = \alpha$$

Under the null hypothesis H_o, $\frac{\sigma_o^2}{\sigma_v^2}$ has an F distribution with $(n-1)$, $(n-1)$ degrees of freedom, which results in $k = F[(n-1), (n-1); (1-\frac{\alpha}{2})]$ and the rejection regions are:

(4)
$$\frac{\sigma_o^2}{\sigma_v^2} \geq F[(n-1), (n-1); (1-\frac{\alpha}{2})]$$

$$\frac{\sigma_o^2}{\sigma_v^2} \geq F[(n-1, (n-1); \frac{\alpha}{2}] \qquad \text{for } \alpha = .01$$

If these inequalities are satisfied by S_o^2 and S_v^2, then we can conclude that the estimated variances are significantly different at $\alpha = .01$ level of significance. That is to say, H_o is rejected when:

$$(5) \qquad F[(n-1),\ (n-1);\ \tfrac{\alpha}{2}] \geq \frac{\sigma_o^2}{\sigma_v^2} \geq F[(n-1),\ (n-1);\ (1-\tfrac{\alpha}{2})]$$

By letting S_o^2 and S_v^2, the sample variances, estimate σ_o^2 and σ_v^2, we form the F ratio:[1]

$$(6) \qquad F = \frac{\dfrac{S_o^2}{\sigma_o^2}}{\dfrac{S_v^2}{\sigma_v^2}}$$

which has the F distribution with $(n-1)$, $(n-1)$ degrees of freedom.

F involves the ratio $\dfrac{\sigma_o^2}{\sigma_v^2}$ but is independent of σ_o^2 and σ_v^2, therefore:

$$(7) \qquad \Pr\left[F[(n-1),(n-1);\tfrac{\alpha}{2}] \leq \left(\frac{\dfrac{S_o^2}{\sigma_o^2}}{\dfrac{S_v^2}{\sigma_v^2}}\right) \leq F[(n-1),(n-1);(1-\tfrac{\alpha}{2})]\right] = 1-\alpha$$

In testing the means the observations were grouped into 85 values for each experiment:

$$(8) \qquad E_1 = \{\bar{d}_1, \bar{d}_2, \ldots\ldots, \bar{d}_{85}\}$$

where $d_i = x_o - y_v$, and $\bar{d} = (1/n)\sum_1^9 d_i$, $i = 1, \ldots\ldots, 9$ per time slot.

$$(9) \qquad E_2 = \{\delta_1, \delta_2, \ldots\ldots, \delta_{85}\} \text{ where: } \delta_j = \bar{x}_{o_j} - \bar{y}_{v_j} \text{ per time slot}$$

and, $\bar{x}_{o_j} = (1/n) \sum_1^9 x_{o_i}$ per time slot

$\bar{y}_{v_j} = (1/n) \sum_1^9 y_{v_i}$ per time slot

These sets consist of the mean values of the differences between OI and VI, (E_1), and the differences between the means of the two populations, (E_2), repeatedly taken at the same time daily for nine days. These two sets are assumed to be normally distributed with variance σ_d^2 and σ_p^2 so that the means of differences, $\bar{d}_i$, and the difference between the means, δ_i, are normally distributed. Since σ_d^2 and σ_p^2 are unknown, take estimates of the variances for each time slot for E_1 and E_2 are:

(10) $$Sd_i^2 = \frac{1}{n-1} \sum_{i=1}^{9} (d_i - \bar{d})^2 \text{ for } E_1, \text{ where}$$

$n = 9$ samples per time slot.

$\bar{d}$ = mean of the differences between OI and VI per time slot.

d_i = difference between f_{OI} and f_{VI} = $x_{o_i} - y_{v_i}$.

(11) For E_2, $$S_p^2 = \frac{(n_1-1)\,S_o^2 + (n_2-1)\,S_v^2}{n_1+n_2-2} = \frac{S_{x_o}^2 + S_{y_v}^2}{2}$$

where $n_1 = n_2$, and:

$$S_{x_o}^2 = \frac{1}{n-1} \sum_1^9 (x_{o_i} - \bar{x}_o)^2$$

$$S_{y_v}^{2} = \frac{1}{n-1} \sum_{i} (y_{v_i} - \bar{y}_v)^2$$

S_p^2 = pooled estimate of variance for E_2

n = 9 samples per time slot

$\bar{x}_o$, $\bar{y}_v$ = means of OI and VI populations per time slot.

x_{o_i}, y_{v_i} = OI and VI data per time slot.

The t-statistics employed are:[2]

(12a) $$t_{n-1} = \frac{\bar{d}_i \sqrt{n}}{S_{d_i}} \text{ for } E_1, \text{ and}$$

(12b) $$t_{n*} = \frac{\bar{x}_o - \bar{y}_v}{S_p\sqrt{\frac{2}{n}}} = \frac{\delta_{ov}}{S_p\sqrt{\frac{2}{n}}} \quad \text{for } E_2,$$

where n=9 samples per time slot, and $n^* = n_1 + n_2 - 2 = 16$ degrees of freedom

$S_p = \sqrt{S_p^2}$, the pooled standard deviation for E_2,

$S_{d_i} = \sqrt{S_{d_i}^2}$, the standard deviation for E_1 .

Therefore, the populations are t-distributed with (n-1), and n* degrees of freedom. The first experiment or "paired" difference test, E_1, concerned itself with analyzing the <u>means of the differences between OI and VI data</u>. The second experiment or "paired" comparison test, E_2 was concerned with analyzing the difference between the means of the two populations. For E_1 the following hypothesis was formulated:

(13a) $$H_0: \mu_o = \mu_v \quad \text{vs.} \quad H_1: \mu_o \neq \mu_v$$

where $\bar{d}_i = (1/n)\sum_{i=1}^{9} (x_{o_i} - y_{v_i})$ for each time slot.

For E_2 the following hypothesis was formulated:

(13b) $$H_0: \mu_o = \mu_v \quad \text{vs.} \quad H_1: \mu_o \neq \mu_v$$

where $\delta = \bar{x}_o - \bar{y}_v$ for each time slot,

and $\bar{x}_o = (1/n) \sum_{i=1}^{9} x_{o_i}$,

$\bar{y}_v = (1/n) \sum_{i=1}^{9} y_{v_i}$,

and x_o and y_v are oblique and vertical incidence data respectively. That is to say, we will test a null hypothesis H_0, (that $\bar{d}_i$ or $\delta_i = 0$) vs. H_1 . If we accept the hypothesis, this would, of course, indicate that the difference between OI - VI = 0 for each time slot at $\alpha = .01$. If we assume the alternate hypothesis H_1 to be true, then the OI and VI data would be significantly different.

The critical region of these tests are:(2)

(14) $$\frac{\bar{d}_i \sqrt{n}}{S_{d_i}} > t\,[(n-1);\, \alpha/2]$$ for E_1, which can

be written as:

$$t[(n-1);\ \alpha/2] > \frac{\bar{d}_i \sqrt{n}}{S_{d_i}} > t[(n-1);\ (1-\alpha/2)]$$ for a two-tailed test, and

(15) $$\frac{\bar{x}_o - \bar{y}_v}{S_p \sqrt{2/n}} > t[(n^*);\ \alpha/2]$$ for E_2 which can be written as:

$$t[(n');\ \alpha/2] > \frac{\bar{x}_o - \bar{y}_v}{S_p\sqrt{2/n}} > t[(n');\ (1-\alpha/2)]\ ,$$

where $n=9$, $n^*=(2n-2)=16$ degrees of freedom, and with $n=9$, $(n-1)=8$ degrees of freedom. This indicates that if (14) is satisfied by $\bar{d}_i$ and S_{d_i}, and (15) is satisfied by δ_i and S_p, the tests are rejected under the null hypothesis H_o and that $\bar{d}_i$ and δ_i does differ significantly from "0" in the critical region (region of rejection).

The critical regions can be explained by the following probabilities:(5)

(16) $$\Pr\left[\frac{|\bar{d}_i \sqrt{n}|}{S_{d_i}} > t[(n-1);\ \alpha/2]\right] = \alpha$$ for E_1 which can be written:

$$\Pr\left[t[(n-1);\ \alpha/2] > \frac{\bar{d}_i \sqrt{n}}{S_{d_i}} > t[(n-1);\ (1-\alpha/2)]\right] = \alpha\ ,$$

and

(17) $$\Pr\left[\frac{|\bar{x}_o - \bar{y}_v|}{S_p\sqrt{2/n}} > t[n^*;\alpha/2]\right] = \alpha$$ for E_2 which can be written:

$$\Pr\left[t[n^*; \alpha/2] < \frac{\bar{x}_0 - \bar{y}_y}{S_p\sqrt{2/n}} > t[n^*; (1-\alpha/2)]\right] = \alpha ,$$

where $\alpha = .01$ is the pre-determined critical region, or region of rejection. Regarding the comparison of means and of variances, if the null hypothesis H_0 is found to be false, then the power function, π, would be used to find the probability that the alternate hypothesis H_1 will fall completely in the critical region. Let β = region of acceptance of the alternate hypothesis H_1 . Normally $\pi = 1-\beta$ should be very large or β very small. To illustrate the concept of a rejection region, suppose we have the following hypothetical probability density function of a variable X :

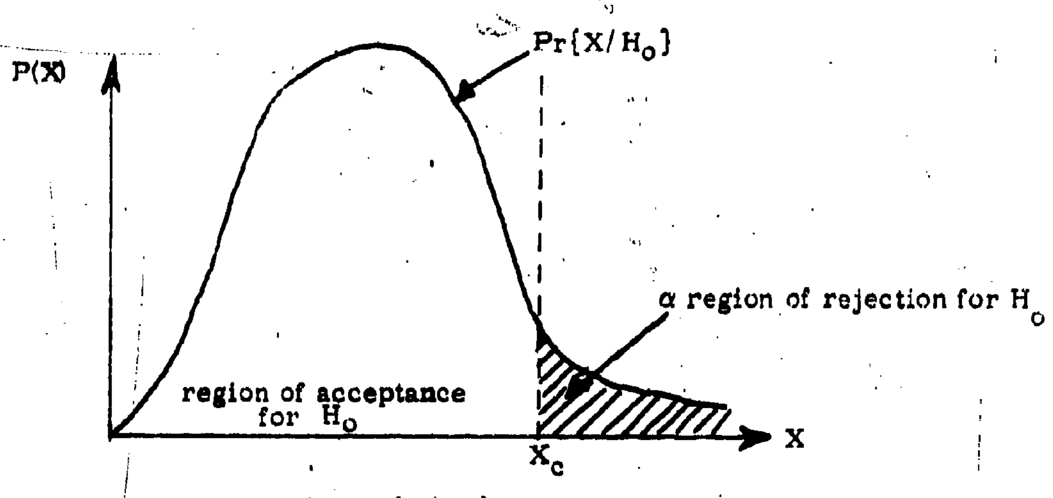


Illustration (a), Pr{X/H$_0$} vs. X

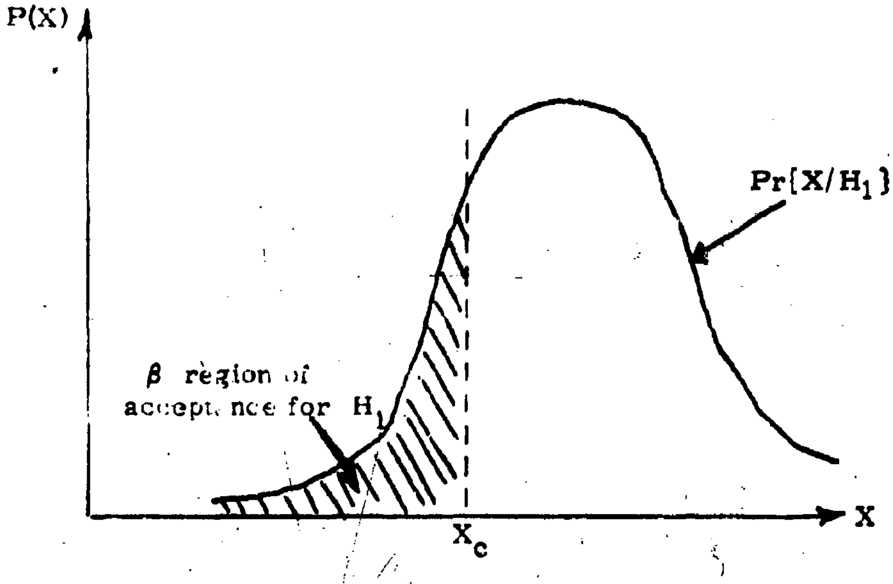


Illustration (b), $Pr\{X/H_1\}$ vs. X

In illustration (a):

(18) $$\int_{X_c}^{\infty} Pr\{X/H_o\}\ d_X = \alpha$$

Therefore, if H_o is true (so that X has the probability distribution $Pr\{X/H_o\}$), the probability of a random observation falling in the critical or rejection region, $X > X_c$ is α,(4) that is: X_c satisfies illustration (a). Now consider H_1 true and X having the density function $Pr\{X/H_1\}$. The probability of a random observation falling in the acceptance region (illustration (b)), $X < X_c$ is β, that is:

$$(19) \qquad \int_{-\infty}^{X_c} Pr\{X/H_1\}\, d_X = \beta$$

The probability of correctly rejecting H_o, is called the power of test where:

$$(20) \qquad \pi = 1 - \beta = \int_{X_c}^{\infty} Pr\{X/H_1\}\, d_X$$

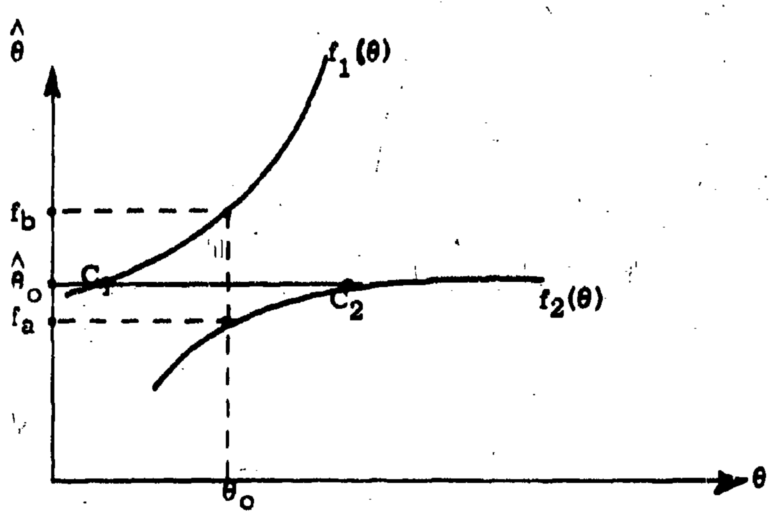
In addition to hypothesis testing, the analysis estimates intervals I and I' for which we can expect, with 99% confidence, that $\bar{d}_i \in I$, and $\frac{S_o^2}{S_v^2} \in I'$.

That is, we utilize the information at each time slot using the t tests described in equations 12, and place a 99% confidence bound on the true state of nature at these points, i.e. $d_i = (x_{o_i} - y_{v_i})$ for E_1 and $\delta_i = (\bar{x}_{o_i} - \bar{y}_{v_i})$ for E_2 respectively. This means that if experiments E_1 and E_2 were to be performed say, 100 times, we could be confident that 99% of such intervals will contain the true state of nature at each time slot. Thus by putting confidence bounds on each set of data points, we would have 85 upper and lower bounds which would generate an envelope. From this envelope we can conclude that for the spectrum of data generated in this experiment, we are 99% confident that the data will be contained with the envelope.

Interval estimation aids in obtaining limits C_1 and C_2 which are functions of the sample values $\{f_c\}$ or functions of the sample values and known population parameters $\{\bar{d}_i\}$, $\{\delta_i\}$ and $\left\{\frac{S_o^2}{S_v^2}\right\}$. The limits are determined so that the probability:[5]

(21) $$\Pr(c_1 < \theta < c_2) \geq 1 - \alpha$$

where θ is the parameter being estimated and $(1-\alpha)$ is the confidence probability. Consider the problem graphically, where $f_1(\theta)$ and $f_2(\theta)$ are drawn so that $c_1 \in f_1(\theta)$, $c_2 \in f_2(\theta)$, $\Pr[f_1(\theta) < \theta < f_2(\theta)] = 1$ and $\hat{\theta}$ is a sufficient estimator of θ obtained from the data.



The line segment (c_1, c_2) will intersect $\theta = \theta_o$ (true value of parameter) if and only if $f_a \leq \hat{\theta}_o \leq f_b$. This is to say that $\Pr(f_a \leq \hat{\theta} \leq f_b) = 1 - \alpha$; this is also the probability that (c_1, c_2) includes θ_o.

It will be shown subsequently that we can be 99% confident, $(1-\alpha = .99)$, that F^* and t^* will be between the calculated upper and lower limits of the confidence interval. In the paired difference test, the probability of accepting H_o:

(22a) $$Pr\left[t[(n-1);\alpha/2] < t^*_1 < t[(n-1);(1-\alpha/2)]\right] = 1-\alpha$$

where $t^*_1 = \dfrac{\bar{d}_i\sqrt{n}}{S_{d_i}}$. From this equation and that of (16) we obtain the confidence interval:

(23a) $$\left[\bar{d}_i \pm t[(n-1);\alpha/2]\left(\frac{S_{\bar{d}_i}}{\sqrt{n}}\right)\right]$$

This means that we can be $100(1-\alpha)$ % confident that this interval contains $\bar{d}_i = 0$ under H_o. (The critical region is: $t^*[(n-1);\alpha/2] > t^* > t^*[(n-1);(1-\alpha/2)]$). Likewise, for the paired comparison test, the probability of accepting H_o is:

(22b) $$Pr\left[t[n';\alpha/2] < t^*_2 < t[n';(1-\alpha/2)]\right] = 1-\alpha$$

where $$t^*_2 = \frac{\bar{x}_o - \bar{y}_v}{S_p\sqrt{2/n}}$$

This equation and equation (17) lead to the confidence interval:[2]

(23b) $\left[(\bar{x}_o - \bar{y}_v) \pm t[n, \alpha/2]\, S_p\sqrt{2/n}\right]$

The F test for the variances as given in equation (7) can now be rewritten:

(24) $$Pr\left[\left(\frac{S_o^2}{S_v^2}\right) F[(n-1),\ (n-1);\ \alpha/2] < \frac{\sigma_o^2}{\sigma_v^2} < \left(\frac{S_o^2}{S_v^2}\right) F[(n-1), (n-1); (1-\alpha/2)]\right] = 1 - \alpha$$

where the confidence interval is:[1]

(25) $$\left[\left(\frac{S_o^2}{S_v^2}\right) F[(n-1), (n-1); \alpha/2]; \left(\frac{S_o^2}{S_v^2}\right) F[(n-1), (n-1), (1-\alpha/2)]\right]$$

The probability that $\left(\frac{S_o^2}{S_v^2}\right)$ will be contained within this interval, under H_o is $(1-\alpha)$.

CONCLUSIONS:

For E_2, the paired comparison test, the computed value of t used to test the hypothesis $\mu_o = \mu_1$ at 10:00 is 0.0174, (see Figure (16)). The corresponding confidence interval for the same time slot is: $I = \{-0.9405, 0.9293\}$. Note that the interval is quite wide considering the small difference between the sample means (for 10:00 hours). Examination of the data, Figures 8, 9, 10, 11, show a marked consistency with this conclusion. The VI measurements (fixed station) are generally smaller than the corresponding value for the OI (mobile station) measurements. Their differences are recorded as: $d_1 = f_o - f_v = x_o - y_v$.

The paired comparison test, E_2, Figures 16, 17, 18, 19, requires that two random samples be independent. The data shows, however, that a pair of measurements for VI and OI for any particular time are of approximately the same magnitude. In other words, the variance within the blocks is small compared to the variance between the blocks. The following data from Figure (9) were taken at 11:30 hours:

Day	x_o	Δx_o	y_v	Δy_v	$x_o - y_v$
1	10.80		10.80		0
2	10.00	0.80	9.80	1.00	0.20
3	9.00	1.00	9.20	0.60	0.20
4	9.50	0.50	9.50	0.30	0
5	10.00	0.50	10.00	0.50	0
6	10.50	0.50	10.50	0.50	0
7	9.50	1.00	9.60	0.90	0.10
8	9.50	0	9.50	0.10	0
9	10.50	1.00	10.40	0.90	0.10

In other words Δx_o, $\Delta y_v > x_o - y_v$; as a result of the homogeneity within the blocks, the new experimental design, that of Paired Differences, utilizes the nine difference measurements, di, per time slot to test the hypothesis:

$$H_o: \quad \mu_V = \mu_I \quad \text{vs.} \qquad H_1: \quad \mu_V \neq \mu_I$$

with the t test and confidence intervals as shown in equations 12a, 13a, 14, 23a. This statistical design is a simple example of a randomized block design. The test is commonly called a paired difference test. It is emphasized that the pairing was part of the planning of the experiments, and was not done after the data were collected. Each of the blocks consists of the two observations x_o and y_v for the same day at a specific time. (See Figures 8, 9, 10, and 11.)

By comparing the computed confidence intervals for the 85 time slots for the paired difference model with those of the unpaired model, see Figures 14, 15, 18, 19 we see a decided gain in information favoring the randomized block design. The gain of information is reflected in the difference in the width of the confidence intervals. Again using data at 10:00 hours, in Figure 20, the interval for the paired comparison test $I_{pc}=(-.9405, .9293)$. The interval for the paired difference test $I_{pd}=(-.1859, .1961)$, and $I_{pd} < I_{pc}$.

The I_{pd_i}, $i=1, \ldots, 85$ are much narrower as a result of blocking in this experiment. Figures 20, 21, 22, and 23 show the comparison of the confidence limits for both methods, as well as a large reduction in the standard deviation S_d as compared to the pooled standard deviation, S_p, of the unpaired observations. Variances are presented graphically in Figures 24, 25.

No appreciable difference exists between the data of the fixed and mobile ionosondes for a distance of 60 Km. The data of the fixed terminal is very nearly identical to those of the mobile terminal for this distance. This means that only one terminal is needed at this distance to provide useful ionospheric data under these given conditions. The result bears out the expectation. Experimentation is planned for investigating critical frequencies at distances beyond 60 Km, (up to 500 Km), to determine the distance within 500 Km where the conclusion becomes invalid. This would provide insight as to an extreme distance limit for the usefulness of vertical incidence ionospheric soundings with respect to critical frequency.

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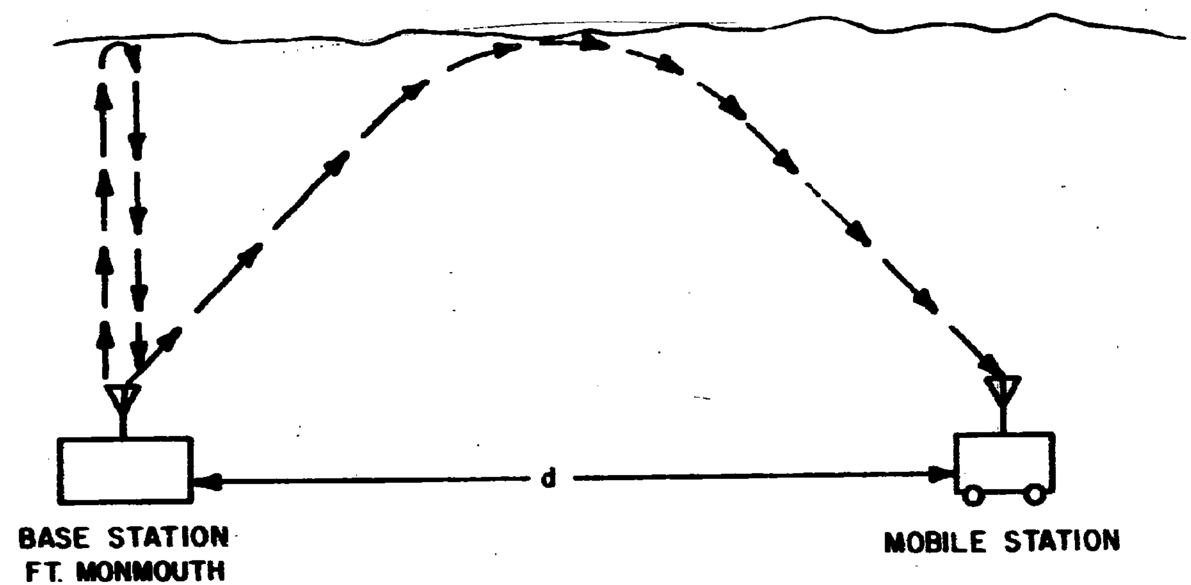


FIELD TESTS FOR A NEAR-REAL TIME
IONOSPHERIC FORCASTING SCHEME

FIG. 1

RAW IONOSONDE DATA

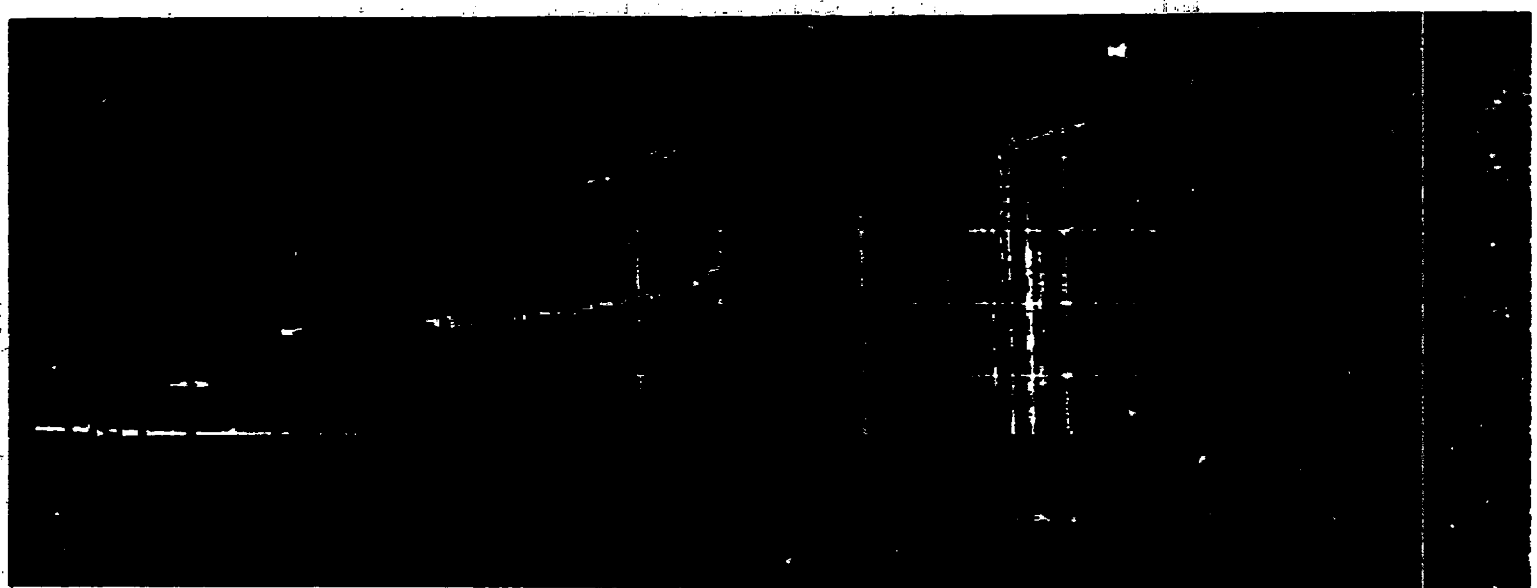


FIG. 2

RAW IONOSONDE DATA

FIG. 3

RAW IONOSONDE DATA

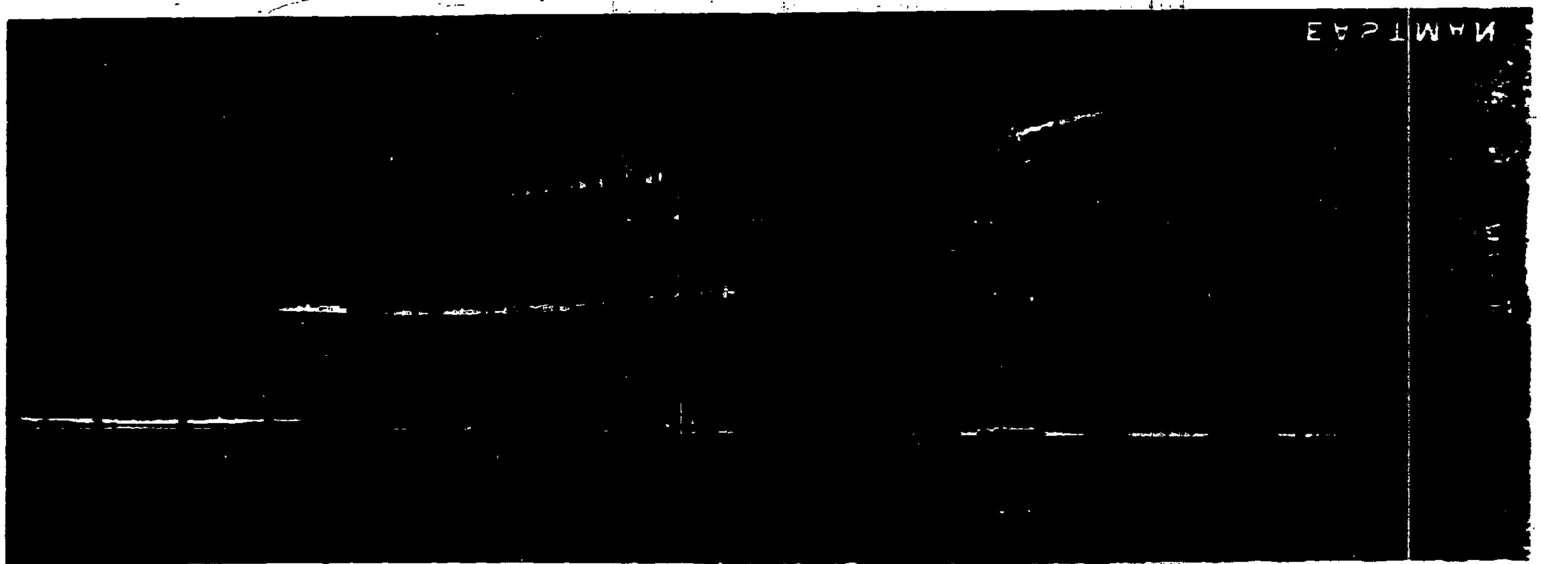


FIG. 4

RAW IONOSONDE DATA

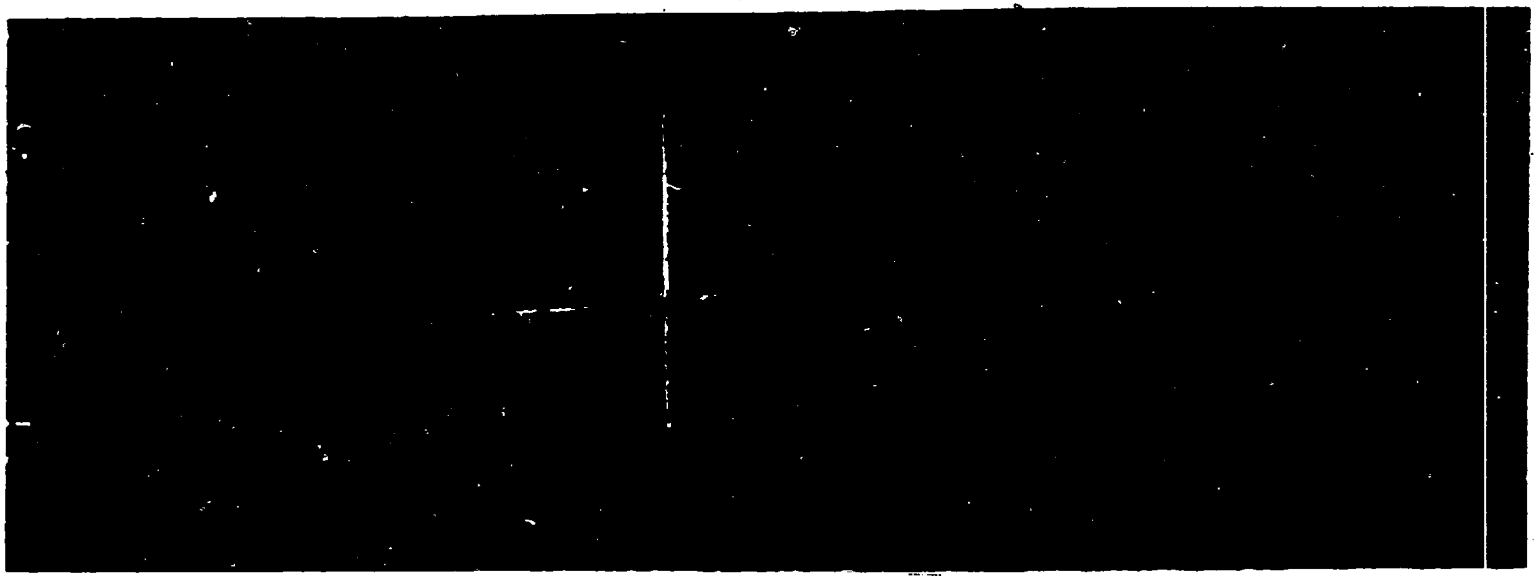


FIG. 5

TIME OF DAY	ORDERED DAYS OF EXPERIMENTATION									EXPERIMENTAL MEAN	STANDARD DEVIATION
	1	2	3	4	5	6	7	8	9		
0530	$f_{1,1}$	$f_{1,2}$	$f_{1,3}$	$f_{1,4}$	$f_{1,5}$	$f_{1,6}$	$f_{1,7}$	$f_{1,8}$	$f_{1,9}$	$\overline{f_1}$	S_1
0540	$f_{2,1}$	$f_{2,2}$	$f_{2,3}$	$f_{2,4}$	$f_{2,5}$	$f_{2,6}$	$f_{2,7}$	$f_{2,8}$	$f_{2,9}$	$\overline{f_2}$	S_2
0550	$f_{3,1}$	$f_{3,2}$	$f_{3,3}$	$f_{3,4}$	$f_{3,5}$	$f_{3,6}$	$f_{3,7}$	$f_{3,8}$	$f_{3,9}$	$\overline{f_3}$	S_3
0600	$f_{4,1}$	$f_{4,2}$	$f_{4,3}$	$f_{4,4}$	$f_{4,5}$	$f_{4,6}$	$f_{4,7}$	$f_{4,8}$	$f_{4,9}$	$\overline{f_4}$	S_4
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1910	$f_{83,1}$	$f_{83,2}$	$f_{83,3}$	$f_{83,4}$	$f_{83,5}$	$f_{83,6}$	$f_{83,7}$	$f_{83,8}$	$f_{83,9}$	$\overline{f_{83}}$	S_{83}
1920	$f_{84,1}$	$f_{84,2}$	$f_{84,3}$	$f_{84,4}$	$f_{84,5}$	$f_{84,6}$	$f_{84,7}$	$f_{84,8}$	$f_{84,9}$	$\overline{f_{84}}$	S_{84}
1930	$f_{85,1}$	$f_{85,2}$	$f_{85,3}$	$f_{85,4}$	$f_{85,5}$	$f_{85,6}$	$f_{85,7}$	$f_{85,8}$	$f_{85,9}$	$\overline{f_{85}}$	S_{85}

1. $\overline{f}_i = \frac{1}{n} \sum_{j=1}^{9} f_{ij} \qquad i = 1, 2, \quad 85 \; ; \; j = 1, 2, \quad , 9$

2. $S_i^2 = \frac{1}{n-1} \sum_{j=1}^{9} (f_{ij} - \overline{f}_i)^2 \quad i = 1, 2, \quad 85 \; ; \; j = 1, 2, \quad , 9$

FIG. 6 PRELIMINARY DATA SCHEME – PAIRED COMARISON

TIME OF DAY	ORDERED DAYS OF EXPERIMENTATION									EXPERIMENTAL MEAN	STANDARD DEVIATION
	1	2	3	4	5	6	7	8	9		
0530	$d_{1,1}$	$d_{1,2}$	$d_{1,3}$	$d_{1,4}$	$d_{1,5}$	$d_{1,6}$	$d_{1,7}$	$d_{1,8}$	$d_{1,9}$	$\bar{d}_1$	S_{d1}
0540	$d_{2,1}$	$d_{2,2}$	$d_{2,3}$	$d_{2,4}$	$d_{2,5}$	$d_{2,6}$	$d_{2,7}$	$d_{2,8}$	$d_{2,9}$	$\bar{d}_2$	S_{d2}
0550	$d_{3,1}$	$d_{3,2}$	$d_{3,3}$	$d_{3,4}$	$d_{3,5}$	$d_{3,6}$	$d_{3,7}$	$d_{3,8}$	$d_{3,9}$	$\bar{d}_3$	S_{d3}
0600	$d_{4,1}$	$d_{4,2}$	$d_{4,3}$	$d_{4,4}$	$d_{4,5}$	$d_{4,6}$	$d_{4,7}$	$d_{4,8}$	$d_{4,9}$	$\bar{d}_4$	S_{d4}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1910	$d_{83,1}$	$d_{83,2}$	$d_{83,3}$	$d_{83,4}$	$d_{83,5}$	$d_{83,6}$	$d_{83,7}$	$d_{83,8}$	$d_{83,9}$	$\bar{d}_{83}$	S_{d83}
1920	$d_{84,1}$	$d_{84,2}$	$d_{84,3}$	$d_{84,4}$	$d_{84,5}$	$d_{84,6}$	$d_{84,7}$	$d_{84,8}$	$d_{84,9}$	$\bar{d}_{84}$	S_{d84}
1930	$d_{85,1}$	$d_{85,2}$	$d_{85,3}$	$d_{85,4}$	$d_{85,5}$	$d_{85,6}$	$d_{85,7}$	$d_{85,8}$	$d_{85,9}$	$\bar{d}_{85}$	S_{d85}

1. $d_{ij} = (f_{oi} - f_{vi})_{ij} \quad i = 1,2,3, \cdots 85 \; ; \; j = 1,2, \cdots 9$
2. $\bar{d}_i = \frac{1}{n} \sum_{j=1}^{9} d_{ij} \quad i = 1,2,3 \cdots 85$
3. $S^2_{di} = \frac{1}{n-1} \sum_{j=1}^{9} (d_{ij} - \bar{d}_i)^2 \quad i = 1,2,3 \cdots 85 \; ; \; j = 1,2, \cdots 9$

FIG. 7 PRELIMINARY DATA SCHEME-PAIRED DIFFERENCE

REDUCED IONOSONDE DATA

TIME OF DAY (24 HOUR CLOCK)	CRITICAL FREQUENCY VERTICAL INCIDENCE	CRITICAL FREQUENCY OBLIQUE INCIDENCE	DIFFERENCE BETWEEN VERTICAL AND OBLIQUE
530.	3.850	3.950	0.100
530.	3.500	3.600	0.100
530.	3.700	3.700	0.000
530.	4.700	4.700	0.00
530.	3.600	3.800	0.200
530.	3.850	3.950	0.100
530.	3.850	3.950	0.100
530.	3.850	3.950	0.100
530.	3.850	3.950	0.100
540.	3.400	3.500	0.100
540.	3.500	3.600	0.100
540.	3.000	4.600	1.600
540.	4.700	3.600	-1.100
540.	3.600	3.500	-0.100
540.	3.400	2.600	-0.800
540.	2.500	3.500	1.000
540.	3.200	3.500	0.300
540.	3.350	3.200	-0.150
550.	3.500	3.500	0.000
550.	3.700	3.500	-0.200
550.	3.500	3.400	-0.100
550.	4.600	4.600	0.000
550.	3.600	3.600	0.000
550.	3.500	3.500	0.000
550.	2.550	2.600	0.050
550.	3.550	3.500	-0.050
550.	3.100	3.200	0.100
600.	3.300	3.500	0.200
600.	2.800	3.500	0.700
600.	3.400	3.400	0.000
600.	4.300	4.300	0.000
600.	3.600	3.600	0.000
600.	3.300	3.500	0.200
600.	2.500	3.100	0.600
600.	3.300	3.500	0.200
600.	3.100	3.200	0.100
610.	3.400	3.300	-0.100
610.	3.000	2.800	-0.200
610.	3.400	3.400	0.000
610.	4.200	4.000	-0.200
610.	3.700	3.700	0.000
610.	3.400	3.300	-0.100
610.	3.000	2.600	-0.400
610.	3.400	3.300	-0.100
610.	3.200	3.100	-0.100
620.	3.500	3.100	-0.400
620.	3.000	2.700	-0.300
620.	3.300	3.300	0.00
620.	4.300	3.100	-1.200
620.	3.600	3.700	0.100
620.	3.500	3.100	-0.400

FIG. 8

REDUCED IONOSONDE DATA

TIME OF DAY (24 HOUR CLOCK)	CRITICAL FREQUENCY VERTICAL INCIDENCE	CRITICAL FREQUENCY OBLIQUE INCIDENCE	DIFFERENCE BETWEEN VERTICAL AND OBLIQUE
1110.	10.500	10.800	-0.300
1110.	9.400	9.600	-0.200
1110.	9.000	8.600	-0.400
1110.	10.500	10.200	-0.300
1110.	10.700	10.800	0.100
1110.	11.200	11.200	0.000
1110.	9.500	9.500	0.000
1110.	9.200	9.000	-0.200
1110.	10.200	10.400	0.200
1120.	10.800	10.800	0.000
1120.	10.200	9.200	-1.000
1120.	9.400	9.600	0.200
1120.	10.000	10.200	0.200
1120.	10.400	10.500	0.100
1120.	10.800	11.000	0.200
1120.	9.700	9.500	-0.200
1120.	9.500	9.500	0.000
1120.	10.200	10.000	-0.200
1130.	10.800	10.800	0.000
1130.	9.800	10.000	0.200
1130.	9.200	9.000	-0.200
1130.	9.500	9.500	0.000
1130.	10.000	10.000	0.000
1130.	10.500	10.500	0.000
1130.	9.600	9.500	-0.100
1130.	9.500	9.500	0.000
1130.	10.400	10.500	0.100
1140.	10.400	10.800	0.200
1140.	9.800	9.800	0.000
1140.	8.800	8.800	0.000
1140.	9.300	9.300	0.000
1140.	9.850	10.200	0.350
1140.	10.800	10.800	0.000
1140.	9.500	9.600	0.100
1140.	9.600	9.700	0.100
1140.	10.300	10.400	0.100
1150.	10.200	10.200	0.000
1150.	9.700	9.800	0.100
1150.	9.200	9.000	-0.200
1150.	9.300	9.500	0.200
1150.	10.500	10.500	0.000
1150.	10.800	10.700	-0.100
1150.	9.500	9.300	-0.200
1150.	9.500	9.500	0.000
1150.	10.200	10.200	0.000
1200.	10.000	8.900	-1.100
1200.	9.200	9.200	0.000
1200.	8.800	9.200	0.400
1200.	10.200	9.400	-0.800
1200.	10.400	10.500	0.100
1200.	10.800	10.800	0.000

FIG. 9

REDUCED IONOSONDE DATA

TIME OF DAY (24 HOUR CLOCK)	CRITICAL FREQUENCY VERTICAL INCIDENCE	CRITICAL FREQUENCY OBLIQUE INCIDENCE	DIFFERENCE BETWEEN VERTICAL AND OBLIQUE
1450.	9.200	9.000	-0.200
1450.	9.400	9.400	0.000
1450.	10.800	10.800	0.000
1500.	10.200	10.200	0.000
1500.	9.000	9.000	0.000
1500.	9.900	10.000	0.100
1500.	9.000	9.000	0.000
1500.	11.000	11.400	0.400
1500.	11.000	11.000	0.000
1500.	9.600	9.400	-0.200
1500.	9.000	9.600	0.600
1500.	10.600	10.600	0.000
1510.	9.800	10.400	0.400
1510.	9.800	9.600	-0.200
1510.	10.850	10.850	0.000
1510.	9.000	9.000	0.000
1510.	10.800	11.000	0.200
1510.	11.200	11.200	0.000
1510.	9.400	9.000	-0.600
1510.	9.800	9.800	0.000
1510.	10.400	10.400	0.000
1520.	10.000	10.000	0.000
1520.	9.000	9.600	0.600
1520.	10.000	10.000	0.000
1520.	9.400	9.000	-0.400
1520.	11.400	10.400	-1.000
1520.	11.200	11.200	0.000
1520.	9.200	9.500	0.300
1520.	10.000	10.000	0.000
1520.	10.200	10.400	0.200
1530.	9.800	10.000	0.200
1530.	9.000	9.600	0.600
1530.	9.950	9.950	0.000
1530.	9.400	9.000	-0.400
1530.	10.300	10.600	0.300
1530.	11.400	11.400	0.000
1530.	9.400	9.100	-0.300
1530.	10.000	10.000	0.000
1530.	9.800	10.000	0.200
1540.	9.600	9.600	0.000
1540.	9.800	9.000	-0.800
1540.	10.200	10.000	-0.200
1540.	9.600	9.400	-0.200
1540.	10.800	10.600	-0.200
1540.	11.400	11.200	-0.200
1540.	10.200	10.000	-0.200
1540.	10.000	9.800	-0.200
1540.	10.200	10.200	0.000
1550.	9.800	9.800	0.000
1550.	9.400	9.600	0.000
1550.	9.900	9.900	0.000

FIG. 10

REDUCED IONOSONDE DATA

TIME OF DAY (24 - HR CLOCK)	CRITICAL FREQUENCY VERTICAL INCIDENCE	CRITICAL FREQUENCY OBLIQUE INCIDENCE	DIFFERENCE BETWEEN VERTICAL AND OBLIQUE
1840.	6.400	6.500	-0.100
1840.	8.600	8.800	-0.200
1840.	7.500	7.600	-0.100
1840.	7.700	7.500	-0.200
1840.	6.600	6.600	0.000
1840.	7.400	7.200	-0.200
1850.	7.100	6.900	-0.200
1850.	6.000	6.200	0.200
1850.	7.000	6.400	-0.600
1850.	6.800	6.850	0.050
1850.	8.600	8.600	0.000
1850.	7.400	6.400	-1.000
1850.	7.300	7.400	0.100
1850.	6.400	6.400	0.000
1850.	7.200	7.200	0.000
1900.	6.800	6.800	0.000
1900.	5.700	5.700	0.000
1900.	6.500	6.600	0.100
1900.	6.500	6.800	0.200
1900.	8.200	8.400	0.200
1900.	7.500	7.400	-0.100
1900.	6.500	6.600	0.100
1900.	6.200	6.000	-0.200
1900.	7.000	7.000	0.000
1910.	6.450	6.450	0.000
1910.	5.300	5.400	0.100
1910.	5.600	5.600	0.000
1910.	6.600	6.600	0.000
1910.	7.600	7.800	0.200
1910.	7.400	7.400	0.000
1910.	6.500	6.000	-0.500
1910.	6.200	6.200	0.000
1910.	6.450	6.450	0.000
1920.	6.100	6.050	-0.050
1920.	5.000	5.200	0.200
1920.	5.500	5.400	-0.100
1920.	6.400	6.400	0.000
1920.	7.000	6.800	-0.200
1920.	6.800	6.600	-0.200
1920.	6.000	6.000	0.000
1920.	5.900	5.900	0.000
1920.	6.100	6.050	-0.050
1930.	5.600	5.900	0.300
1930.	4.800	4.800	0.000
1930.	5.400	5.000	-0.400
1930.	5.800	5.900	0.100
1930.	6.200	6.500	0.300
1930.	5.600	7.500	1.900
1930.	5.800	5.800	0.000
1930.	5.600	5.900	0.300
1930.	5.600	5.900	0.300

FIG. 11

PRELIMINARY CALCULATIONS

TIME-OF DAY	MEAN CRITICAL FREQUENCY: VI=VERTICAL INCIDENCE	MEAN CRITICAL FREQUENCY: OI=OBLIQUE INCIDENCE	MEAN OF DIFFERENCES BETWEEN OI AND VI CRITICALS	ESTIMATE OF VARIANCE: VI	ESTIMATE OF VARIANCE: OI	ESTIMATE OF VARIANCE: OF DIFFERENCES	STANDARD DEVIATION: VI	STANDARD DEVIATION: OI	STANDARD DEVIATION: OF DIFFERENCES
530.	3.441	3.750	0.0889	0.116	0.096	0.004	0.341	0.310	0.060
540.	3.406	3.511	0.1056	0.344	0.266	0.682	0.587	0.516	0.826
550.	3.511	3.489	-0.0222	0.290	0.256	0.008	0.539	0.516	0.097
600.	3.289	3.511	0.2222	0.258	0.114	0.087	0.504	0.337	0.259
610.	3.411	3.278	-0.1333	0.136	0.179	0.015	0.369	0.424	0.122
620.	3.489	3.089	-0.4000	0.129	0.116	0.197	0.359	0.341	0.444
630.	3.111	3.189	0.0778	0.237	0.297	0.097	0.487	0.545	0.311
640.	3.417	3.378	-0.0389	0.050	0.102	0.059	0.224	0.319	0.229
650.	3.933	3.450	-0.4833	0.118	0.105	0.032	0.344	0.324	0.180
700.	4.328	4.211	-0.1167	0.062	0.104	0.104	0.249	0.322	0.322
710.	5.300	5.300	-0.0000	0.145	0.072	0.047	0.381	0.269	0.218
720.	5.422	5.433	0.0111	0.154	0.148	0.271	0.393	0.384	0.521
730.	5.878	5.922	0.0444	0.264	0.094	0.108	0.514	0.307	0.328
740.	6.511	6.478	-0.0333	0.096	0.087	0.013	0.310	0.295	0.112
750.	6.389	6.250	-0.1389	0.046	0.117	0.072	0.215	0.342	0.268
800.	7.289	7.111	-0.1778	0.109	0.109	0.099	0.330	0.330	0.315
810.	7.411	7.433	0.0222	0.079	0.068	0.018	0.280	0.260	0.120
820.	7.444	7.389	-0.0556	0.170	0.169	0.027	0.412	0.411	0.165
830.	7.378	7.511	0.1333	0.284	0.116	0.148	0.533	0.341	0.384
840.	7.700	7.700	-0.0000	0.225	0.270	0.025	0.474	0.520	0.158
850.	8.356	8.367	0.0111	0.225	0.145	0.026	0.475	0.381	0.162
900.	8.100	8.178	0.0778	0.183	0.349	0.042	0.427	0.591	0.205
910.	8.389	8.433	0.0444	0.479	0.487	0.169	0.692	0.698	0.410
920.	8.644	8.633	-0.0111	0.388	0.515	0.026	0.623	0.718	0.162
930.	8.933	8.956	0.0222	0.365	0.333	0.014	0.604	0.577	0.120
940.	9.022	9.156	0.1333	0.149	0.320	0.070	0.387	0.566	0.265
950.	9.250	9.211	-0.0389	0.300	0.374	0.027	0.548	0.611	0.165
1000.	9.328	9.333	0.0056	0.397	0.525	0.029	0.630	0.725	0.170
1010.	9.528	9.522	-0.0056	0.269	0.459	0.042	0.519	0.678	0.205
1020.	9.578	9.650	0.0722	0.212	0.207	0.027	0.460	0.456	0.164
1030.	9.600	9.522	-0.0778	0.128	0.112	0.012	0.357	0.335	0.109
1040.	9.789	9.650	-0.1389	0.264	0.362	0.115	0.513	0.602	0.339
1050.	9.933	9.922	-0.0111	0.535	0.447	0.018	0.731	0.669	0.135
1100.	9.989	9.967	-0.0222	0.634	0.770	0.029	0.796	0.877	0.172
1110.	10.367	10.311	-0.0556	0.545	0.786	0.055	0.738	0.887	0.235
1120.	10.133	10.033	-0.1000	0.233	0.403	0.135	0.482	0.634	0.367
1130.	9.922	9.922	0.0000	0.292	0.354	0.012	0.540	0.595	0.112
1140.	9.839	9.933	0.0944	0.407	0.459	0.014	0.638	0.676	0.118
1150.	9.628	9.556	-0.0722	0.319	0.333	0.017	0.565	0.577	0.132
1200.	9.667	9.522	-0.1444	0.498	0.472	0.235	0.705	0.687	0.485
1210.	9.622	9.550	-0.0722	0.432	0.460	0.017	0.657	0.578	0.130
1220.	9.456	9.567	-0.0889	0.495	0.460	0.056	0.704	0.678	0.237
1230.	9.567	9.511	0.0444	0.725	0.719	0.013	0.851	0.848	0.113
1240.	9.689	9.756	0.0667	0.811	0.928	0.160	0.901	0.963	0.400
1250.	9.600	9.733	0.1333	1.100	1.032	0.050	1.049	1.015	0.224

FIG. 12

PRELIMINARY CALCULATIONS

TIME OF DAY	[illegible] CRITICAL FRE[illegible]: [illegible]	[illegible]	MEAN OF DIFFERENCES BETWEEN DI AND VI CRITICALS	ESTIMATE OF VARIANCE: VI	DI	OF DIFFERENCES	STANDARD DEVIATION: VI	DI	OF DIFFERENCES
1300.	9.722	9.689	-0.0333	0.879	0.741	0.020	0.938	0.861	[illegible].141
1310.	9.178	9.711	-0.0667	0.934	1.[illegible]	0.030	0.967	1.003	[illegible].173
1320.	9.754	9.800	0.0888	1.218	1.230	0.028	1.104	1.109	0.167
1330.	10.[illegible]22	9.989	-0.1333	0.904	1.031	0.060	0.951	1.015	0.245
1340.	9.395	1[illegible].[illegible]22	0.0333	1.101	0.904	0.235	1.049	0.951	0.485
1350.	10.1[illegible]	1[illegible].[illegible]22	-0.0778	1.030	1.[illegible]64	0.139	1.015	1.032	0.373
1400.	10.372	1[illegible].422	0.0500	1.104	0.874	0.098	1.051	0.935	0.312
1410.	10.122	10.378	0.0556	0.779	0.694	0.038	0.883	0.846	0.194
1420.	10.3[illegible]	10.350	0.0500	0.870	0.918	0.039	0.933	0.958	0.198
1430.	10.[illegible]54	10.150	0.0044	0.603	0.717	0.089	0.776	0.847	0.298
1440.	10.172	1[illegible].150	-0.0222	0.564	0.678	0.054	0.751	0.823	0.233
1450.	10.1[illegible]	1[illegible].[illegible]50	-0.1000	0.418	0.657	0.030	0.786	0.811	0.173
1500.	9.222	1[illegible].[illegible]29	0.1000	0.689	0.734	0.060	0.830	0.857	0.245
1510.	10.[illegible]5[illegible]	1[illegible].[illegible]50	-0.0000	0.437	0.617	0.106	0.661	0.786	0.315
1520.	10.[illegible]47	1[illegible].[illegible]11	-0.0556	0.650	0.396	0.228	0.806	0.629	0.477
1530.	9.3[illegible]	9.961	0.0111	0.487	0.535	0.108	0.698	0.731	0.328
1540.	10.222	1[illegible].000	-0.2222	0.314	0.410	0.054	0.561	0.640	0.233
1550.	9.[illegible]	9.911	0.0111	0.270	0.374	0.051	0.520	0.611	0.226
1600.	9.722	9.650	-0.0722	0.299	0.348	0.029	0.547	0.589	0.172
1610.	9.622	9.650	0.0278	0.274	0.327	0.054	0.524	0.572	0.233
1620.	9.344	9.411	0.0667	0.268	0.276	0.023	0.517	0.525	0.150
1630.	9.156	9.100	-0.0556	0.628	0.525	0.098	0.792	0.725	0.313
1640.	8.922	8.933	-0.0[illegible]	0.769	0.525	0.124	0.877	0.725	0.352
1650.	8.750	8.678	-0.0722	0.808	0.897	0.042	0.899	0.947	0.205
1700.	8.55[illegible]	8.489	-0.0611	0.898	0.879	0.092	0.947	0.937	0.304
1710.	8.249	8.328	0.0389	0.494	0.612	0.042	0.703	0.782	0.206
1720.	8.200	8.25[illegible]	0.0500	0.418	0.370	0.060	0.646	0.608	0.245
1730.	8.[illegible]11	8.039	0.0278	0.298	0.370	0.017	0.546	0.608	0.130
1740.	7.3[illegible]	7.980	-0.0111	0.526	0.489	0.011	0.726	0.699	0.105
1750.	7.922	7.072	-0.0500	0.752	0.934	0.045	0.867	0.967	0.212
1800.	7.45[illegible]	7.750	-0.1000	1.093	0.652	0.110	1.045	0.808	0.332
1810.	7.741	7.761	-0.0000	0.737	0.852	0.017	0.859	0.923	0.132
1820.	7.636	7.700	0.0611	0.675	0.893	0.045	0.821	0.945	0.212
1830.	7.322	7.378	0.0556	0.564	0.572	0.008	0.751	0.756	0.088
1840.	7.289	7.189	-0.1000	0.601	0.509	0.007	0.775	0.713	0.087
1850.	7.[illegible]89	6.922	-0.1667	0.524	0.559	0.15[illegible]	0.724	0.748	0.387
1900.	6.778	6.411	0.0333	0.534	0.611	0.014	0.731	0.782	0.132
1910.	6.[illegible]56	6.433	-0.0222	0.585	0.604	0.037	0.738	0.777	0.192
1920.	6.[illegible]95	6.[illegible]44	-0.0444	0.376	0.270	0.015	0.613	0.519	0.121
1930.	5.[illegible]	5.911	0.3111	0.140	0.519	[illegible]	0.374	0.787	0.639

FIG. 13

PAIRED DIFFERENCE TEST

TIME OF DAY	CONFIDENCE INTERVALS OF PAIRED DIFFERENCES			MEAN VALUES OF DIFFERENCES BETWEEN VI AND CI	T-TEST	STANDARD DEVIATION OF DIFFERENCES BETWEEN VI AND CI
	LOWER LIMIT	MEAN VALUES OF DIFFERENCES	UPPER LIMIT			
530.	0.0217	0.0889	0.1561	0.089	4.1836	.08609252
540.	-0.8127	0.1056	1.0288	0.106	0.3016	.82554494
550.	-0.1195	-0.0222	0.0751	-0.022	0.7224	.08700255
600.	-0.0671	0.2222	0.5116	0.222	2.4293	.25873424
610.	-0.2703	-0.1333	0.0036	-0.133	3.0792	.12247449
620.	-0.8070	-0.4000	0.0070	-0.400	2.5458	.44441972
630.	-0.2704	0.0778	0.4260	0.078	0.7065	.31135903
640.	-0.2948	-0.0389	0.2170	-0.039	0.4807	.27082557
650.	-0.3849	-0.1833	0.0183	-0.183	2.8744	.19627756
700.	-0.4769	-0.1167	0.2436	-0.117	1.0295	.37210287
710.	-0.2437	-0.0000	0.2437	-0.000	0.0000	.21794495
720.	-0.5712	0.0111	0.5934	0.011	0.0694	.57068331
730.	-0.3227	0.0444	0.4116	0.044	0.3829	.32829524
740.	-0.1584	-0.0333	0.0917	-0.033	0.9433	.11180340
750.	-0.4384	-0.1389	0.1606	-0.139	1.4667	.26783598
800.	-0.5304	-0.1778	0.1749	-0.178	1.5945	.31534813
810.	-0.1122	0.0222	0.1566	0.022	0.5230	.12018504
820.	-0.2398	-0.0556	0.1287	-0.056	0.9536	.15478160
830.	-0.2962	0.1333	0.5628	0.133	0.9819	.38405729
840.	-0.1768	-0.0000	0.1768	-0.000	0.0000	.15811388
850.	-0.1694	0.0111	0.1916	0.011	0.1945	.16158933
900.	-0.1513	0.0778	0.3068	0.078	1.0741	.20480343
910.	-0.4134	0.0444	0.5025	0.044	0.3259	.40960406
920.	-0.1918	-0.0111	0.1696	-0.011	0.1945	.16158933
930.	-0.1122	0.0222	0.1566	0.022	0.5230	.12018504
940.	-0.1625	0.1333	0.4292	0.133	1.4254	.26457513
950.	-0.2239	-0.0389	0.1461	-0.039	0.4650	.16541194
1000.	-0.1850	0.0056	0.1961	0.006	0.0922	.170375[illegible]
1010.	-0.2300	-0.0056	0.2189	-0.006	0.0753	.20069324
1020.	-0.1113	0.0722	0.2556	0.072	1.2445	.16414763
1030.	-0.2000	-0.0778	0.0444	-0.078	2.0129	.1[illegible]929[illegible]
1040.	-0.5179	-0.1389	0.2401	-0.139	1.1591	.33691164
1050.	-0.1637	-0.0111	0.1415	-0.011	0.2304	.13842255
1100.	-0.2141	-0.0222	0.1697	-0.022	0.3663	.17159364
1110.	-0.3185	-0.0556	0.2074	-0.056	0.6693	.23511227
1120.	-0.5109	-0.1000	0.3109	-0.100	0.7498	.34742334
1130.	-0.1250	0.0000	0.1250	0.000	0.0000	.111[illegible]
1140.	-0.0380	0.0944	0.2269	0.094	2.2554	.1164[illegible]
1150.	-0.1678	-0.0222	0.1234	-0.022	0.4829	.1301[illegible]
1200.	-0.6969	-0.1444	0.3990	-0.144	0.8423	.495[illegible]
1210.	-0.1678	-0.0222	0.1234	-0.022	0.4829	.13017[illegible]
1220.	-0.3539	-0.0889	0.1761	-0.089	1.0414	.230[illegible]
1230.	-0.0820	0.0444	0.1709	0.044	1.1121	.113[illegible]
1240.	-0.3807	0.0667	0.5140	0.067	0.4714	.40000000
1250.	-0.1167	0.1333	0.3834	0.133	1.5955	.2234[illegible]

FIG. 14

PAIRED DIFFERENCE TEST

ITEM NO.	RANGE OF T TEST VALUES OF PAIRED DIFFERENCES LOWER LIMIT	MEAN VALUE OF DIFFERENCES	UPPER LIMIT	MEAN VALUES OF DIFFERENCES BETWEEN VI AND CI	T-TEST	STANDARD DEVIATION OF DIFFERENCES BETWEEN VI AND CI
[illegible]	-.[illegible]	-.[illegible]	[illegible]	-0.033	[illegible]	.14142136
1120.	-.2[illegible]	-.[illegible]	[illegible]	-0.047	[illegible]	.17320508
1130.	-.1[illegible]	.0[illegible]	[illegible]	0.044	[illegible]	.1[illegible]
1330.	-.4[illegible]73	-0.1333	0.1476	-0.133	1.5396	.24494897
1340.	-.5[illegible]94	0.0333	0.5755	0.033	0.[illegible]5	.48476799
1350.	-.4[illegible]5	-0.0778	0.3398	-0.078	0.5891	.37342201
1400.	-.[illegible]97	0.0500	0.3992	0.050	0.4529	.31224990
1410.	-.1515	0.0556	0.2729	0.056	0.4085	.19436506
1420.	-.1444	0.0500	0.2656	0.050	0.7333	.19364917
1430.	-.2187	0.0286	0.4291	0.094	0.4953	.29037523
1440.	-.2432	-0.0222	0.2389	-0.022	0.2694	.23333333
1450.	-.2737	-0.1000	0.0937	-0.100	1.4330	.17320508
1500.	-.1739	0.1000	0.3739	0.100	1.1547	.24494897
1510.	-.3534	-0.0000	0.3536	-0.000	0.0000	.31622777
1520.	-.5891	-0.0556	0.4782	-0.056	0.3292	.47726070
1530.	-.3532	0.0111	0.3754	0.011	0.1045	.32574700
1540.	-.4832	-0.2222	0.0387	-0.222	2.4937	.23333333
1550.	-.2417	0.0111	0.2639	0.011	0.1390	.22607767
1600.	-.2441	-0.0722	0.1197	-0.072	1.1905	.17159384
1610.	-.2332	0.0278	0.2887	0.028	0.3367	.23333333
1620.	-.1011	0.0667	0.2344	0.067	1.2571	.15000000
1630.	-.4053	-0.0556	0.2941	-0.056	0.5025	.31269438
1640.	-.4821	-0.0889	0.3043	-0.089	0.7151	.35158372
1650.	-.3013	-0.0722	0.1568	-0.072	0.9974	.20480343
1700.	-.4010	-0.0611	0.2788	-0.061	0.5687	.30390971
1710.	-0.1913	0.0389	0.2691	0.039	0.5384	.20501815
1720.	-0.2239	0.0500	0.3239	0.050	0.5774	.24494897
1730.	-0.1178	0.0278	0.1734	0.028	0.6036	.13017083
1740.	-0.1290	-0.0111	0.1068	-0.011	0.2981	.10540926
1750.	-0.2872	-0.0500	0.1872	-0.050	0.6667	.21213203
1800.	-0.4709	-0.1000	0.2709	-0.100	0.9528	.33166248
1810.	-0.1479	-0.0000	0.1479	-0.000	0.0000	.13228757
1820.	-0.1758	0.0611	0.2980	0.061	0.9151	.21180442
1830.	-0.0431	0.0556	0.1542	0.056	1.7917	.09819171
1840.	-0.1969	-0.1000	0.0031	-0.100	3.2660	.08660254
1850.	-0.5398	-0.1667	0.2665	-0.167	1.2172	.38729833
1900.	-.1146	0.0333	0.1813	0.033	0.7[illegible]27	.13228757
1910.	-.2372	-0.0222	0.1927	-0.022	0.3270	.19220938
1920.	-0.1798	-0.0444	0.0909	-0.044	1.[illegible]3[illegible]5	.12104866
1930.	-0.4038	0.3111	1.0260	0.311	1.3[illegible]46	.63922696

FIG. 15

PAIRED COMPARISON TEST

TIME OF DAY	ESTIMATED VARIANCES OF CRITICAL FREQUENCIES: VERTICAL INCIDENCE	OBLIQUE INCIDENCE	PAIRED VARIANCE	MEAN CRITICAL FREQUENCIES: OBLIQUE INCIDENCE	VERTICAL INCIDENCE	DIFFERENCE	F RATIO OF VARIANCES	T-TEST
530.	0.1141	0.3952	0.1062	3.861	3.950	-0.089	1.2043	0.5787
540.	0.3441	0.2661	0.3051	3.406	3.511	-0.106	1.2925	0.4054
550.	0.2905	0.2641	0.2783	3.511	3.489	0.022	1.0916	0.0894
600.	0.2536	0.1136	0.1836	3.289	3.511	-0.222	2.2323	1.1001
610.	0.1361	0.1794	0.1578	3.411	3.278	0.133	0.7585	0.7121
620.	0.1286	0.1161	0.1224	3.489	3.069	0.420	1.1077	2.4257
630.	0.2367	0.2967	0.2667	3.111	3.189	-0.078	0.7978	0.3195
640.	0.0500	0.1019	0.0760	3.417	3.378	0.039	0.4905	0.2993
650.	0.1181	0.1050	0.1116	3.833	3.650	0.183	1.1250	1.1644
700.	0.0619	0.1036	0.0828	4.328	4.211	0.117	0.5979	0.8652
710.	0.1451	0.0725	0.1087	5.000	5.000	0.000	2.0000	0.0000
720.	0.1544	0.1475	0.1519	5.822	5.833	-0.011	1.0471	0.0607
730.	0.2644	0.0944	0.1794	5.878	5.922	-0.044	2.8000	0.2220
740.	0.0961	0.0869	0.0915	6.511	6.478	0.033	1.1054	0.2337
750.	0.0461	0.1169	0.0815	6.989	6.850	0.139	0.3945	1.0321
800.	0.1085	0.1086	0.1086	7.289	7.111	0.178	1.0000	1.1443
810.	0.0784	0.0675	0.0731	7.411	7.433	-0.022	1.1645	0.1744
820.	0.1697	0.1686	0.1691	7.444	7.389	0.056	1.0062	0.2886
830.	0.2844	0.1161	0.2003	7.378	7.511	-0.133	2.4498	0.6320
840.	0.2251	0.2700	0.2475	7.700	7.700	0.000	0.8333	0.0000
850.	0.2253	0.1451	0.1851	8.056	8.067	-0.011	1.5536	0.0548
900.	0.1825	0.3494	0.2460	8.100	8.178	-0.078	0.5223	0.3199
910.	0.4785	0.4875	0.4831	8.389	8.433	-0.044	0.9816	0.1357
920.	0.3879	0.5152	0.4514	8.444	8.433	0.011	0.7530	0.0351
930.	0.3651	0.3329	0.3489	8.933	8.956	-0.022	1.0949	0.0796
940.	0.1494	0.3203	0.2349	9.022	9.156	-0.133	0.4666	0.5836
950.	0.3003	0.3736	0.3369	9.250	9.211	0.039	0.8030	0.1421
1000.	0.3963	0.5257	0.4610	9.328	9.333	-0.006	0.7551	0.0174
1010.	0.2694	0.4594	0.3644	9.528	9.522	0.006	0.5865	0.0195
1020.	0.2119	0.2075	0.2097	9.578	9.650	-0.072	1.0214	0.3345
1030.	0.1275	0.1119	0.1197	9.600	9.522	0.078	1.1390	0.4749
1040.	0.2636	0.3625	0.3131	9.789	9.650	0.139	0.7272	0.5246
1050.	0.5351	0.4469	0.4910	9.933	9.922	0.011	1.1970	[illegible]
1100.	0.5334	[illegible]	0.7018	9.989	9.957	0.022	[illegible]	[illegible]
[illegible]	0.545	[illegible]	0.6656	10.067	[illegible]	[illegible]	[illegible]	[illegible]
1120.	[illegible]	[illegible]	0.3175	10.133	[illegible]	[illegible]	[illegible]	[illegible]
1130.	[illegible]	0.3564	0.3232	9.922	9.922	[illegible]	[illegible]	[illegible]
1140.	[illegible]	[illegible]	0.4324	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1150.	[illegible]	[illegible]	0.3261	9.878	[illegible]	[illegible]	[illegible]	[illegible]
1200.	[illegible]	[illegible]	[illegible]	9.667	9.522	[illegible]	[illegible]	[illegible]
1210.	[illegible]	[illegible]	0.4860	9.622	[illegible]	[illegible]	[illegible]	[illegible]
1220.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1230.	0.725	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1240.	[illegible]	[illegible]	0.8694	9.689	[illegible]	[illegible]	[illegible]	0.1517
1250.	[illegible]	[illegible]	1.0450	[illegible]	[illegible]	[illegible]	[illegible]	.2741

FIG. 16

PAIRED COMPARISON TEST

[illegible]	ESTIMATE OF VARIANCES OF CRITICAL FREQUENCIES			MEAN CRITICAL FREQ.				
[illegible]	OBLIQUE INCIDENCE	VERTICAL INCIDENCE	PAIRED VARIANCE	OBLIQUE INCIDENCE	VERTICAL INCIDENCE	[illegible]	F RATIO OF VARIANCES	[illegible]
130	[illegible]	[illegible]	[illegible]	9.722	[illegible]	[illegible]	1.1947	[illegible]
131	0.2964	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.9248	[illegible]
132	[illegible]	[illegible]	1.2235	9.756	9.500	-0.044	0.9901	0.0352
133	0.9444	1.0312	0.9678	10.022	9.489	-0.133	0.8772	[illegible]
134	[illegible]	0.9044	1.0028	9.949	10.022	-0.033	1.2174	0.0708
135	[illegible]	1.1684	1.0477	10.100	10.022	0.078	0.9676	0.1612
140	[illegible]	0.9744	0.9494	10.372	10.422	-0.050	1.2030	0.1180
141	0.7774	0.9984	0.8369	10.322	10.378	-0.056	0.8714	0.1285
142	0.87	0.9175	0.8337	10.300	10.350	-0.050	0.9492	0.1122
143	0.5624	0.7175	0.6601	10.056	10.150	-0.094	0.8421	0.2466
144	0.5634	0.6775	0.6216	10.172	10.150	0.022	0.9331	0.0396
145	0.5175	0.5575	0.6375	10.150	10.050	0.100	0.9392	0.2657
150	0.4894	0.7344	0.7119	9.922	10.022	-0.100	0.8387	0.2514
151	0.4375	0.6175	0.5279	10.050	10.050	0.000	0.7085	0.0000
152	0.450	0.3961	0.5231	10.047	10.011	0.036	1.0013	0.1632
153	0.4875	0.5349	0.5112	9.950	9.961	-0.011	0.9115	0.0332
154	0.3184	0.4100	0.3527	10.222	10.000	0.222	0.7669	0.7833
155	0.2709	0.3734	0.3218	9.900	9.911	-0.011	0.7297	0.0415
160	0.2994	0.3475	0.3235	9.722	9.650	0.072	0.8617	0.2094
161	0.2743	0.3275	0.3010	9.422	9.450	-0.028	0.8350	0.1074
162	0.2674	0.2741	0.2719	9.344	9.411	-0.067	0.9699	0.2712
163	0.6274	0.5250	0.5764	9.156	9.100	0.056	1.1958	0.1597
164	0.7694	0.5750	0.6472	8.922	8.833	0.089	1.4636	0.2344
165	0.9075	0.8943	0.8922	8.750	8.678	0.072	0.9003	0.1660
170	0.8974	0.9786	0.8881	8.550	8.489	0.061	1.0215	0.1370
171	0.4935	0.6119	0.5528	8.289	8.328	-0.039	0.8066	0.1110
172	0.4175	0.3703	0.3938	8.700	8.750	-0.050	1.1284	0.1690
173	0.2951	0.3699	0.3330	8.011	8.039	-0.028	0.9006	0.1021
174	0.5275	0.4884	0.5081	7.900	7.889	0.011	1.0796	0.0331
175	0.7514	0.9344	0.8432	7.922	7.872	0.050	0.8047	0.1155
180	1.0025	0.4575	0.8795	7.850	7.750	0.100	1.6743	0.2271
181	0.7374	0.9524	0.7989	7.761	7.761	0.000	0.8691	0.0000
182	0.8744	0.5925	0.7837	7.639	7.700	-0.061	0.7561	0.1464
183	0.5644	0.5719	0.5682	7.322	7.378	-0.056	0.9869	0.1563
184	0.4011	0.5084	0.5549	7.289	7.189	0.100	1.1818	0.2348
185	0.5235	0.5594	0.5415	7.089	6.922	0.167	0.9359	0.4814
190	0.5344	0.6111	0.5728	6.778	6.811	-0.033	0.8745	0.0934
191	0.5447	0.6044	0.5745	6.456	6.433	0.022	0.9012	0.0622
192	0.3761	0.2697	0.3229	6.089	6.144	[illegible]	1.3948	0.1859
193	[illegible]	0.4184	0.3793	5.400	5.511	[illegible]	0.2243	1.0716

FIG. 17

PAIRED COMPARISON TEST

TIME	CONFIDENCE INTERVALS FOR MEANS (T DISTRIBUTION) LOWER CONFIDENCE LIMIT	DIFFERENCE BETWEEN THE CI AND VI MEANS	UPPER CONFIDENCE LIMIT	CONFIDENCE INTERVALS FOR VARIANCES (F DISTRIBUTION) LOWER CONFIDENCE LIMIT	RATIO OF VARIANCES	UPPER CONFIDENCE LIMIT
530.	-0.5374	-0.0889	0.3599	0.1363	1.2063	9.0474
540.	-0.8661	-0.1056	0.6553	0.1461	1.2920	9.698[illegible]
550.	-0.7047	0.0222	0.7486	0.1234	1.0916	8.1876
600.	-0.8123	-0.2292	0.3678	0.2522	2.2323	16.7421
610.	-0.4136	0.1333	0.6803	0.0857	0.7585	5.6880
620.	-0.0817	0.4000	0.8817	0.1252	1.1077	8.3079
630.	-0.7889	-0.0778	0.6334	0.0902	0.7978	5.9839
640.	-0.3406	0.0389	0.4184	0.0554	0.4905	3.6785
650.	-0.2756	0.1833	0.6433	0.1271	1.1250	8.4375
700.	-0.2795	0.1147	0.5128	0.0676	0.5979	4.4839
710.	-0.4541	0.0000	0.4541	0.2260	2.0000	15.0000
720.	-0.5461	-0.0111	0.5239	0.1183	1.0471	7.8531
730.	-0.6277	-0.0444	0.5389	0.3154	2.8000	21.0000
740.	-0.3812	0.0333	0.4479	0.1249	1.1054	8.2907
750.	-0.2542	0.1389	0.5320	0.0446	0.3945	2.959[illegible]
800.	-0.2760	0.1778	0.6316	0.1130	1.0000	7.5000
810.	-0.3944	-0.0222	0.35[illegible]	0.1316	1.1646	8.7348
820.	-0.51[illegible]7	0.0556	0.6218	0.1137	1.0062	7.5461
830.	-0.7496	-0.1333	0.4829	0.2767	2.4495	18.3732
840.	-0.6850	0.0000	0.6850	0.0942	0.8333	6.2500
850.	-0.6834	-0.0111	0.5814	0.1756	1.5536	11.6523
900.	-0.7879	-0.0778	0.6324	0.0590	0.5223	3.9169
910.	-1.0015	-0.0444	0.9126	0.1109	0.9818	7.3632
920.	-0.913[illegible]	0.0111	0.935[illegible]	0.0851	0.7530	5.6472
930.	-0.8356	-0.0222	[illegible]	0.1239	1.0960	8.22[illegible]
940.	-0.8[illegible]4	-0.1333	0.534[illegible]	0.0527	0.4666	3.4996
950.	-0.7412	0.0389	[illegible]	0.0907	0.8030	6.0221
1000.	-0.9315	-0.0056	0.92[illegible]	0.0854	0.7561	5.67[illegible]
1010.	-0.8257	0.0056	[illegible]	0.[illegible]043	0.5865	4.198[illegible]
1020.	-0.7978	-0.0722	0.553[illegible]	0.1154	1.0214	7.66[illegible]
1030.	-0.3987	0.0778	0.5582	0.1287	1.1390	[illegible]
1040.	-[illegible]315	0.1389	[illegible]	[illegible]	[illegible]	[illegible]
1050.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1100.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1110.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1120.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1130.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1140.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1150.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1230.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	-0.1333	[illegible]	[illegible]	[illegible]	[illegible]

FIG. 18

PAIRED COMPARISON TEST

TIME	LOWER CONFIDENCE LIMIT	DIFFERENCE BETWEEN THE TWO MEANS	UPPER CONFIDENCE LIMIT	LOWER CONFIDENCE LIMIT	RATIO OF VARIANCES	UPPER CONFIDENCE LIMIT
	CONFIDENCE INTERVALS FOR STUDENT DISTRIBUTION			CONFIDENCE INTERVALS FOR VARIANCE DISTRIBUTION		
1300.	-1.2062	.2333	1.2728	0.1391	1.1987	[illegible]
1310.	-1.2867	.0667	1.4233	0.1150	1.0298	6.7398
1320.	-1.5474	-0.0444	1.4709	0.1119	0.9901	7.4255
1330.	-1.2213	0.1333	1.4879	0.0991	0.8772	6.5787
1340.	-1.6122	-0.0333	1.3456	0.1376	1.2174	9.1308
1350.	-1.3313	0.0778	1.4869	0.1093	0.9676	7.2573
1400.	-1.4197	-0.0500	1.3197	0.1427	1.2630	9.4727
1410.	-1.3151	-0.0555	1.2042	0.0985	0.8714	6.5357
1420.	-1.3518	-0.0500	1.2518	0.1071	0.9482	7.1117
1430.	-1.2132	-0.0944	1.0243	0.0949	0.8401	6.3008
1440.	-1.0629	0.0222	1.1073	0.0941	0.8331	6.2485
1450.	-0.9994	0.1000	1.1994	0.1061	0.9392	7.0437
1500.	-1.2418	-0.1000	1.0418	0.1061	0.9387	7.0405
1510.	-1.0001	0.0000	1.0001	0.0801	0.7085	5.3138
1520.	-0.9403	0.0556	1.0514	0.1854	1.6410	12.3072
1530.	-0.9954	-0.0111	0.9734	0.1030	0.9115	6.8350
1540.	-0.6065	0.2222	1.0510	0.0867	0.7669	5.7520
1550.	-0.7922	-0.0111	0.7700	0.0817	0.7227	5.4201
1600.	-0.7119	0.0722	0.8554	0.0974	0.8617	6.4628
1610.	-0.7832	-0.0278	0.7276	0.0947	0.8380	6.2850
1620.	-0.7847	-0.0667	0.6514	0.1096	0.9698	7.2736
1630.	-0.9899	0.0556	1.1010	0.1351	1.1950	8.9603
1640.	-1.0189	0.0889	1.1967	0.1656	1.4656	10.9921
1650.	-1.1989	0.0722	1.3434	0.1017	0.9003	6.7521
1700.	-1.2365	0.0611	1.3587	0.1154	1.0215	7.6612
1710.	-1.0627	-0.0389	0.9849	0.0911	0.8066	6.0497
1720.	-0.9140	-0.0500	0.8140	0.1275	1.1284	8.4628
1730.	-0.8224	-0.0278	0.7668	0.0905	0.8006	6.0045
1740.	-0.9704	0.0111	0.9926	0.1220	1.0796	8.0969
1750.	-1.2144	0.0500	1.3144	0.0909	0.8047	6.0358
1800.	-1.1862	0.1000	1.3862	0.1892	1.6743	12.5575
1810.	-1.2274	0.0000	1.2276	0.0978	0.8651	6.4881
1820.	-1.2981	-0.0611	1.1579	0.0854	0.7561	5.6711
1830.	-1.0935	-0.0556	0.9824	0.1115	0.9869	7.4017
1840.	-0.9257	0.1000	1.1257	0.1336	1.1819	8.8640
1850.	-0.8666	0.1667	1.1800	0.1058	0.9359	7.0196
1900.	-1.0755	-0.0333	1.0088	0.0988	0.8745	6.5591
1910.	-1.0215	0.0222	1.0659	0.1018	0.9012	6.7589
1920.	-0.7380	0.0444	0.8269	0.1576	1.3948	10.4613
1930.	-1.1592	-0.3111	0.5369	0.0256	0.2263	1.6974

FIG. 19

GAIN OF INFORMATION

TIME OF DAY	PAIRED COMPARISON TEST: CONFIDENCE INTERVALS, LOWER LIMIT	DIFFERENCE BETWEEN THE OI AND VI MEANS	UPPER LIMIT	POOLED STANDARD DEVIATION	PAIRED DIFFERENCE TEST: CONFIDENCE INTERVALS, LOWER LIMIT	MEAN VALUES OF DIFFERENCES BETWEEN OI AND VI	UPPER LIMIT	STANDARD DEVIATION OF DIFFERENCES
530.	-0.5375	-0.0889	0.3598	0.3259	0.0217	0.0089	0.1551	0.0451
540.	-0.8661	-0.1058	0.6550	0.5523	-0.8177	0.1055	1.0297	0.8255
550.	-0.7042	0.0222	0.7484	0.5775	-0.1195	-0.0222	0.0751	0.0870
600.	-0.8123	-0.2222	0.3678	0.4785	-0.0671	0.2222	0.5114	0.2587
610.	-0.4135	0.1333	0.6803	0.3972	-0.2703	-0.1333	0.0036	0.1225
620.	-0.0817	0.4000	0.8817	0.3498	-0.8970	-0.4000	0.097	0.4444
630.	-0.7893	-0.0778	0.6338	0.5165	-0.2704	0.0778	0.4261	0.3114
640.	-0.3408	0.0389	0.4186	0.2756	-0.2948	-0.0389	0.2171	0.2280
650.	-0.2765	0.1833	0.6433	0.3340	-0.3849	-0.1833	0.0183	0.1803
700.	-0.2795	0.1167	0.5128	0.2877	-0.4769	-0.1167	0.2435	0.3221
710.	-0.4581	0.0000	0.4581	0.3298	-0.2437	-0.0000	0.2437	0.2179
720.	-0.5461	-0.0111	0.5239	0.3886	-0.5712	0.0111	0.5934	0.5207
730.	-0.4277	-0.0444	0.5389	0.4236	-0.3227	0.0444	0.4116	0.3283
740.	-0.3832	0.0333	0.4499	0.3025	-0.1584	-0.0333	0.0917	0.1118
750.	-0.2542	0.1389	0.5320	0.2855	-0.4384	-0.1389	0.1605	0.2478
800.	-0.2761	0.1778	0.6316	0.3296	-0.5304	-0.1778	0.1749	0.3153
810.	-0.3944	-0.0222	0.3500	0.2703	-0.1122	0.0222	0.1566	0.1202
820.	-0.5107	0.0556	0.6218	0.4113	-0.2398	-0.0556	0.1287	0.1448
830.	-0.7495	-0.1333	0.4829	0.4475	-0.2902	0.1333	0.5624	0.3841
840.	-0.6851	0.0000	0.6850	0.4875	-0.1765	-0.0000	0.1755	0.1581
850.	-0.6035	-0.0111	0.5814	0.4303	-0.1696	0.0111	0.1917	0.1415
900.	-0.7879	-0.0778	0.6374	0.5157	-0.1513	0.0778	0.3068	0.2048
910.	-1.0015	-0.0444	0.9126	0.6050	-0.4136	0.0444	0.5025	0.4404
920.	-0.9143	0.0111	0.9369	0.6719	-0.1919	-0.0111	0.1696	0.1416
930.	-0.9354	-0.0222	0.7911	0.5907	-0.1122	0.0222	0.1566	0.1207
940.	-0.9005	-0.1333	0.5396	0.4886	-0.1625	0.1333	0.42[illegible]	0.2448
950.	-0.7802	0.0389	0.8396	0.5923	-0.2239	-0.0369	0.1551	0.1454
1000.	-0.9405	-0.0056	1.0293	0.5788	-0.1850	0.0055	0.1561	0.1774
1010.	-0.8257	0.0056	0.8368	0.6037	-0.2300	-0.0056	0.21[illegible]	0.2[illegible]7
1020.	-0.7024	-0.0722	0.5578	0.4582	-0.1113	0.0722	0.255[illegible]	0.1441
1030.	-0.3987	0.0778	0.5542	0.3467	-0.2760	-0.0779	0.0444	0.1193
1040.	-0.6315	0.1389	0.9093	0.5995	-0.5179	-0.1389	0.24[illegible]	0.3169
1050.	-0.9537	0.0111	0.9758	0.7[illegible]7	-0.1937	-0.0111	0.1417	0.1169
1100.	-1.1313	0.0222	1.1754	0.8372	-0.2141	-0.0222	[illegible]	[illegible]
1110.	-1.0877	0.0556	1.1796	0.8188	-0.3105	-0.0556	[illegible]	0.2151
1120.	-0.6793	0.1000	0.8756	0.5635	-0.51[illegible]	[illegible]	[illegible]	0.3471
1130.	-0.78[illegible]	[illegible]	0.7828	0.5685	-0.1751	0.0[illegible]	0.12[illegible]	0.1115
1140.	-0.8895	-0.0944	0.8117	0.6575	-0.0957	0.0944	0.2[illegible]	0.1104
1150.	-0.7641	0.0222	0.8084	0.5711	-0.1479	-0.0222	[illegible]	0.17
1200.	-0.8143	0.1444	1.1631	0.6062	-0.6869	-0.1044	0.3[illegible]	0.4051
1210.	-0.8873	0.0722	0.9418	0.6678	-0.1475	-0.0222	0.1[illegible]	0.1512
1220.	-0.8621	0.0899	1.0415	0.6511	-0.3538	-0.0899	0.1[illegible]	0.2169
1230.	-1.2143	-0.0444	1.1258	0.8255	-0.0826	0.0444	0.1[illegible]	0.1137
1240.	-1.39[illegible]	-0.0667	1.2179	0.9320	-0.3[illegible]7	[illegible]	[illegible]	0.4[illegible]
1250.	-1.5546	-0.1333	1.2877	1.0320	-0.1107	0.1111	0.[illegible]	0.2235

FIG. 20

GAIN OF INFORMATION

TIME	LOWER LIMIT	DIFFERENCE BETWEEN	UPPER LIMIT	POOLED STANDARD	LOWER LIMIT	MEAN VALUE OF DIFFERENCE BETWEEN	UPPER LIMIT	STANDARD DEVIATION OF DIFFERENCES
130.	-1.2[illegible]	0.0333	1.2724	[illegible]	-0.1715	-0.0333	0.1249	0.1118
131.	-1.2977	0.0457	1.4230	0.8850	-0.2604	-0.0667	0.1270	0.1732
132.	-1.5675	-0.0444	1.4780	1.1063	-0.1419	0.0444	0.2318	0.1487
133.	-1.2213	0.1333	1.4879	0.9838	-0.4073	-0.1333	0.1406	0.2449
134.	-1.4172	-0.0333	1.3455	1.0014	-0.5088	0.0333	0.5755	0.4848
135.	-1.3313	0.0778	1.4889	1.0233	-0.4954	-0.0778	0.3398	0.3734
140.	-1.4197	-0.0500	1.3197	0.9947	-0.2992	0.0500	0.3992	0.3122
141.	-1.3153	-0.0556	1.2042	0.9148	-0.1618	0.0556	0.2729	0.1944
142.	-1.3514	-0.0500	1.2518	0.9454	-0.1666	0.0500	0.2666	0.1936
143.	-1.2132	-0.0944	1.0289	0.8125	-0.2392	0.0944	0.4281	0.2984
144.	-1.0622	0.0222	1.1073	0.7880	-0.2532	-0.0222	0.2387	0.2333
145.	-0.8991	0.1000	1.1894	0.7984	-0.2937	-0.1000	0.0937	0.1732
150.	-1.2611	-0.1000	1.0618	0.8438	-0.1739	0.1000	0.3739	0.2449
151.	-1.0001	0.0000	1.0001	0.7263	-0.3536	-0.0000	0.3536	0.3102
1520.	-0.8403	0.0556	1.0514	0.7232	-0.5893	-0.0556	0.4782	0.4773
1530.	-0.9955	-0.0111	0.9734	0.7150	-0.3532	0.0111	0.3754	0.3257
154.	-0.6045	0.2222	1.0510	0.6019	-0.4832	-0.2222	0.0387	0.2333
1550.	-0.7922	-0.0111	0.7700	0.5673	-0.2417	0.0111	0.2639	0.2261
1600.	-0.7133	0.0722	0.8554	0.5887	-0.2641	-0.0722	0.1197	0.1716
1610.	-0.7833	-0.0278	0.7276	0.5406	-0.2332	0.0278	0.2887	0.2333
1620.	-0.7867	-0.0457	0.6514	0.5215	-0.1911	0.0667	0.2344	0.1500
1630.	-0.8894	0.0555	1.1010	0.7592	-0.4053	-0.0556	0.2941	0.3127
1640.	-1.0183	0.0889	1.1947	0.8045	-0.4821	-0.0889	0.3043	0.3916
1650.	-1.1983	0.0722	1.3434	0.9232	-0.3013	-0.0722	0.1568	0.2048
1700.	-1.2365	0.0611	1.3597	0.9424	-0.4010	-0.0611	0.2788	0.3039
1710.	-1.0627	-0.0389	0.9849	0.7435	-0.1913	0.0389	0.2691	0.2058
1720.	-0.9183	-0.0500	0.8140	0.6275	-0.2239	0.0500	0.3239	0.2449
173.	-0.8226	-0.0278	0.7648	0.5770	-0.1176	0.0278	0.1734	0.1302
1740.	-0.9734	0.0111	0.9926	0.7125	-0.1290	-0.0111	0.1068	0.1054
1750.	-1.2188	0.0500	1.3144	0.9183	-0.2872	-0.0500	0.1872	0.2121
1800.	-1.1847	0.1000	1.3867	0.9341	-0.4769	-0.1000	0.2709	0.3317
1810.	-1.2275	0.0000	1.2276	0.8915	-0.1479	-0.0000	0.1479	0.1323
1820.	-1.2801	-0.0611	1.1579	0.8853	-0.1756	0.0611	0.2980	0.2110
1830.	-1.0935	-0.0556	0.9824	0.7539	-0.0431	0.0556	0.1542	0.0882
1940.	-0.9258	0.1000	1.1257	0.7449	-0.1969	-0.1000	-0.0031	0.0866
195.	-0.5465	0.1447	1.1800	0.7359	-0.5958	-0.1667	0.2465	0.3873
190.	-1.0755	-0.0333	1.0068	0.7568	-0.1146	0.0333	0.1813	0.1323
191.	-1.0215	0.0222	1.0659	0.7580	-0.2372	-0.0222	0.1927	0.1922
192.	-0.7384	0.0444	1.0260	0.5652	-0.1798	-0.0444	0.0[illegible]	0.1910
193.	-1.1533	-0.3111	0.5390	0.4159	-0.4035	0.3111	1.[illegible]	0.6392

FIG. 21

GAIN OF INFORMATION

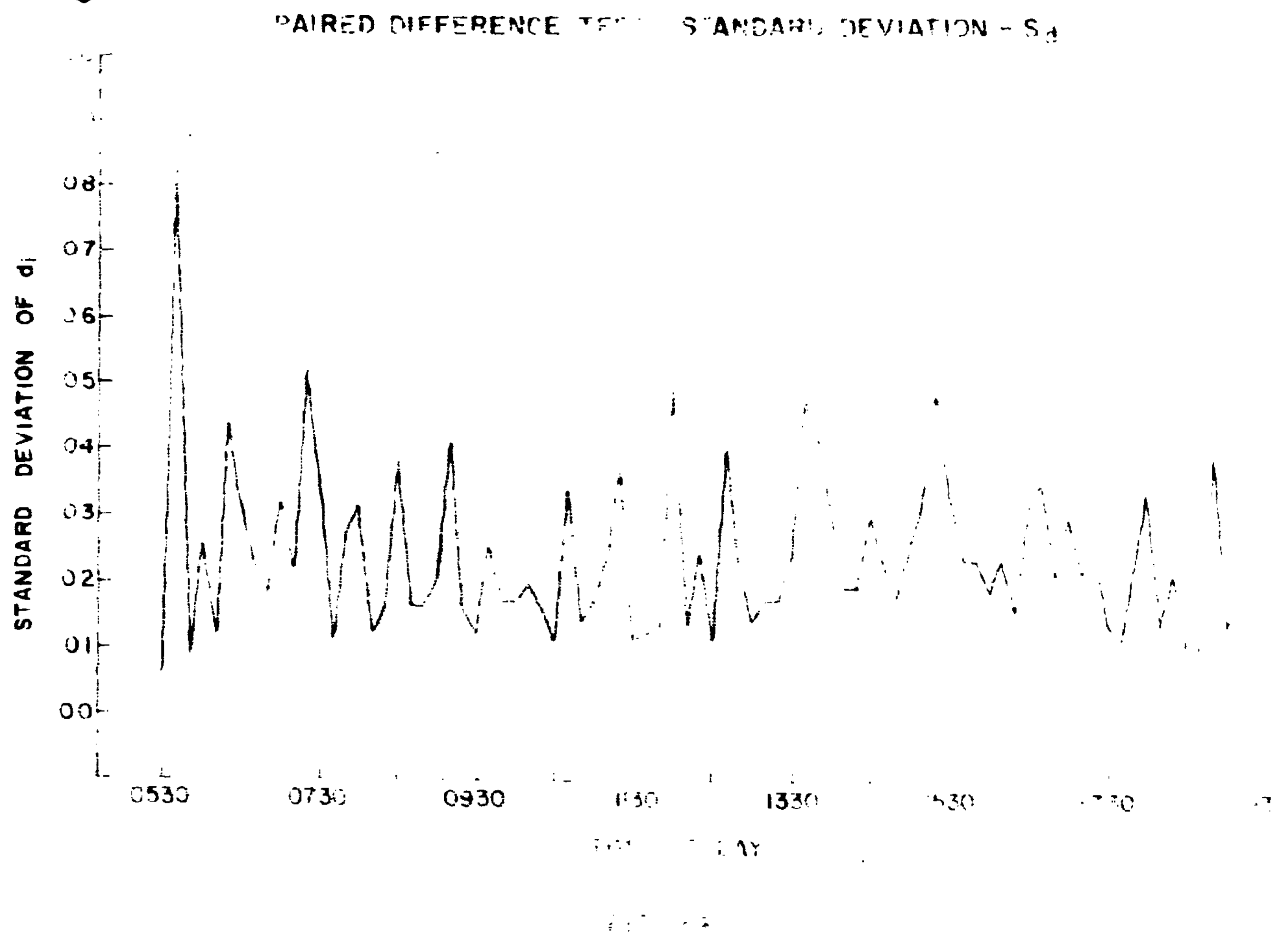
TIME OF DAY	POOLED VARIANCE	VARIANCE OF DI-VI DIFFERENCES	DIFFERENCE	POOLED STANDARD DEVIATION	STANDARD DEVIATION OF DI-VI DIFFERENCES	DIFFERENCE
530.	0.1062	0.0036	0.1024	0.3250	0.0601	0.2658
540.	0.3091	0.0815	-0.3765	0.5573	0.2255	-0.2732
550.	0.2783	0.0076	0.2707	0.5275	0.0870	0.4405
600.	0.1836	0.0069	0.1167	0.4285	0.2587	0.1698
610.	0.1578	0.0152	0.1428	0.3072	0.1225	0.2747
620.	0.1224	0.1975	-0.0751	0.3498	0.4444	-0.0946
630.	0.2667	0.0969	0.1698	0.5165	0.3114	0.2051
640.	0.0760	0.0524	0.0238	0.2756	0.2288	0.0468
650.	0.1114	0.0325	0.0791	0.3340	0.1803	0.1537
700.	0.0829	0.1037	-0.0210	0.2877	0.3221	-0.0344
710.	0.1087	0.0475	0.0613	0.3298	0.2179	0.1118
720.	0.1512	0.2711	-0.1201	0.3886	0.5207	-0.1321
730.	0.1794	0.1078	0.0717	0.4236	0.3283	0.0953
740.	0.0915	0.0125	0.0795	0.3025	0.1118	0.1907
750.	0.0815	0.0717	0.0098	0.2855	0.2678	0.0176
800.	0.1055	0.0994	0.0092	0.3296	0.3153	0.0142
810.	0.0731	0.0146	0.0586	0.2703	0.1202	0.1501
820.	0.1691	0.0272	0.1423	0.4113	0.1640	0.2465
830.	0.2003	0.1475	0.0528	0.4475	0.3841	0.0635
840.	0.2475	0.0250	0.2225	0.4975	0.1581	0.3394
850.	0.1951	0.0261	0.1590	0.4303	0.1616	0.2687
900.	0.2841	0.0419	0.2240	0.5157	0.2048	0.3109
910.	0.4831	0.1673	0.3153	0.6950	0.4096	0.2854
920.	0.4514	0.0261	0.4253	0.6719	0.1616	0.5103
930.	0.3489	0.0144	0.3344	0.5907	0.1202	0.4705
940.	0.2349	0.0700	0.1649	0.4846	0.2646	0.2200
950.	0.3368	0.0274	0.3094	0.5803	0.1654	0.4149
1000.	0.4611	0.0290	0.4319	0.6789	0.1704	0.5386
1010.	0.3644	0.0403	0.3242	0.6037	0.2007	0.4030
1020.	0.2097	0.0269	0.1828	0.4580	0.1641	0.2935
1030.	0.1197	0.0119	0.1078	0.3465	0.1093	0.2367
1040.	0.3131	0.1149	0.1982	0.5595	0.3389	0.2206
1050.	0.4911	0.0186	0.4723	0.7007	0.1364	0.5643
1100.	0.7018	0.0294	0.4724	0.8377	0.1716	0.6661
1110.	0.6654	0.0553	0.6103	0.8158	[illegible]	[illegible]
1120.	0.3175	0.1350	0.1825	0.5635	[illegible]	[illegible]
1130.	0.3212	0.0125	0.3107	[illegible]	[illegible]	0.4567
1140.	0.4324	[illegible]	0.4138	0.6476	[illegible]	[illegible]
1150.	0.3261	[illegible]	0.3092	0.5711	[illegible]	[illegible]
1200.	0.4807	0.2353	0.2494	0.6362	[illegible]	0.2112
1210.	0.4445	[illegible]	0.4297	0.6575	[illegible]	0.5376
1220.	0.4774	[illegible]	0.4215	0.4911	[illegible]	0.4562
1230.	0.7219	[illegible]	0.7092	0.8496	[illegible]	0.7966
1240.	0.3369	[illegible]	0.7054	0.5124	[illegible]	0.5324
1250.	1.0665	[illegible]	[illegible]	1.0325	[illegible]	0.8084

FIG. 22

GAIN OF INFORMATION

[illegible]	[illegible]	[illegible]	[illegible]	[illegible] DEVIATION	[illegible] DIFFERENCES	DIFFERENCE
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
131.	[illegible]	[illegible]	[illegible]	.9950	[illegible]	[illegible]
132.	1.[illegible]	[illegible]	[illegible]	1.1963	[illegible]	[illegible]
133.	[illegible]	[illegible]	0.9070	0.9338	0.2440	0.7388
134.	1.[illegible]	0.235[illegible]	0.7678	1.0314	0.4848	0.5166
135.	1.0472	0.1394	0.9078	1.0233	0.3734	0.6499
140.	[illegible]	[illegible]	0.8958	0.9947	0.3122	0.6825
141.	[illegible]	[illegible]	0.7962	0.9148	[illegible]	0.7205
142.	[illegible]	[illegible]	0.8562	0.9454	0.1936	0.7517
143.	[illegible]	[illegible]	0.5711	0.8125	0.2984	0.5141
144.	0.521[illegible]	[illegible]	0.5665	0.7880	0.2333	0.5547
145.	0.6375	[illegible]	0.6075	0.7984	0.1732	0.6252
150.	0.7115	[illegible]	0.6519	0.8450	0.2449	0.5988
151.	0.5275	[illegible]	0.4275	0.7263	0.3162	0.4101
152.	0.5231	0.2270	0.2953	0.7232	0.4773	0.2460
153.	0.5112	0.1301	0.4051	0.7150	0.3257	0.3892
154.	0.3522	[illegible]	0.3079	0.6018	0.2333	0.3685
155.	0.3219	0.0511	0.2707	0.5673	0.2281	0.3412
160.	0.3235	0.0294	0.2949	0.5687	0.1716	0.3972
161.	[illegible]	0.0544	0.2465	0.5486	0.2333	0.3153
162.	0.2719	[illegible]	0.2494	0.5215	0.1500	0.3715
163.	0.5764	[illegible]	0.4784	0.7592	0.3127	0.4465
164.	0.6472	0.1235	0.5234	0.8045	0.3516	0.4529
165.	0.6522	0.0419	0.6103	0.9232	0.2049	0.7184
170.	[illegible]	[illegible]	0.7957	0.9424	0.3039	0.6385
171.	[illegible]	0.0424	0.5104	0.7435	0.2058	0.5377
172.	0.3938	[illegible]	0.3337	0.6275	0.2449	0.3825
173.	[illegible]	[illegible]	0.3160	0.5770	0.1302	0.4469
174.	[illegible]	[illegible]	[illegible]	0.7128	0.1054	0.6074
175.	0.8032	[illegible]	0.7792	0.9183	0.2121	0.7081
180.	0.8725	[illegible]	0.7625	0.9341	0.3317	0.6024
181.	[illegible]	[illegible]	0.7774	0.8915	0.1323	0.7593
182.	0.7837	[illegible]	0.7368	0.8853	0.2118	0.6735
183.	0.5692	[illegible]	0.5604	0.7538	0.0882	0.6656
184.	0.5549	[illegible]	0.5474	0.7449	0.0866	0.6583
185.	[illegible]	[illegible]	0.3915	0.7359	0.3873	0.3486
190.	0.5729	0.0175	0.5553	0.7559	0.1323	0.4245
191.	[illegible]	[illegible]	[illegible]	0.7559	[illegible]	0.5658
192.	[illegible]	[illegible]	[illegible]	0.5692	[illegible]	0.4472

FIG. 23


PAIRED DIFFERENCE TEST STANDARD DEVIATION - S_d
STANDARD DEVIATION OF d_i
.08
.07
.06
.05
.04
.03
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.01
.00
0530
0730
0930
1330

PAIRED COMPARISON TEST - POOLED STANDARD DEVIATION - Sp

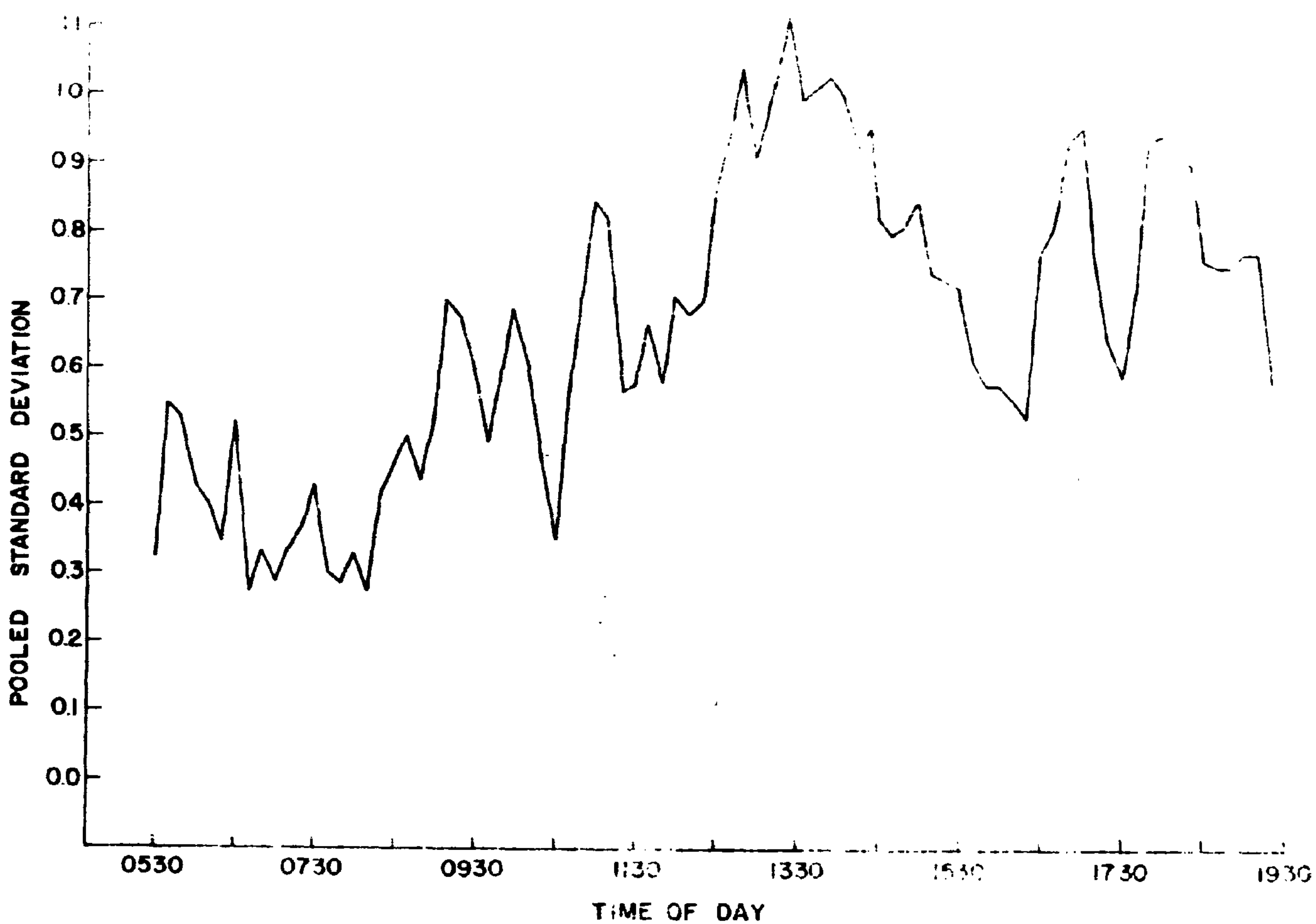

FIG. 25

POSITION LOCATION VIA MULTIPLE TRIANGULATION*

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1. INTRODUCTION. Classical triangulation in the plane involves locating an unknown position by measuring its direction from two known points and finding the intersection point of the two location lines. If, more generally, there are n known points reporting directions--and there are errors in the observed directions--then the n lines cannot be expected to intersect in a common point. Two different methods of obtaining a closed form estimate of the true position, with variations on each, will be derived and discussed, along with an error analysis of each method.

2. ESTIMATION METHODS.

a. Least Squares (LSQ). The n known positions are denoted by P_i (x_i, y_i), $i = 1, 2, \ldots, n$, and the observed directions by the respective angles ϕ_i. This yields an equation for the i th direction line L_i:

$$y - y_i = \tan \phi_i \ (x - x_i).$$

The perpendicular distance d from an arbitrary point P (x, y) to the line L_i is given by:

$$d\ (P, L_i) = |\ (x - x_i) \sin \phi_i - (y - y_i) \cos \phi_i\ |.$$

The LSQ method, roughly, determines a point that is close to all the lines L_i, in the least squares sense. Specifically, define the function

$$\begin{aligned} f\ (P) &= f\ (P\ (x, y)) \\ &= \sum_{i=1}^{n} (d\ (P, L_i))^2 \\ &= \Sigma\ d^2\ (P, L_i) \\ &= \Sigma[(x - x_i)^2 \sin^2\phi_i + (y - y_i)^2 \cos^2\phi_i \\ &\quad -2\ (x - x_i)\ (y - y_i) \sin\phi_i \cos\phi_i]. \end{aligned}$$

The (unweighted) LSQ estimate is the point P that minimizes this function. A slightly more general function is

$$g\ (P) = \Sigma\lambda_i d^2\ (P, L_i),$$

where $\{\lambda_i\}$ is a set of fixed, but arbitrary, nonnegative numbers. Physically, the minimization of this function corresponds to weighting some of the

*This article has been reproduced photographically from the author's manuscript.

information more heavily than the rest. Setting each ω_i equal to 1 reduces the problem to the unweighted case.

Minimization of g (F) is straightforward: compute $\partial g/\partial x$ and $\partial g/\partial y$, set both equal to 0, and solve the pair of equations. The estimates derived are:

$$x_{LSQ} = \frac{\begin{vmatrix} \Sigma(\omega_i x_i \sin^2\phi_i - \omega_i y_i \sin\phi_i \cos\phi_i) & -\Sigma\omega_i \sin\phi_i \cos\phi_i \\ \Sigma(\omega_i x_i \sin\phi_i \cos\phi_i - \omega_i y_i \cos^2\phi_i) & -\Sigma\omega_i \cos^2\phi_i \end{vmatrix}}{\begin{vmatrix} \Sigma\omega_i \sin^2\phi_i & -\Sigma\omega_i \sin\phi_i \cos\phi_i \\ \Sigma\omega_i \sin\phi_i \cos\phi_i & -\Sigma\omega_i \cos^2\phi_i \end{vmatrix}}$$

$$y_{LSQ} = \frac{\begin{vmatrix} \Sigma\omega_i \sin^2\phi_i & \Sigma(\omega_i x_i \sin^2\phi_i - \omega_i y_i \sin\phi_i \cos\phi_i) \\ \Sigma\omega_i \sin\phi_i \cos\phi_i & \Sigma(\omega_i x_i \sin\phi_i \cos\phi_i - \omega_i y_i \cos^2\phi_i) \end{vmatrix}}{\begin{vmatrix} \Sigma\omega_i \sin^2\phi_i & -\Sigma\omega_i \sin\phi_i \cos\phi_i \\ \Sigma\omega_i \sin\phi_i \cos\phi_i & -\Sigma\omega_i \cos^2\phi_i \end{vmatrix}}$$

B. Point of Intersection (POI). Each pair of direction lines intersects in a unique point, and there are $\binom{n}{2} = n(n-1)/2$ such points. The intersection point of lines L_i and L_j, P_{ij} (x_{ij}, y_{ij}), is derived easily from their equations and is given by:

$$x_{ij} = \frac{\cos\phi_i (y_j \cos\phi_j - x_j \sin\phi_j) - \cos\phi_j (y_i \cos\phi_i - x_i \sin\phi_i)}{\sin(\phi_i - \phi_j)}$$

$$y_{ij} = \frac{\sin\phi_i (y_j \cos\phi_j - x_j \sin\phi_j) - \sin\phi_j (y_i \cos\phi_i - x_i \sin\phi_i)}{\sin(\phi_i - \phi_j)}$$

The POI method of estimation simply involves computing a weighted or unweighted average of the $\binom{n}{2}$ points P_{ij}. That is, a set of nonnegative numbers ω_{ij}, $1 \le i < j \le n$, is selected, and the coordinates of the estimated point are:

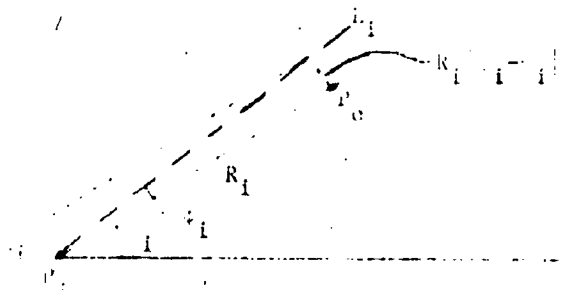
$$x_{POI} = \Sigma_i \Sigma_j \omega_{ij} x_{ij} / \Sigma_i \Sigma_j \omega_{ij}$$

$$y_{POI} = \Sigma_i \Sigma_j \omega_{ij} y_{ij} / \Sigma_i \Sigma_j \omega_{ij}.$$

Easiest of course is to choose the weights $\{\omega_{ij}\}$ so that they sum to 1.

3. APPROXIMATION USED IN ERROR ANALYSES. If one assumes that the errors in measuring the true angles from the respective known points P_i are small, then the following approximation is a very sound one and aids considerably in the error analyses. Note that the error in locating an unknown position is in general a function of the position, so that in this sense the errors derived are conditional errors, conditioned by the actual (unknown) position.

Let (R_i, θ_i) denote the true distance and direction of the unknown point P_o from the ith known point P_i. The line L_i goes through P_i and misses P_o by a distance of approximately $R_i|\phi_i - \theta_i|$ (see figure). The



geometry of the error analyses is eased considerably if line L_i is replaced by line Λ_i, parallel to the line from P_i to P_o and displaced by an amount $R_i|\phi_i-\theta_i|$ in the appropriate direction. Lines L_i and Λ_i are virtually indistinguishable, and both miss P_o by the same amount.

4. ERROR ANALYSES. If P_o (x_o, y_o) denotes the true point to be located and P (x, y) denotes the computed estimate (via either LSQ or POI), then the basic criterion used to assess the quality of the respective estimates is $E[(x - x_o)^2 + (y - y_o)^2] = E[|P - P_o|^2]$, the mean squared radial error. The only assumptions made about the random variables ϕ_i are that they be independent, unbiased, and have common variance σ^2.

a. LSQ Method. Using the counterclockwise convention for positive angles, one can show that the distance from an arbitrary point P (x, y) to Λ_i is given by:

$$d(P, \Lambda_i) = |(x - x_o)\sin\theta_i - (y - y_o)\cos\theta_i + R_i(\phi_i - \theta_i)|.$$

Given P_o--and thus $\{R_i\}$ and $\{\theta_i\}$--the point P (x_{LSQ}, y_{LSQ}), because of the way it was computed, also minimizes the function $h(P) = \Sigma_i d^2(P, \Lambda_i)$.

Therefore, minimizing E(P) expresses x_{LSQ} and y_{LSQ} (conditionally) in terms of the ϕ_i's and θ_i's; the expressions are:

$$x_{LSQ} = \frac{(\Sigma \ell_i R_i(\phi_i - \theta_i)\cos\theta_i)(\Sigma \ell_i \sin\theta_i \cos\theta_i) - (\Sigma \ell_i R_i(\phi_i - \theta_i)\sin\theta_i)(\Sigma \ell_i \cos^2\theta_i)}{(\Sigma \ell_i \sin^2\theta_i)(\Sigma \ell_i \cos^2\theta_i) - (\Sigma \ell_i \sin\theta_i \cos\theta_i)^2} + x_o$$

and similarly for y_{LSQ}. Note immediately that x_{LSQ} and y_{LSQ} are linear functions of $(\phi_i - \theta_i)$, so that the mean squared radial error is in fact a function only of the variance of $(\phi_i - \theta_i)$ (and the geometry).

The error expression, $E[(x_{LSQ} - x_o)^2 + (y_{LSQ} - y_o)^2]$, is still relatively simple to obtain, by the fact that the random variables ϕ_i are independent with common variance σ^2, and is given by:

$$\frac{\left\{ \begin{array}{l} [(\Sigma \ell_i \cos^2\theta_i)^2 + (\Sigma \ell_i \cos\theta_i \sin\theta_i)^2](\Sigma \ell_i^2 R_i^2 \sin^2\theta_i) \\ + [(\Sigma \ell_i \sin^2\theta_i)^2 + (\Sigma \ell_i \cos\theta_i \sin\theta_i)^2](\Sigma \ell_i^2 R_i^2 \cos^2\theta_i) \\ -2\,(\Sigma \ell_i \cos\theta_i \sin\theta_i)(\Sigma \ell_i)(\Sigma \ell_i^2 R_i^2 \sin\theta_i \cos\theta_i) \end{array} \right\}}{[(\Sigma \ell_i \sin^2\theta_i)(\Sigma \ell_i \cos^2\theta_i) - (\Sigma \ell_i \sin\theta_i \cos\theta_i)^2]^2} \, \sigma^2 .$$

b. POI Method. The coordinates of the point P_{ij} are determined approximately by finding the intersection of Λ_i and Λ_j. This yields:

$$x_{ij} = x_o + \frac{(R_i(\phi_i - \theta_i))\cos\theta_j - (R_j(\phi_j - \theta_j))\cos\theta_i}{\sin(\theta_i - \theta_j)}$$

$$y_{ij} = y_o + \frac{(R_i(\phi_i - \theta_i))\sin\theta_j - (R_j(\phi_j - \theta_j))\sin\theta_i}{\sin(\theta_i - \theta_j)}$$

Since the estimate (x_{POI}, y_{POI}) is given by

$$\left. \begin{array}{l} x_{POI} = \sum_{i<j} \ell_{ij} x_{ij} \\ y_{POI} = \sum_{i<j} \ell_{ij} y_{ij} \end{array} \right\} \quad \sum_{i<j} \ell_{ij} = 1,$$

the error function $E[(x_{POI} - x_o)^2 + (y_{POI} - y_o)^2]$ is derived by combining the ϕ_1 terms, ϕ_2 terms, etc., squaring, and careful bookkeeping, noting again the independence and common variance σ^2 of the random variables. For convenience of notation, define $\ell_{ij} = \ell_{ji}$, if $i>j$, and define $\lambda_{ij} = \ell_{ij}/\sin(\theta_i - \theta_j)$, for all $i \neq j$. Then the mean squared radial error is given by:

$$\left\{ \sum_{i=1}^{n} R_i^{\,2} \left[\left(\sum_{j \neq i} \lambda_{ij} \cos\theta_j \right)^2 + \left(\sum_{j \neq i} \lambda_{ij} \sin\theta_j \right)^2 \right] \right\} \sigma^2.$$

5. IMPROVEMENTS BY ADJUSTING THE WEIGHTS. The derived error expressions are too complex to permit many general observations to be made. Extensive study of examples indicates that the LSQ method leads to smaller error than does the POI method. In particular, it is conjectured that unweighted LSQ is always better than unweighted POI. However, either method can be considerably improved through the use of even imperfect information about the unknown location. In the following subsections idealized weights are derived for each method, weights that minimize the respective error expressions but are unattainable because they require perfect information about the unknown locations. In later sections these idealized weights are interpreted as yielding lower bounds on the error expressions, bounds that cannot be attained but can be approached by various iterative schemes.

a. LSQ Method. The intention here is to find the set of nonnegative weights $\{\ell_i\}$ that minimizes the (conditional) error expression, E, for a <u>particular</u> unknown location P_o. Of course the set is different for each P_o and thus cannot be derived, even in theory, without perfect knowledge of P_o itself. However, the mere existence of such a minimal set indicates a lower limit on how much improvement can be expected even with partial information about P_o.

Note first that E is homogeneous in the ℓ_i's, that is, multiplying the ℓ_i's by a common factor leaves E unchanged. Note also that a minimum could not occur along a boundary (one or more ℓ_i's equal to 0), since this means ignoring some of the data. Thus, a <u>necessary</u> condition for a <u>local</u> minimum to occur is that all the partials, $\partial E/\partial \ell_i$, be equal to 0 at some point (or any multiple thereof). One solution (and, it is conjectured, the unique one) is:

$$\ell_i = R_i^{-2}, \text{ for all } i.$$

Again, this solution was suggested through study of numerous examples, and it can be checked, through straightforward but tedious computation, that it does indeed satisfy $\partial E/\partial \ell_i = 0$, for all i. The simplicity and plausibility of this solution, once attained, make it a most likely candidate for unique global minimum. In particular, the data from more remote points P_i should obviously be weighted less heavily. Interestingly, the minimal weights do not depend on the angles $\{\theta_i\}$. The error expression, E_{MIN}, for $\ell_i = R_i^{-2}$, is given by:

$$\Sigma R_i^{-2}/[(\Sigma R_i^{-2}\cos^2\theta_i)(\Sigma R_i^{-2}\sin^2\theta_i) - (\Sigma R_i^{-2}\cos\theta_i\sin\theta_i)^2],$$

a relatively simple expression.

b. POI Method. Since E is a simpler expression in the POI case, it would be expected that minimization is also easier and this is true. In fact, since E is a quadratic function of the ℓ_{ij}'s (or λ_{ij}'s) and the constraint, $\sum_{i<j}\ell_{ij} = 1$, is linear, the minimization problem--via partial differentiation and Lagrange multipliers--reduces to solving a set of simultaneous linear equations in ℓ_{ij}. Once more, study of examples suggested a solution, hence the solution, which is given by:

$$\ell_{ij} = R_i^{-2}R_j^{-2}\sin^2(\theta_i-\theta_j)/\sum_{i<j}R_i^{-2}R_j^{-2}\sin^2(\theta_i-\theta_j),$$

for all i<j. As noted, the optimal weights for POI involve both the R_i's and the θ_i's, perhaps because the points of intersection are so intimately tied to both. The most striking fact is that the error expression, E_{MIN}, is precisely the same as that derived in the previous subsection for LSQ. (In fact, the "optimal" estimates themselves are identical as well.) Certainly this fact is more than coincidence, and some of its unifying implications will be discussed in a later section.

6. INTERPRETATION OF "OPTIMAL" WEIGHTINGS. In Section 5, best possible weightings were derived for each estimation method, given a particular P_0. As stated, these weights are unattainable, requiring omniscience, but they indicate directions for improvement of the respective unweighted methods. Simply stated, some information about the location of P_0 is better than none at all. This suggests that an initial (unweighted) estimate, $P_{(1)}$, be computed, via either method, and the distances R_{i1} (and angles θ_{i1}) from each P_i to $P_{(1)}$ be determined. Then these R_{i1}'s (and θ_{i1}'s, if applicable) can be used to compute a second, weighted estimate, $P_{(2)}$. Since $P_{(2)}$ is

presumed better than $P_{(1)}$, its associated R_{i2}'s and θ_{i2}'s can be used to compute a third, weighted estimate, $P_{(3)}$. Naturally this iterative scheme can be continued as long as desired. Unfortunately, very little work has been done at this time to investigate convergence, and rate of convergence, of the iterations to any type of best estimate, but this is certainly an area for continued research. It is felt that the iterations for the LSQ method probably converage rather well, whereas the POI iterations, because they involve angles as well as distances and the data is in terms of angles, allow the possibility of circularity and instability.

7. CONCLUSIONS. Two methods have been discussed for estimating the location of an unknown point, given direction data from n known points. Closed form formulas for the estimates were derived, as well as general expressions for the mean squared radial error of the two methods, least squares (LSQ) and point of intersection (POI). In addition, idealized "optimal" weights were derived for each method, weights that reduce the respective errors to their smallest possible values, conditioned by the true location of the unknown point.

It was noted in Section 6 that the optimally weighted solutions for the two methods are identical. To understand the underlying reason for this, consider the following estimation method: for an arbitrary point P (x, y), let γ_i be the angle it makes with P_i and let $F(P) = \sum_{i=1}^{n} (\theta_i - \gamma_i)^2$. Find, if possible, the point P that minimizes F (P). To the degree of approximation used throughout the paper, this P is identical to the optimal LSQ or POI point. To see this, rewrite F (P) as:

$$F(P) = \sum_{i=1}^{n} R_i^{-2}[R_i^2(\theta_i-\gamma_i)^2] \approx \sum w_i d^2(P, A_i),$$

where $\{w_i\}$ is the optimal set of weights for LSQ. This method, finding a least squares fit to the raw data itself, $\{\theta_i\}$, was not used until late in the investigation and still is not preferable to the others merely because it does not lend itself to a closed form solution. Conceivably, with sufficient computational facilities, this method may be preferable to either LSQ or POI, particularly if the latter require a large number of iterations. Note that this estimate is in fact the maximum likelihood estimate in the case where the θ_i's are normally distributed.

In summary, the two methods, LSQ and POI, were presented separately as parallel methods, although the preceding paragraph does tend to unify the two under one broad theory. Emphasized was the simplicity of both methods, especially in the absence of unequal weights. The POI method is easier to visualize graphically, while the LSQ method has fewer computations and smaller associated error. Both methods are systematic and easy to apply in many practical situations.

A General Computational Algorithm for Bayesian Confidence Bounds

by Richard W. Clarke

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INTRODUCTION

For anyone unfamiliar with Bayesian analysis this paper should serve as an introduction to this very useful confidencing technique. The aspect of this paper which might be interesting to those already familiar with this subject is simply that I have outlined a computational algorithm which will eliminate the very messy mapping which arises in applying the Bayes formulation.

I found it convenient in what follows to work with a specific example in order to make a few basic points. A more general treatment may be found in a Watervliet Arsenal technical report by the same title.[1]

A BAYESIAN CONFIDENCE BOUND ON RELIABILITY

The basic contention of the Bayesian analysis is that any physical parameter about which we have less than precise knowledge may be treated as a random variable. For instance, the shape and location Weibull parameters might be treated as random variables if we are using the Weibull density to represent a set of failure data. If from the data we can construct the joint density of those parameters, then the reliability density for a given safe life or the safe life density for a given reliability will follow directly.

To arrive at that joint parameter density we must first specify some prior knowledge of those parameters. This consists of stating that, from prior testing of similar items, these parameters are likely to be within certain bounds. If very little information is available we might say only that a certain parameter can take on any value between two limits, and that each value between those bounds is equally likely before testing.

Suppose that five components have been cycled, under actual conditions of field use, to failure and that these failures (cycles or hours, etc.) are $X_1, X_2, \ldots X_5$. From prior testing of similar mechanical components we deduce that the population from which these failures come can be reasonably approximated by the two parameter Weibull density. Then the density of X is given by:

$$f(x/\beta,\tau) = \frac{\beta}{\tau}\left[\frac{x}{\tau}\right]^{\beta-1}\exp\left[-\left(\frac{x}{\tau}\right)^{\beta}\right] \qquad (1)$$

---

in which X is a random variable drawn from $W(\beta, T)$. For any set of Weibull parameters we have that the joint density of the above five independent observations is:

$$g(X_1, X_2 \cdots X_5/\beta, T) = \prod_{i=1}^{5} f(X_i/\beta, T) \qquad (2)$$

or:

$$g(\mathbf{X}/\Theta) = \prod_{i=1}^{5} f(X_i/\Theta) \qquad (3)$$

Now let our prior knowledge of the parameters be represented by:

$$g_1(\Theta) = g_1(\beta, T) \qquad (4)$$

In this case we might say, for instance, that:

$$g_1(\beta, T) = \frac{1}{Q_1 Q_2} \qquad \begin{array}{l} \beta_0 < \beta < \beta_0 + Q_1 \\ T_0 < T < T_0 + Q_2 \end{array}$$

$$= 0 \qquad \text{ELSEWHERE} \qquad (5)$$

Bayes theorem then states that the posterior knowledge available on the parameter space for these five observations is:

$$g_2(\hat{\Theta}/\mathbf{X}) = \frac{g(\mathbf{X}/\Theta)\, g_1(\Theta)}{\int_{\Theta} g(\mathbf{X}/\Theta)\, g_1(\Theta)\, d\Theta} \qquad (6)$$

In the example we've been following, noting that g_1 is a constant:

$$g_2(\hat{\Theta}/\mathrm{X}) = \frac{\prod_{i=1}^{5} f(x_i/\Theta)}{\int_0^{Q_1}\int_0^{Q_2}\prod_{i=1}^{5} f(x_i/\Theta)\,dT\,d\beta} \qquad (7)$$

We now have an expression which assigns a probability density to each point in the parameter space.

At any point in that space we can related reliability (R) to safe life (X_s).

$$R = 1 - \int_0^{X_s} f(x/\Theta)\,dx \qquad (8)$$

or for the Weibull example:

$$R = \exp\left[-\left(\frac{X_s}{T}\right)^{\beta}\right] \qquad (9)$$

Then for a given safe life the density of the reliability estimator may be found by mapping the parameter joint density (g_2) onto reliability (R) through Eqn. (8). Analytically:

$$r(\hat{R}) = \int_0^{\infty} g_2(\hat{\beta},\hat{T})\left|\frac{\partial(\hat{\beta},\hat{T})}{\partial(\hat{R},\hat{T})}\right| d\hat{T} \qquad (10)$$

In which the Jacobian is evaluated from Eqn. (8). A one sided, (1 $-\alpha$) 100% lower confidence bound on reliability is then the 100αth percentile in the reliability estimator density or:

$$\alpha = \int_0^{\hat{R}_c} r(\hat{R})\,d\hat{R} \qquad (11)$$

Expression (10) is not particularly simple to evaluate. If we happened to be working with a three parameter density the Jacobian would contain three terms instead of two and two of the variables would have to be eliminated by integration instead of one. In general, that is for most two and three parameter densities, the integrations could not be carried out in closed form and some numerical or computer solution would be required.

A SUBSTITUTE FOR THE ANALYTICAL MAPPING

Instead of the usual analytical mapping as defined by Expression (10), we can start directly with the posterior joint parameter density and do a numerical mapping onto reliability as follows.

In specifying the prior parameter density (g_1) choose a rectangular region of definition such as in Eqn. (5) above. Then divide that region into small subregions by dividing the parameter axes into equal intervals. In the Weibull example we have been following the midpoint of a specific subregion would be represented by:

$$\beta_i = \beta_0 + \frac{Q_1}{N_\beta}\left(i - \tfrac{1}{2}\right) \qquad i = 1, 2, \cdots N_\beta \qquad (12)$$

$$T_j = T_0 + \frac{Q_2}{N_T}\left(j - \tfrac{1}{2}\right) \qquad j = 1, 2, \cdots N_T \qquad (13)$$

In which Q_1 and Q_2 represent the ranges over which the parameters β and T are defined (See Eqn. (5)), and N_β and N_T are the number of intervals into which those ranges have been partitioned. (See Figure I)

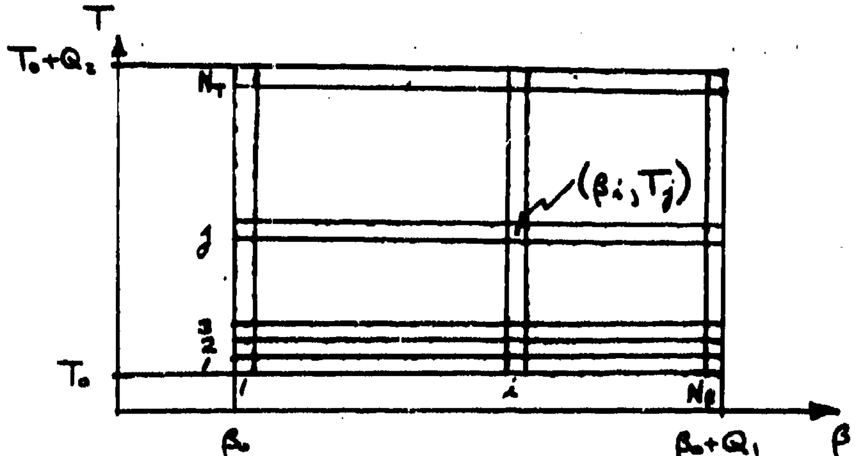

FIGURE I

If these subregions are "small" enough (how small will be discussed later), the joint posterior density (g_2) at the subregion midpoint will be a reasonable approximation throughout the subregion. Then the probability that any subregion contains the actual population parameters can be represented by

$$P_{ij} = g_2(\beta_i, T_j)V \qquad (14)$$

$$V = \frac{Q_1}{N_\beta}\frac{Q_2}{N_T} \qquad (15)$$

This probability (P_{ij}) can then be associated with an interval on the range of possible reliabilities by calculating the reliability for the parameters β_i and T_j:

$$R_{ij} = \exp\left[-\left(\frac{X_s}{T_j}\right)^{\beta_i}\right] \qquad (16)$$

An actual mapping of the ijth subregion onto reliability might look like Figure II below.

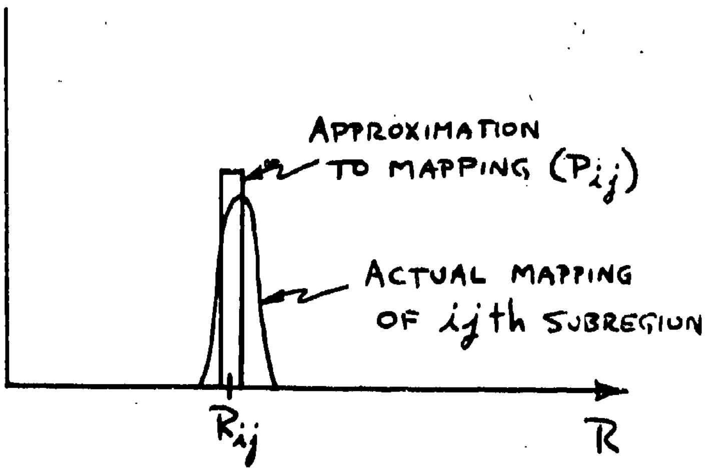

FIGURE II

We will approximate this mapping by dividing the reliability axis into intervals and assigning the entire Pij to the interval in which Rih falls. Mathematically, calculate:

$$\Delta R = \frac{R_{MAX} - R_{MIN}}{M} \qquad (17)$$

with R_{MAX} = Maximum reliability possible

R_{MIN} = Minimum reliability possible

M = Number of intervals on the reliability axis

then:

$$I_{ij} = \frac{R_{ij} - R_{MIN}}{\Delta R} + 1 \qquad (18)$$

Truncating I_{ij} to an integer value then defines the interval number to which Pij is to be assigned.

By running through all the (βi, Tj) combinations and assigning each Pij to an interval on the reliability axis we are constructing a histogram which approximates the reliability estimator density. (See Figure III)

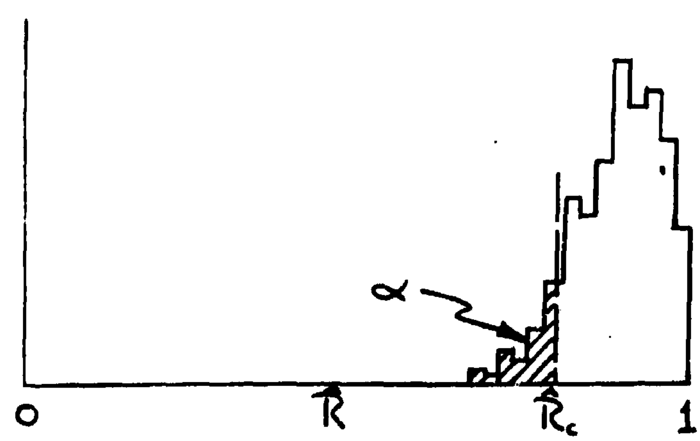

FIGURE III

The accuracy of the process depends only on the interval sizes chosen. We have simply replaced the integral evaluation of the mapping process (Eqn. 10), by a much more straightforward numerical evaluation. Confidenced reliability follows from the histogram by replacing Eqn. (11) with a summation.

Looking back on the process we can note that certain simplications are possible. The evaluation of the posterior parameter joint density could be written:

$$g_2(\hat{\Theta}/X) = K\, g(X/\Theta)\, g_1(\Theta) \qquad (19)$$

in which the constant is:

$$K = \frac{1}{\int_{\Theta} g(X/\Theta)\, g_1(\Theta)\, d\Theta} \qquad (20)$$

Then:

$$P_{ij} = KV\, g(X/\beta_i, T_j)\, g_1(\beta_i, T_j) \qquad (21)$$

But the sum of all Pij should be unity so that:

$$KV = \frac{1}{\sum_{i=1}^{N_\beta} \sum_{j=1}^{N_T} g(X/\beta_i, T_j)\, g_1(\beta_i, T_j)} \qquad (22)$$

In other words we do not have to evaluate the integral in Eqn. (20).

GRID SIZE

The remaining problem then is to determine in any case what interval size is sufficiently "small." No satisfactory solution to this problem is presently available. In applying the technique to actual data sets, however, the following points were noted.

1. One specific application to the lognormal density with a uniform prior parameter density yielded the following confidence bounds on safe life for a given reliability:

Grid Size (N_μ X N_σ)	Bound (Cycles)
15 x 15	800
40 x 40	1150
70 x 70	1150

In this case 15 x 15 was too coarse, but 40 x 40 was agood as 70 x 70.

2. An application of the three parameter Weibull seemed to converge with a grid size of 5 x 5 x 5. Finer grids resulted in a negligible change in the confidence bound.

3. For very high reliabilities (.999, .9999, etc.) the lower confidence bound on safe life seems to increase as the grid is made finer. This would indicate that this method yields, for given grid size, a confidence bound which is on the conservative side of the "exact" Bayesian confidence bound.

CONCLUSION

The point which makes this computational mapping extremely interesting is that it can be extended to any distributional form; it can be extended to system reliability work in which the joint posterior parameter space for all components is mapped onto system reliability, and so on. It's drawback, of course, is that it is completely computer dependent and for large parameter spaces the computations can be expensive.

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Exact Lower Confidence Limits on Normal and Lognormal Reliability

by Royce W. Soanes, Jr.

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WATERVLIET, NEW YORK

This paper is a synopsis of References (8) and (9) which were written in order to document more fully the solution to the problem of concern:

Given a population having a normal or lognormal life distribution, and a representative sample of failures drawn from this population, calculate an exact 100 C % lower confidence limit on population reliability (R) for a given mission life (or calculate the mission life (x) corresponding to a given lower confidence limit on reliability.)

The normal reliability estimator* is given by

$$\hat{R} = 1 - \Phi\left(\frac{x - \hat{\mu}}{\hat{\sigma}}\right) \tag{1}$$

By performing a bivariate change of variable, the joint density of $\hat{R}$ and $\hat{\sigma}$ may be obtained in terms of the joint density of $\hat{\mu}$ and $\hat{\sigma}$.

$$h(\hat{R}, \hat{\sigma}) = f(\hat{\mu}, \hat{\sigma}) \sqrt{2\pi}\, \hat{\sigma}\, e^{\frac{1}{2} \mathfrak{z}_{\hat{R}}^{2}} \tag{2}$$

The joint density of $\hat{\mu}$ and $\hat{\sigma}$ may be determined from the fact that:

(1) $\hat{\mu}$ and $\hat{\sigma}$ are independent random variables

(2) $\frac{n\hat{\sigma}^2}{\sigma^2}$ has a chi-square distribution with n-1 degrees of freedom and

(3) $\hat{\mu}$ is normally distributed with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$

---

*Estimates are maximum likelihood

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The joint density of $\hat{\mu}$ and $\hat{\sigma}$ is therefore:

$$f(\hat{\mu},\hat{\sigma}) = \frac{2}{\sigma^2}\sqrt{\frac{n}{2\pi}}\,\frac{\left(\frac{n}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\left(\frac{\hat{\sigma}}{\sigma}\right)^{n-2} e^{-\frac{n}{2}\left[\left(\frac{\hat{\sigma}}{\sigma}\right)^2+\left(\frac{\hat{\mu}-\mu}{\sigma}\right)^2\right]} \tag{3}$$

The joint density of $\hat{R}$ and $\hat{\sigma}$ is therefore:

$$h(\hat{R},\hat{\sigma}) = \frac{2\sqrt{n}}{\sigma}\,\frac{\left(\frac{n}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\left(\frac{\hat{\sigma}}{\sigma}\right)^{n-1} e^{-\frac{n}{2}\left[\left(\frac{\hat{\sigma}}{\sigma}\right)^2+\left(\frac{\hat{\mu}-\mu}{\sigma}\right)\right]+\frac{1}{2}z_{\hat{R}}^2} \tag{4}$$

but by definition,

$$\hat{\mu} = x + \hat{\sigma} z_{\hat{R}}$$

$$\mu = x + \sigma z_R$$

letting $$K = \frac{2\sqrt{n}\left(\frac{n}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$\therefore\; h(\hat{R},\hat{\sigma}) = \frac{K}{\sigma}\left(\frac{\hat{\sigma}}{\sigma}\right)^{n-1} e^{-\frac{n}{2}\left[\left(\frac{\hat{\sigma}}{\sigma}\right)^2+\left(\frac{\hat{\sigma}}{\sigma}z_{\hat{R}}-z_R\right)^2\right]+\frac{1}{2}z_{\hat{R}}^2} \tag{5}$$

Now $\hat{\sigma}$ is integrated out to obtain the density of $\hat{R}$:

$$h(\hat{R}) = \int_0^\infty \frac{K}{\sigma}\left(\frac{\hat{\sigma}}{\sigma}\right)^{n-1} e^{-\frac{n}{2}\left[\left(\frac{\hat{\sigma}}{\sigma}\right)^2 + \left(\frac{\hat{\sigma}}{\sigma} z_{\hat{R}} - z_R\right)^2\right] + \frac{1}{2} z_{\hat{R}}^2} d\hat{\sigma}$$

letting $s = \dfrac{\hat{\sigma}}{\sigma}$

$$h(\hat{R}) = K \int_0^\infty s^{n-1} e^{-\frac{n}{2}\left[s^2 + \left(s z_{\hat{R}} - z_R\right)^2\right] + \frac{1}{2} z_{\hat{R}}^2} ds \tag{6}$$

Since s is a dummy variable of integration and $\hat{R}$ is the argument of h, the only numbers upon which the form of h is dependent are R and n. The density of the reliability estimator is therefore a one parameter (R) density which is independent of the life density population parameters and mission life.

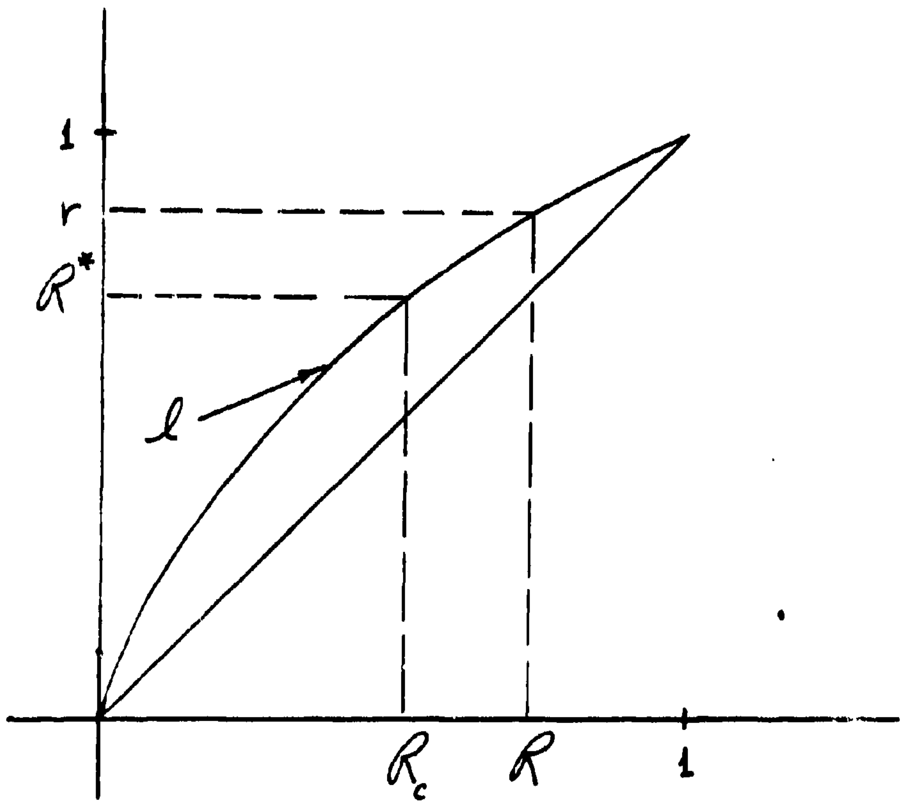
Changing the argument of n to avoid confusion and adding the subscript R to n to indicate its dependence on the population reliability R, the density of $\hat{R}$ is:

$$h_R(v) = K \int_0^\infty s^{n-1} e^{-\frac{n}{2}\left[s^2 + \left(z_R - s z_v\right)^2\right] + \frac{1}{2} z_v^2} ds \tag{7}$$

The distribution function of $\hat{R}$ is therefore given by:

$$H_R(r) = K \int_0^r \int_0^\infty s^{n-1} e^{-\frac{n}{2}\left[s^2 + (z_R - s z_v)^2\right] + \frac{1}{2} z_v^2} \, ds \, dv \qquad (8)$$

The meaning of the Neyman method of finding a one sided confidence interval for R may be explained through the following diagram:



The curve 1 is determined by:

$$P(\hat{R} < r \,;\, R) = C$$

or

$$H_{\hat{R}}(r) = C \tag{9}$$

i.e., 1 is determined such that for any population reliability R, the reliability estimator $\hat{R}$ falls below 1 100 C % of the time. Suppose now that the true value of population reliability is R as shown in the figure. We don't know R or r but we do know 1. If the experiment is now performed and the reliability estimate R* is calculated, the 100 C % lower confidence limit on R is R_c from the diagram. This is so because if the experiment is performed many times, R* will be below r 100 C % of the time and hence R_c will be below R 100 C % of the time.

Confidenced reliability R_c is therefore determined by solving for R_c:

$$H_{R_c}(R^*) = C \tag{10}$$

Before this is done, however, the distribution function of $\hat{R}$ should be simplified. Changing the order of integration in Eqn. (8) and making some appropriate changes of variable, one has:

$$H_{R}(r) = \frac{K}{\sqrt{n}} \int_0^\infty s^{n-2} e^{-\frac{n s^2}{2}} \left\{ 1 - \Phi\left[\sqrt{n}\left(z_R - s z_r\right)\right]\right\} ds \tag{11}$$

Multiplying out in Eqn. (11) and evaluating an integral, one has:

$$H_R(r) = 1 - \frac{2\left(\frac{n}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty s^{n-2} e^{-\frac{ns^2}{2}} \Phi\left[\sqrt{n}\left(z_R - s z_r\right)\right] ds \tag{12}$$

Using Eqn. (12), Eqn. (10) now becomes:

$$\frac{(1-c)\,\Gamma\left(\frac{n-1}{2}\right)}{2\left(\frac{n}{2}\right)^{\frac{n-1}{2}}} = \int_0^\infty s^{n-2} e^{-\frac{ns^2}{2}} \Phi\left[\sqrt{n}\left(z_{R_c} - s z_{R^*}\right)\right] ds \tag{13}$$

If R* were calculated using the desired mission life and the sample parameter estimates, Eqn. (13) could be solved numerically for R_c, but for purposes of calculating tables, it is better to stipulate R_c and solve Eqn. (13) for $z_R{}^*$ instead. This was done for confidence levels of 90% and 95%, confidenced reliabilities of .999, .995, .99, .975, .95, .925, .90, .875, .85 and sample sizes of 2—10, 15, 20, 25, 30.

The equations used with the tables to calculate mission life for the normal and lognormal models are:

$$x = \mu^* - \sigma^* z_{R^*} \tag{14}$$

$$x = e^{\mu_e^* - \sigma_e^* z_{R^*}} \tag{15}$$

The lognormal case is only trivially different from the normal case because the lognormal reliability estimator is

$$\hat{R}_{\ell} = 1 - \Phi\left(\frac{\ln x - \hat{\mu}_{\ell}}{\hat{\sigma}_{\ell}}\right) \tag{16}$$

and the logs of the data are by definition normally distributed.

C=90%

R_c	n = 2	3	4	5	6	7	8	9	10	15	20	25	30
.999	34.764	11.820	8.232	6.833	6.086	5.619	5.297	5.060	4.879	4.363	4.113	3.962	3.85[illegible]
.995	28.972	9.911	6.914	5.742	5.115	4.722	4.451	4.252	4.099	3.663	3.451	3.322	3.23[illegible]
.99	26.163	8.990	6.280	5.217	4.647	4.290	4.044	3.862	3.723	3.325	3.131	3.013	[illegible].93[illegible]
.975	22.042	7.647	5.354	4.451	3.966	3.661	3.450	3.294	3.[illegible]74	2.831	2.664	2.562	2.492
.95	18.512	6.505	4.569	3.801	3.387	3.126	2.944	2.811	2.707	2.411	2.265	2.176	2.115
.925	16.232	5.772	4.064	3.383	3.015	2.781	2.619	2.499	2.407	2.139	2.008	1.927	1.872
.90	14.500	5.215	3.681	3.066	2.732	2.520	2.372	2.262	2.177	1.932	1.811	1.737	1.685
.875	13.083	4.760	3.367	2.806	2.500	2.305	2.168	2.067	1.989	1.762	1.649	1.579	1.532
.85	11.877	4.371	3.099	2.583	2.300	2.120	1.994	1.900	1.827	1.615	1.509	1.444	1.399

R_c	n: 2	3	4	5	6	7	8	9	10	15	20	25	30
							C=95%						
.999	69.687	16.971	10.64	8.387	7.243	6.548	6.080	5.742	[illegible]	4.769	4.430	4.228	4.091
.995	58.084	14.241	8.947	7.058	6.097	5.512	5.118	4.832	4.615	4.010	3.722	3.550	3.434
.99	52.458	12.924	8.132	6.419	5.545	5.014	4.654	4.394	[illegible]	3.644	3.381	3.223	3.116
.975	44.205	11.006	6.945	5.488	4.742	4.288	3.980	3.757	[illegible]	3.110	2.883	2.746	[illegible]
.95	37.137	9.377	5.940	4.699	4.062	3.672	3.407	3.215	3.068	2.656	2.458	2.339	2.259
.925	32.573	8.331	5.295	4.193	3.625	3.276	3.040	2.867	2.735	2.364	2.185	2.076	2.003
.90	29.107	7.539	4.806	3.809	3.293	2.976	2.760	2.603	2.482	2.141	1.976	1.876	1.808
.875	26.273	6.891	4.406	3.495	3.022	2.730	2.531	2.386	2.274	1.958	1.804	1.711	1.647
.85	23.860	6.338	4.064	3.226	2.789	2.519	2.335	2.199	2.096	1.800	1.657	1.569	1.509

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WVT-6937 1969

REAL-TIME SIMULATION TECHNIQUE FOR EVALUATING A GYRO-SEEKER ASSEMBLY

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ABSTRACT. Missile simulations of systems including a gyro-seeker guidance assembly have often excluded the gyro-seeker representation by assuming that some ideal proportional tracking ratio will be achieved. Thus, some of the basic characteristics of the guidance loop are omitted or approximated. This paper develops a real-time simulation technique so as to include the basic functions of the gyro-seeker assembly such as precession, nutation, drift, gain, noise, etc.

In the first section of the paper, the model equations are derived and are used in a discussion of the system parameters and system dynamics.

The second section of the paper presents the analog computer mechanization and results of the simulation, some of which have been verified by system experiments, and some predicted by analytical theory.

SYSTEM DESCRIPTION. The basic gyro-seeker unit consists of a gyroscope, rotating gyro magnet, and stationary induction coils about the gyro. (See Figure 1.) Target source energy is collected and focused to produce a spot image on a reticle centered on the spin axis. When the image spot is off center the reticle pattern produces an error signal which is modulated at spin frequency. The amplitude of the error signal is a function of the radial displacement of the image from the reticle center, while the phase corresponds to azimuthal position about the seeker axis. After being amplified and filtered, the signal is fed to the precession coil which torques the gyro magnet so as to precess the gyro toward a null position with respect to the line of sight. The processed a.c. signal can also be demodulated into orthogonal components using reference coils. The demodulated d.c. components can be used for tracking or guidance signals.

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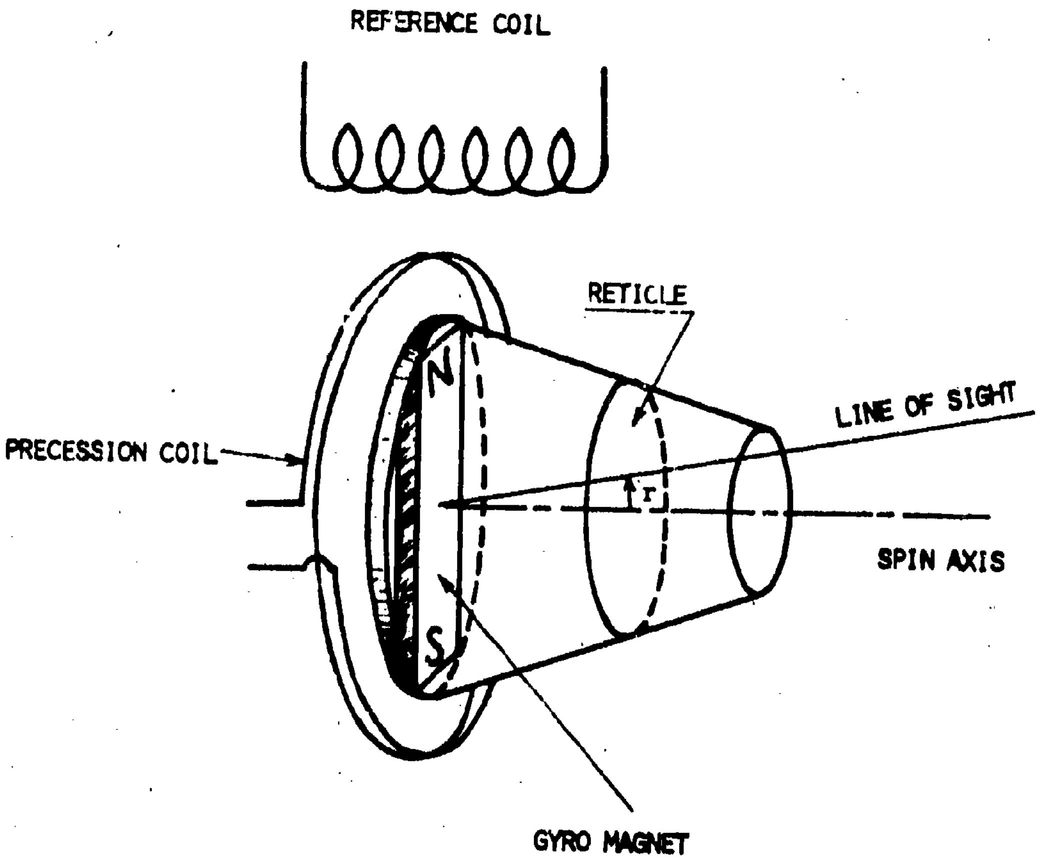

REFERENCE COIL
RETICLE
LINE OF SIGHT
PRECESSION COIL
N
r
SPIN AXIS
S
GYRO MAGNET

FIGURE 1

Mathematical Model

Two coordinate systems will be referenced in the model derivation. The g> system, or ground system, will be fixed at the initial gyro position with $\bar{g}_1$ axis horizontal and pointed at the initial target ground position. The s> coordinate system, or seeker system, will be fixed to the center of gravity of the gyro assembly. The $\bar{s}_1$ axis is along the gyro spin axis and $\bar{s}_2$ is along the North-South axis of the gyro magnet. (See Fig. 2).

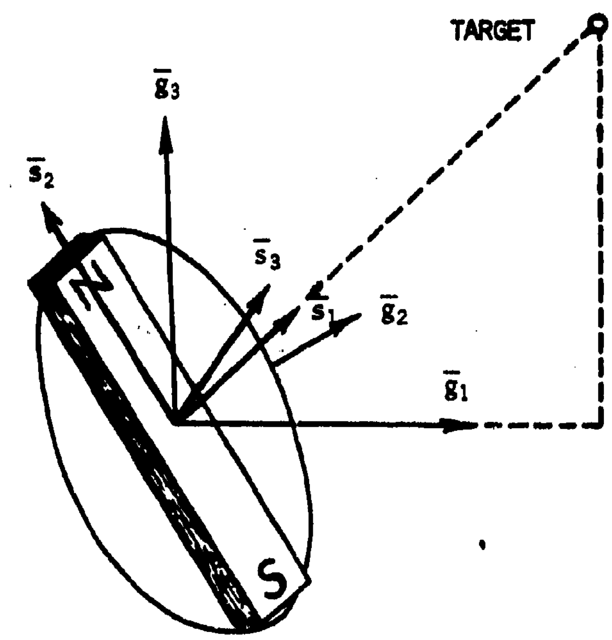


FIGURE 2

The yaw, pitch, roll sequence for the Euler transformation between coordinate systems is given by,

$s> = M(\phi_1, \phi_2, \phi_3)\ g>$, where the subscript denotes rotation about the respective axis. In detail then, the model Euler equations are:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{bmatrix} C\phi_3\ C\phi_2 & S\phi_3\ C\phi_2 & -S\phi_2 \\ C\phi_3\ S\phi_2\ S\phi_1 - S\phi_3\ C\phi_1 & S\phi_3\ S\phi_2\ S\phi_1 + C\phi_3\ C\phi_1 & C\phi_2\ S\phi_1 \\ C\phi_3\ S\phi_2\ C\phi_1 + S\phi_3\ S\phi_1 & S\phi_3\ S\phi_2\ C\phi_1 - C\phi_3\ S\phi_1 & C\phi_2\ C\phi_1 \end{bmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

$$\dot{\phi}_3 = \sec \phi_2\ (\omega_2 \sin \phi_1 + \omega_3 \cos \phi_1)$$

$$\dot{\phi}_2 = \omega_2 \cos \phi_1 - \omega_3 \sin \phi_1$$

$$\dot{\phi}_1 = \omega_1 + \sin \phi_2\ \dot{\phi}_3$$

$$\phi_3 = (\phi_3)_o + \int \dot{\phi}_3\ dt$$

$$\phi_2 = (\phi_2)_o + \int \dot{\phi}_2\ dt$$

$$\phi_1 = (\phi_1)_o + \int \dot{\phi}_1\ dt$$

To derive an equation for the seeker output signal we assume that the reticle is parallel to the $\bar{s}_2 - \bar{s}_3$ plane and centered on the $\bar{s}_1$ axis. The optics produce a target image point on the reticle, as shown in Fig. 3, whenever the $\bar{s}_1$ axis deviates from the line of sight. The radial displacement d of the image point from the reticle center is proportional to the angle ϵ between the $\bar{s}_1$ axis and the line of sight vector to the target. The distance d is now modeled by deriving a distance r in the $\bar{s}_2 - \bar{s}_3$ plane which is proportional to d. Let $\bar{\sigma}$ be the unit

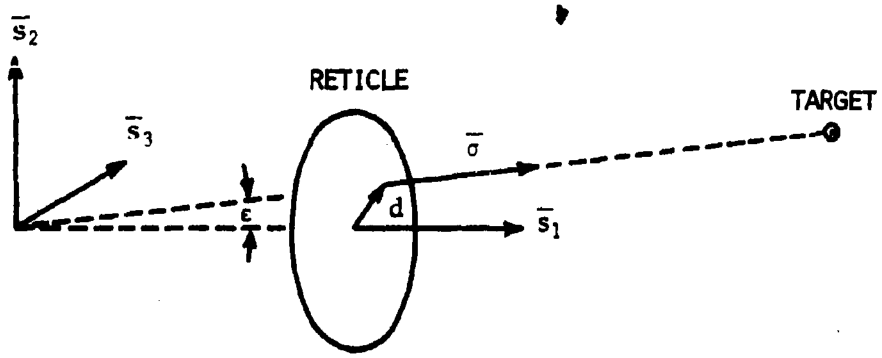


FIGURE 3

vector along line of sight vector in the ground coordinate system. Then,

$$\bar{\sigma} = \frac{X}{R}\bar{g}_1 + \frac{Y}{R}\bar{g}_2 + \frac{Z}{R}\bar{g}_3, \text{ where}$$

$$R = \sqrt{X^2 + Y^2 + Z^2}\ .$$

Let σ_{s_2} and σ_{s_3} denote the components of $\bar{\sigma}$ in the seeker coordinate system. (See Fig. 4). Then $r = \sqrt{\sigma_{s_2}^2 + \sigma_{s_3}^2}$ depends on ε and

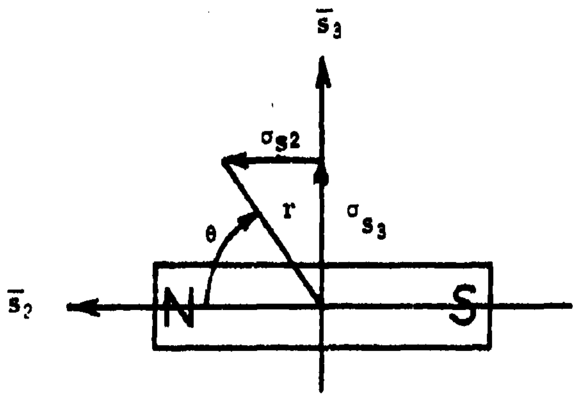


FIGURE 4

not on the length of $\bar{\sigma}$. Also the length r is proportional to the actual displacement d which depends on the optics and other physical parameters. The azimuthal phase determined by the angle θ measured from the $\bar{s}_2$ axis to the image point line is given by $\sin\theta = \frac{\sigma_{s_3}}{r}$. (See Fig. 4). This derivation assumes that reticle modulated output has the form of a sine wave. Other wave shapes could be generated by using various reticle patterns and electronic processing. For a discussion of other wave forms and their effect on gyro precession see reference 1. Then $e_o = Kr\sin\theta = K\sigma_{s_3}$ is an equation which represents the seeker output signal. To compute σ_{s_3} we recall that,

$\bar{\sigma}_s = M(\phi_1, \phi_2, \phi_3)\,\bar{\sigma}g$, so that

$$\sigma_{s_3} = \frac{X}{R}[c\phi_3\, s\phi_2\, c\phi_1 + s\phi_3\, s\phi_1] + \frac{Y}{R}[s\phi_3\, s\phi_2\, c\phi_1 - c\phi_3\, s\phi_1] + \frac{Z}{R} c\phi_2\, c\phi_1.$$

The signal $e_o = Kr\sin\theta$ is amplified and applied to the coil about the gyro whose field acts on the permanent gyro magnet to precess the gyro. The variation of the magnetic field is thus proportional to $r\sin\theta$ and its direction is perpendicular to the plane of the coil as shown in Fig. 5 . Suppose that the magnetic field $\bar{B}_c$ of the coil makes an angle λ with respect to $\bar{s}_1$, then, $|\bar{B}_c \cdot \bar{s}_1|\,\bar{s}_1 = |\bar{B}_c|\cos\lambda\,\bar{s}_1$. The magnetic field of the permanent magnet can be written simply

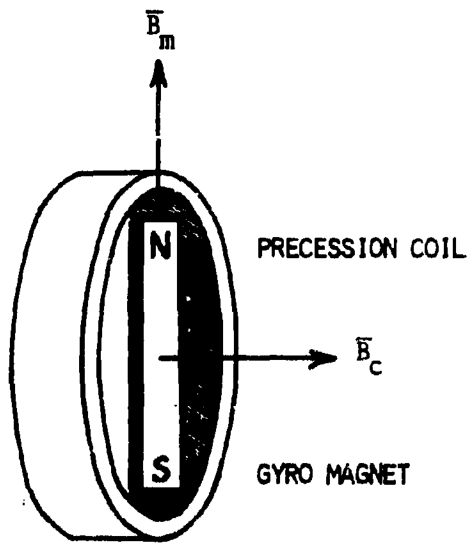


FIGURE 5

as $\bar{B}_m = |\bar{B}_m|\bar{s}_2$. The interaction of the two magnetic fields produces a torque which tends to align the two fields. This torque is given by the vector cross product,

$T_3\bar{s}_3 = |\bar{B}_m|\ \bar{s}_2 \times |\bar{B}_c| \cos\lambda\ \bar{s}_1$ where $|\bar{B}_c| = Kr \sin\theta$ and $|\bar{B}_m|$ is constant. Note that the torque T_3 will be a maximum when the spot line is along the $\bar{s}_3$ axis, then the rotation of the magnet will be in the $\bar{s}_2 - \bar{s}_1$ plane and the precession in the $\bar{s}_3 - \bar{s}_1$ plane. Let $\bar{T} = T_1\bar{s}_1 + T_2 s_2 + T_3\bar{s}_3$ be the total external torque acting on the gyro. Then the angular accelerations are given by,

$$\dot{\omega}_3 = \frac{T_3}{I_3} - \omega_1\omega_2\ \frac{[I_2 - I_1]}{I_3}$$

$$\dot{\omega}_2 = \frac{T_2}{I_2} - \omega_1\omega_3\ \frac{[I_1 - I_3]}{I_2}$$

$$\dot{\omega}_1 = \frac{T_1}{I_1} - \omega_3\omega_2\ \frac{[I_3 - I_2]}{I_1}$$

If we assume that the only torque acting is T_3 as derived above, and that the transverse moments are equal, then the equations reduce to:

$$\dot{\omega}_3 = \frac{T_3}{I_3} - \frac{[I_2 - I_1]}{I_3}$$

$$\dot{\omega}_2 = -\omega_1\omega_3 \frac{[I_1 - I_3]}{I_2}$$

Integrating the components of $\dot{\bar{\omega}}$ yields the angular velocity components of $\bar{\omega}$ of the seeker relative to the seeker coordinate system. Using the seeker to ground transformation, $\bar{\omega}$ can be transformed to $\bar{\omega}_g = \dot{\phi}_1 + \dot{\phi}_2 + \dot{\phi}_3$ by,

$$\dot{\phi}_3 = \sec \phi_2 \ (\omega_2 \sin \phi_1 + \omega_3 \cos \phi_1)$$

$$\dot{\phi}_2 = \omega_2 \cos \phi_1 - \omega_3 \sin \phi_1$$

$$\dot{\phi}_1 = \omega_1 + \sin \phi_2 \ \dot{\phi}_3.$$

Integrating $\dot{\phi}$ yields the angles ϕ of the ground coordinate system which are needed to compute θ_{s_3}. Thus the loop is closed.

Orthagonal components of the seeker signal are obtained by demodulating the signal using two reference signals. Reference signals can be obtained by mounting coils about the gyro 90° apart. (See Fig. 6). A sinusoidal voltage is produced by each coil as the flux lines of the gyromagnet cut the windings of the coil. These signals can simply be modeled as Ksin $(\phi_1+\gamma)$ and Kcos $(\phi_1+\gamma)$ where γ is the angle the coils are rotated

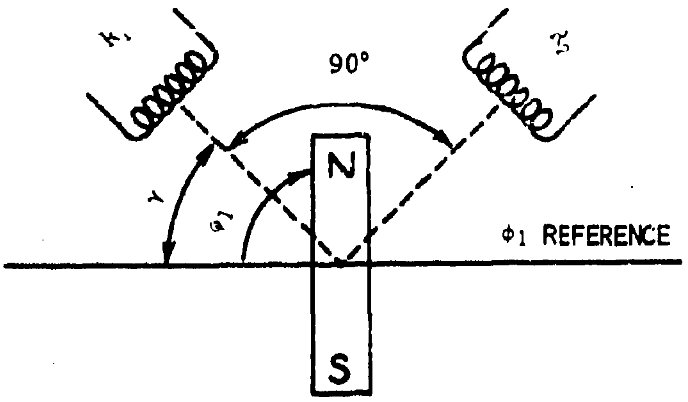
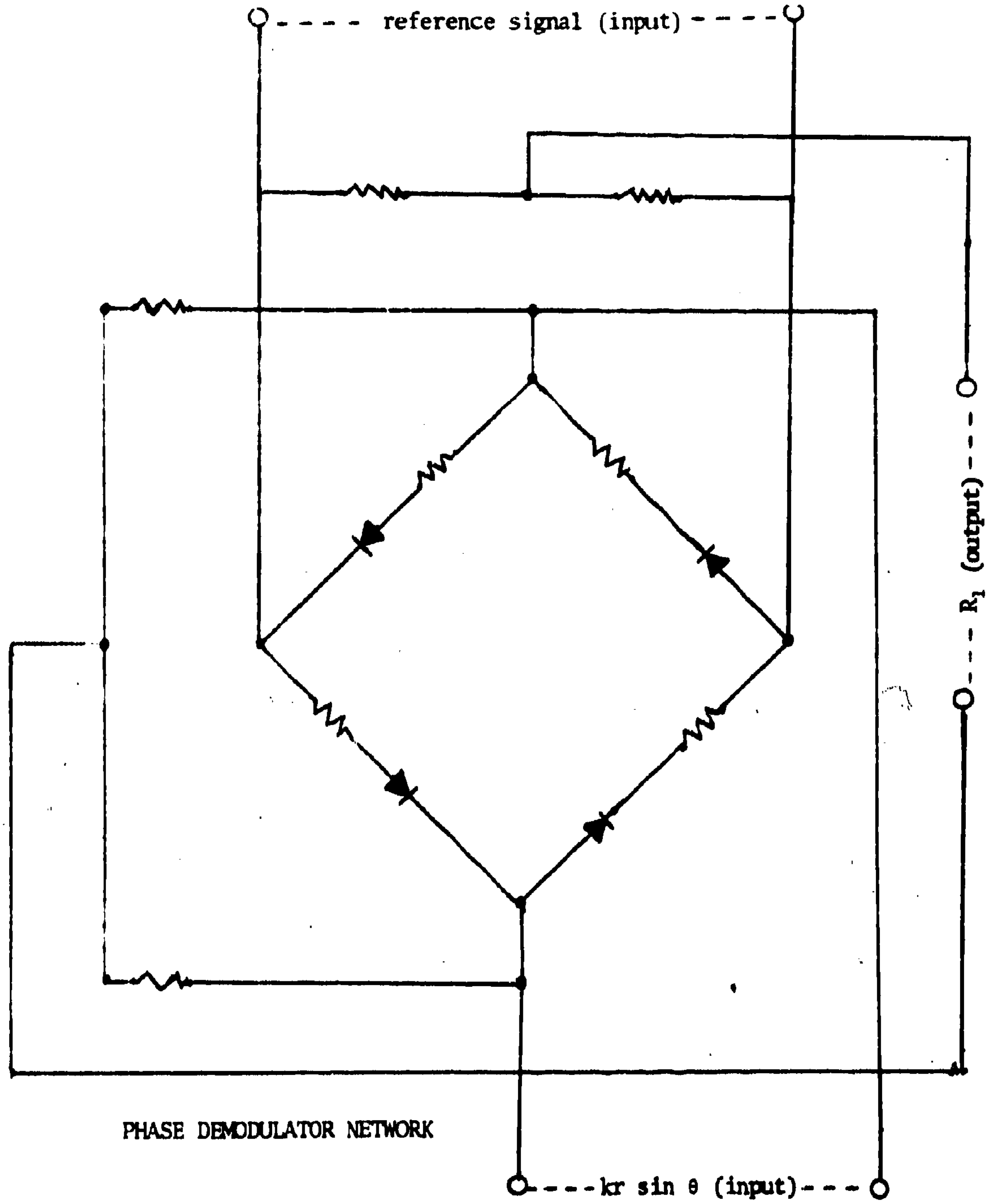


FIGURE 6

from the ϕ_1 reference. The inputs to each demodulator are the seeker signal and one of the reference signals. The filtered output is a d.c. level which is proportional to the error amplitude component in the respective plane or direction determined by the angle γ. The mathematical representation for the phase demodulator is not given since the actual electronic network is easily adapted to the analog computer components. Fig. 7 shows a typical phase demodulator bridge network used for one plane.

The basic gyro-seeker model which has been developed can easily be expanded or modified to include hardware changes or known parameter variations. For example, the actual seeker output signal could be a function of source intensity, target range, noise, filtering and other phase and amplitude



PHASE DEMODULATOR NETWORK

FIGURE 7

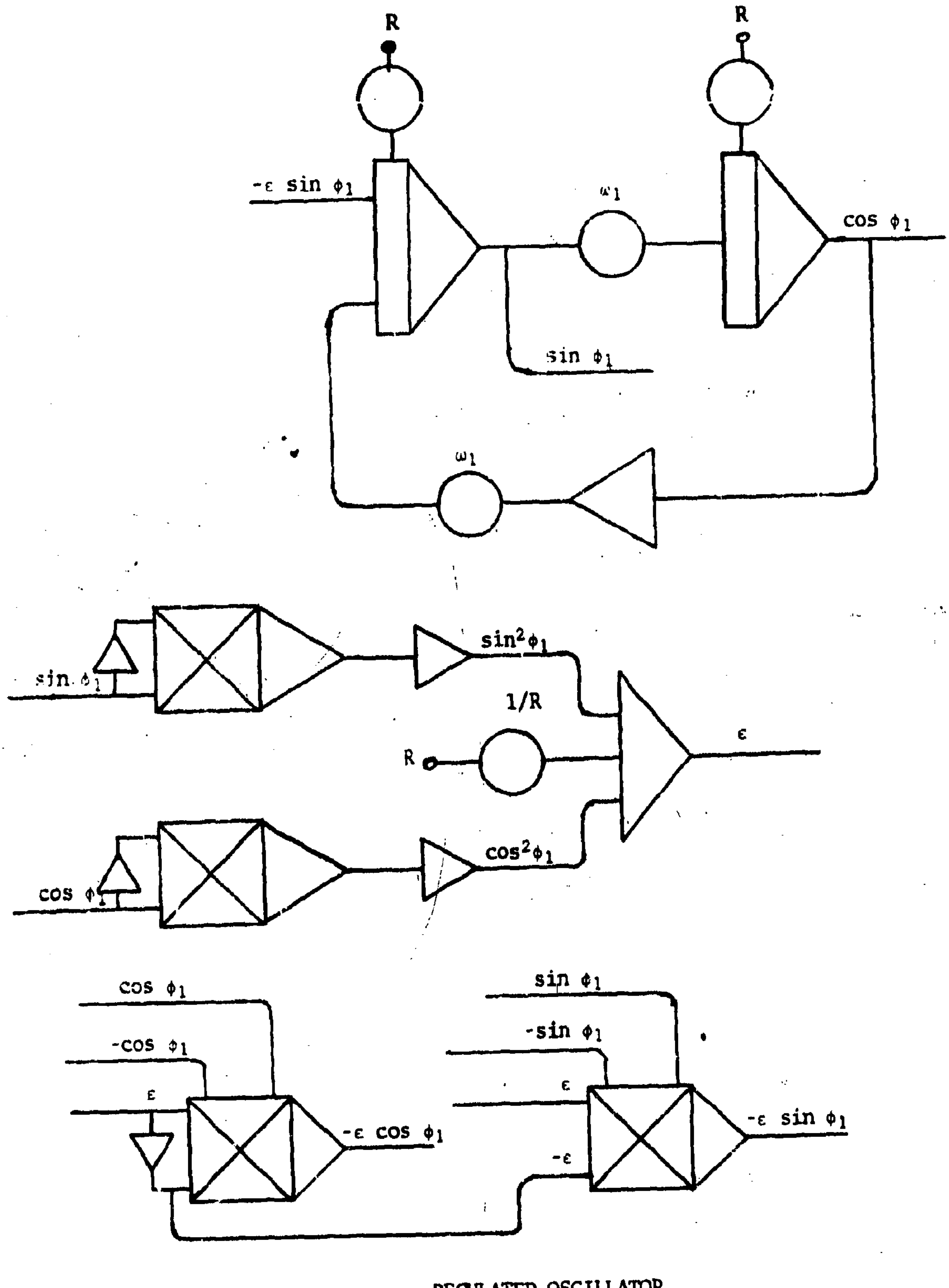
variations. Most of these variations can be modeled as transfer function additions or multiples of the signal $e_o = Kr \sin \theta$.

Simulation

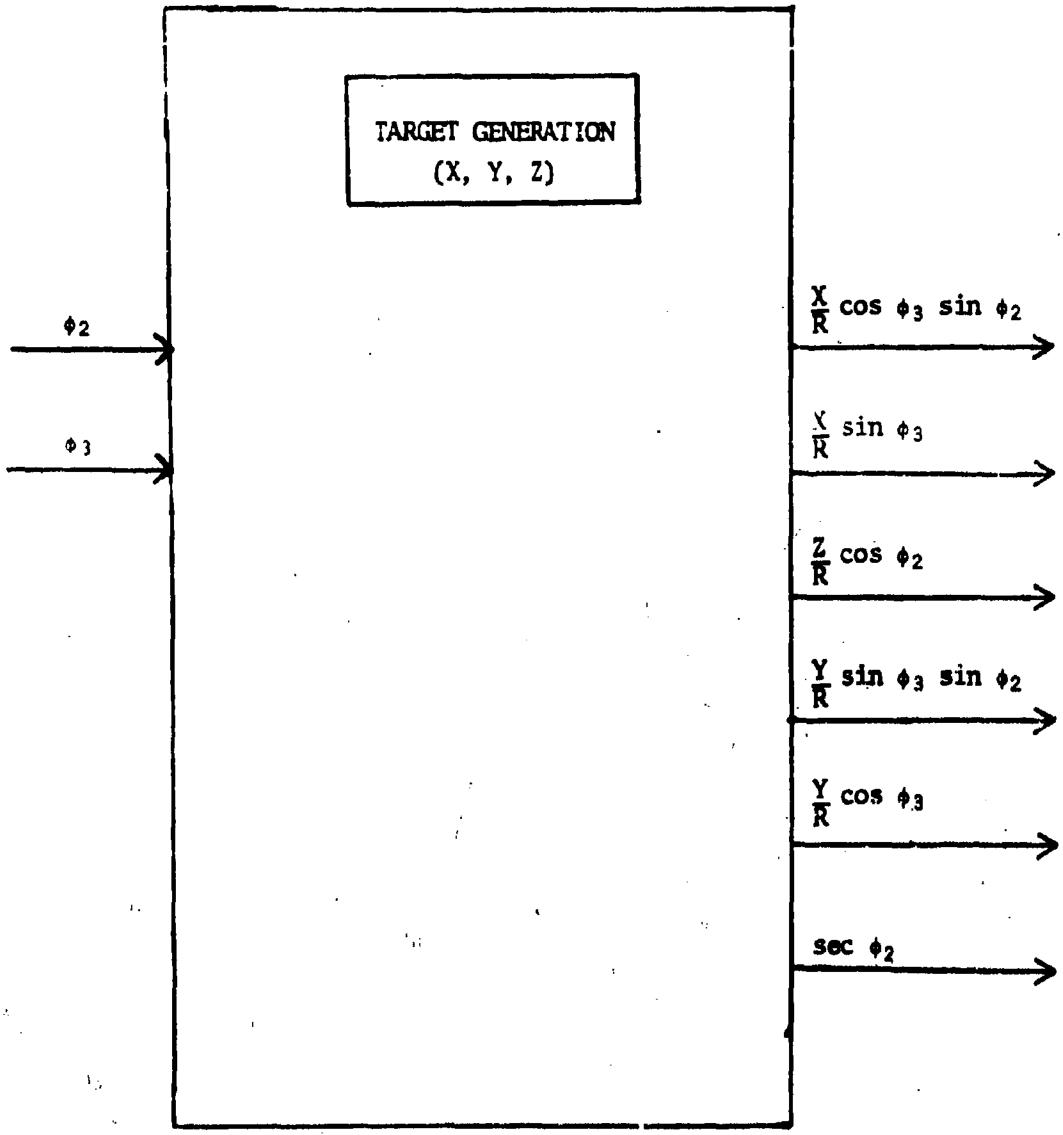
The simulation of the gyro-seeker employs the mathematical equations as derived in the math model with the following exceptions:

1. It was assumed that the gyro speed was constant. This is the case in a gyro when the motor-driving torque just balances the friction torques so the gyro spins at a constant rate. Thus sin ϕ_1 and cos ϕ_1 were obtained by running in oscillator at the required spin frequency.

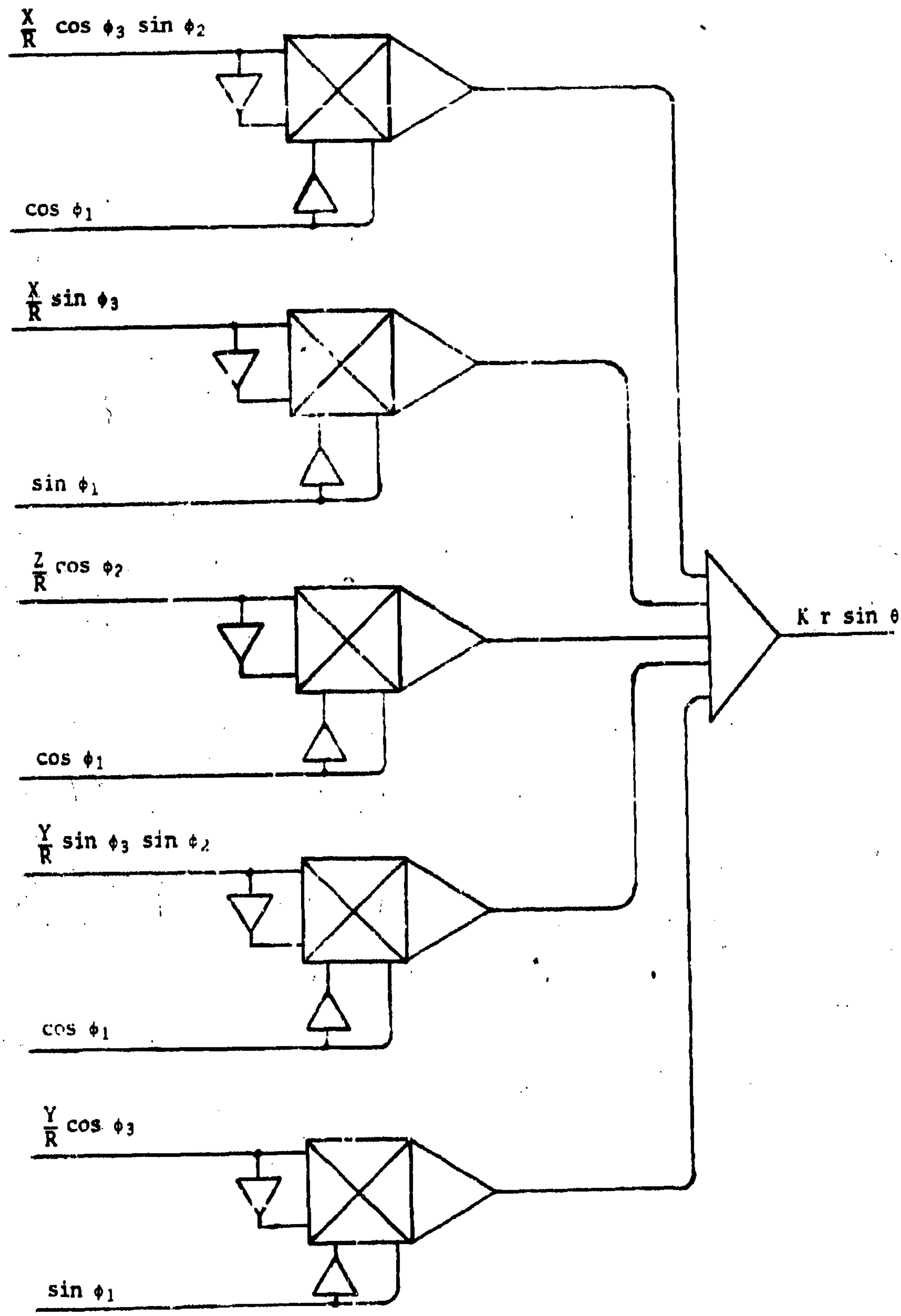
2. The equation $\dot{\phi}_1 = \omega_1 + \sin \phi_2 \dot{\phi}_3$ was approximated by the equation, $\dot{\phi}_1 = \omega_1$, i.e., it was assumed that $\sin \phi_2 \dot{\phi}_3 << \omega_1$. The computer mechanization diagrams are presented on pages 13 to 18. Since this particular mechanization is part of a hybrid missile simulation, some of the computations are shown as digital. It should be clear to the reader how on all-analog simulation could be obtained from the given mechanization. The parameters such as spin frequency, moments of inertia and loop gains were obtained from experimental data taken from the seeker hardware. After all parameters were obtained, the simulation was verified by comparing the response characteristics of the hardware with those of the simulation.

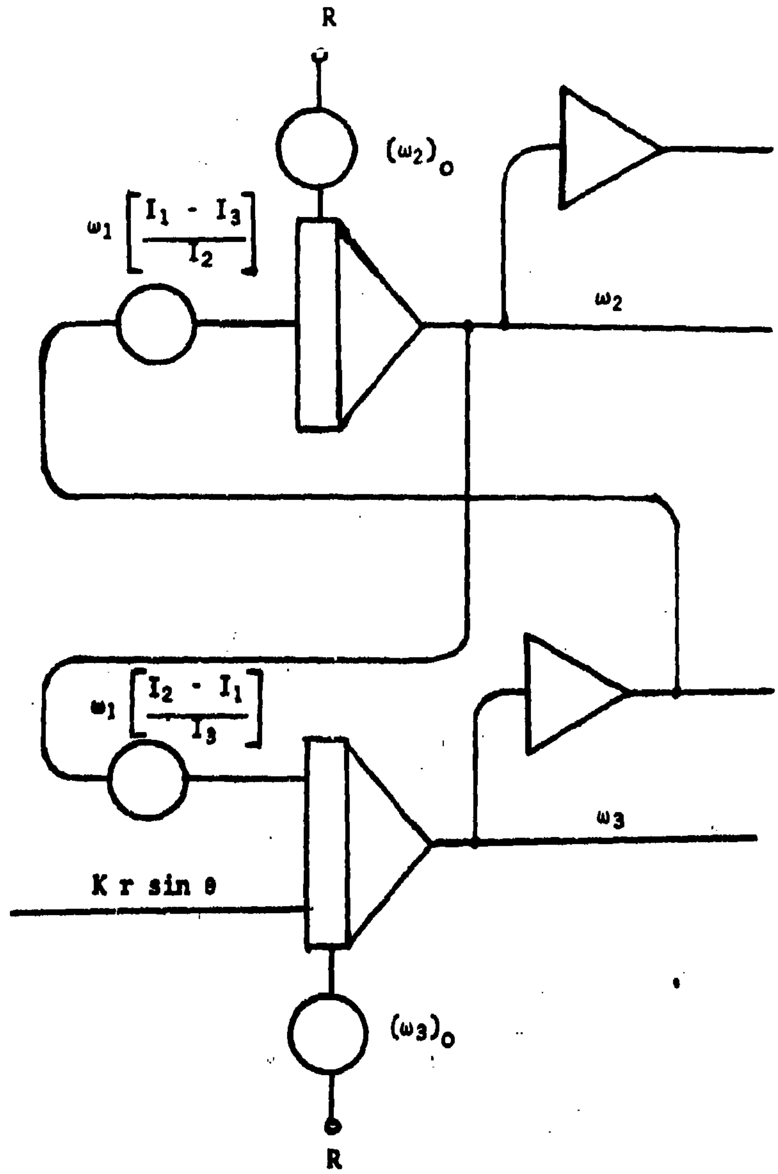


ω_1 REGULATED OSCILLATOR

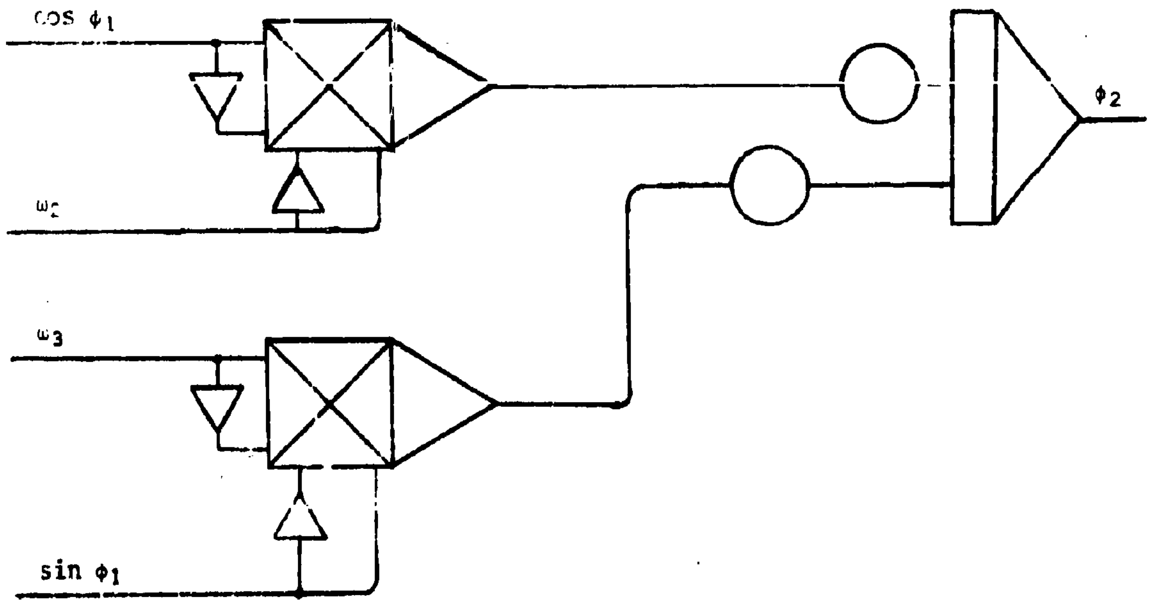


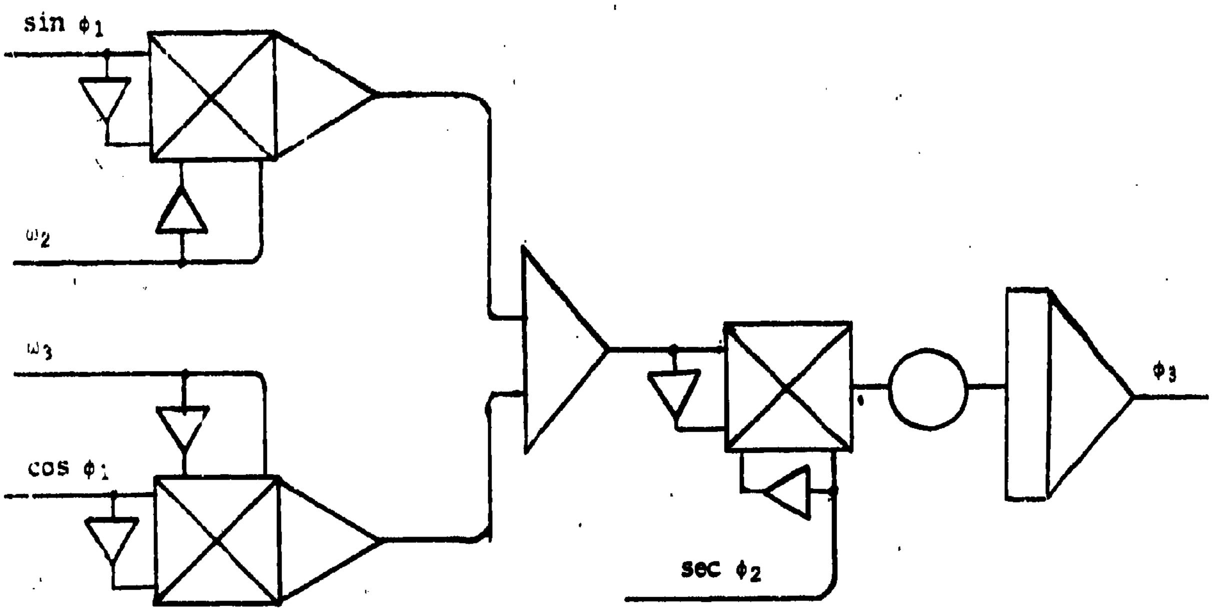
DIGITAL COMPUTATION

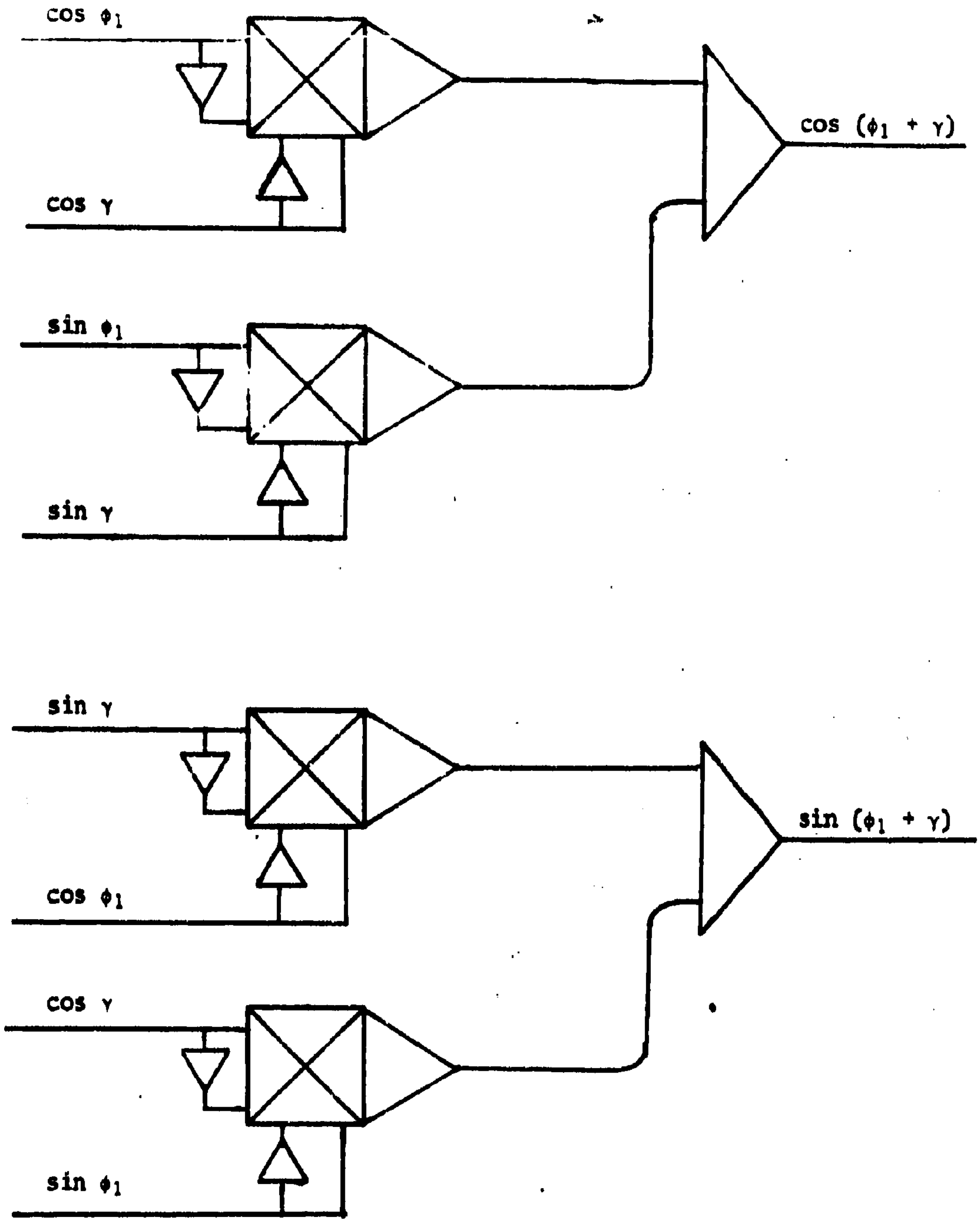

$\frac{X}{R}$ cos ϕ_3 sin ϕ_2
cos ϕ_1
$\frac{X}{R}$ sin ϕ_3
sin ϕ_1
$\frac{Z}{R}$ cos ϕ_2
cos ϕ_1
$\frac{Y}{R}$ sin ϕ_3 sin ϕ_2
cos ϕ_1
$\frac{Y}{R}$ cos ϕ_3
sin ϕ_1
K r sin θ



GYRO NUTATION OSCILLATOR


cos φ1
ω3
sin φ1
φ2


sin φ1
ω2
ω3
cos φ1
sec φ2
φ3



REFERENCE COIL SIGNALS

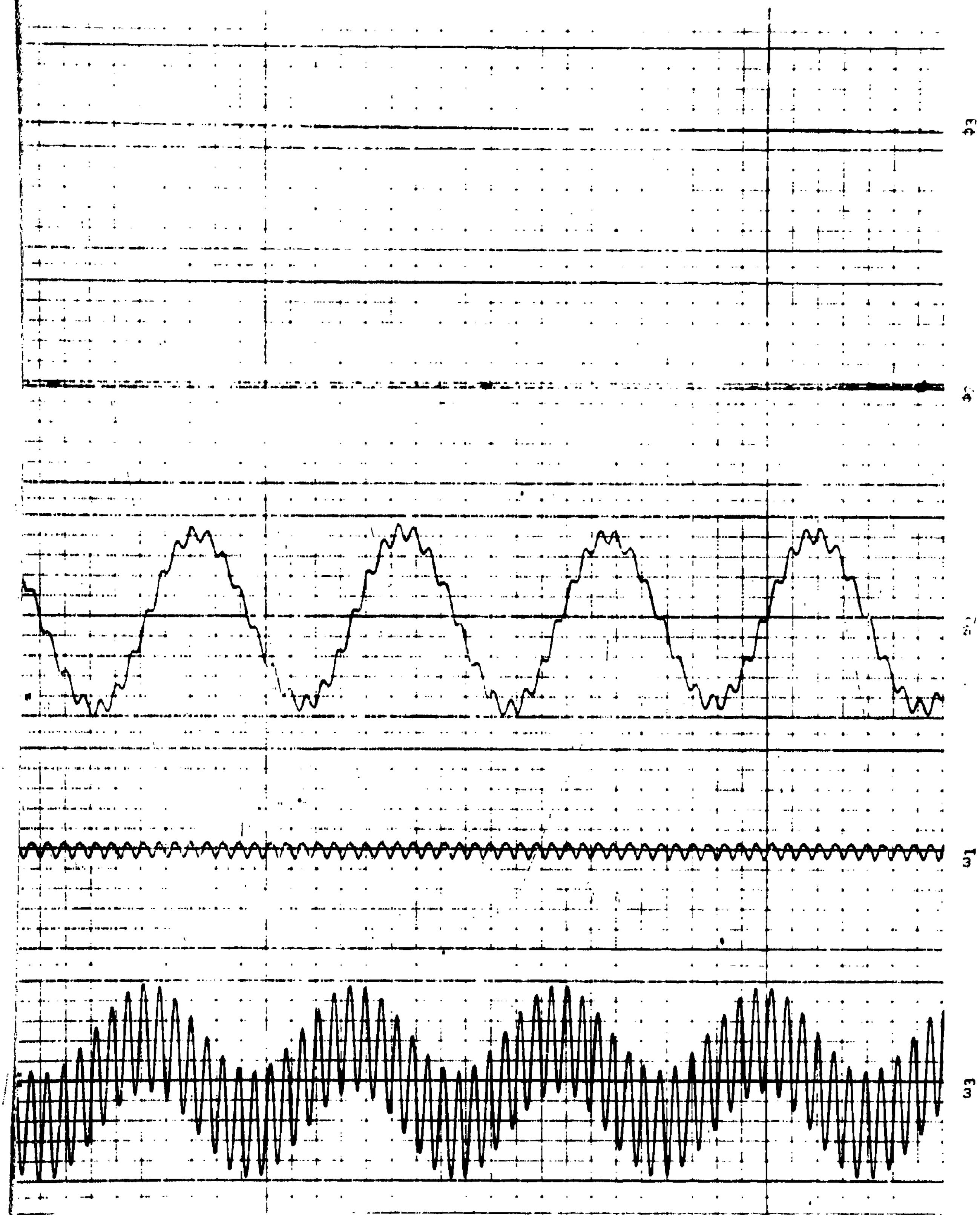
Some simulation recordings and description of results are presented on the following pages.

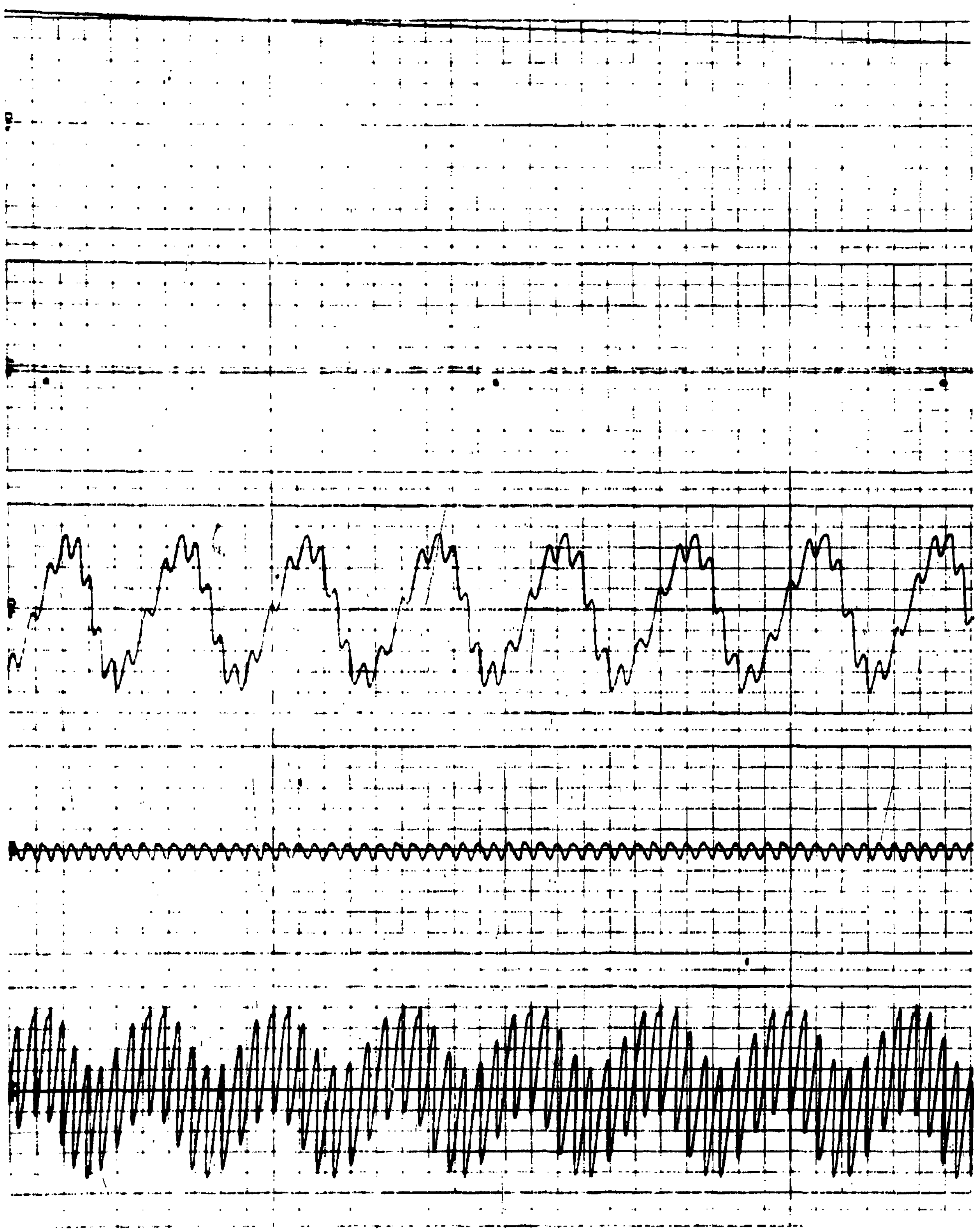
Pages 21 and 22 show the response of the gyro alone to a constant amplitude sinusoidal torque at spin frequency applied about the $\bar{g}_3$ axis and phased so as to precess the gyro in the ϕ_3 plane. The difference in the two recordings is the result of a change in the moment of inertia ratio $\rho = {}^{I_3}/I_1$ which can be seen as a change in the nutation frequency on the ω_2 and ω_3 channels. The frequency of the nutation is determined by ω_1 $(1 - \rho)$ while the amplitude depends on the initial conditions $(\omega_2)_0$ and $(\omega_3)_0$. (See, e.g., reference 3 for an analytical derivation). The gyro precession which is seen as a change in ϕ_3 is proportional to the amplitude of the applied torque.

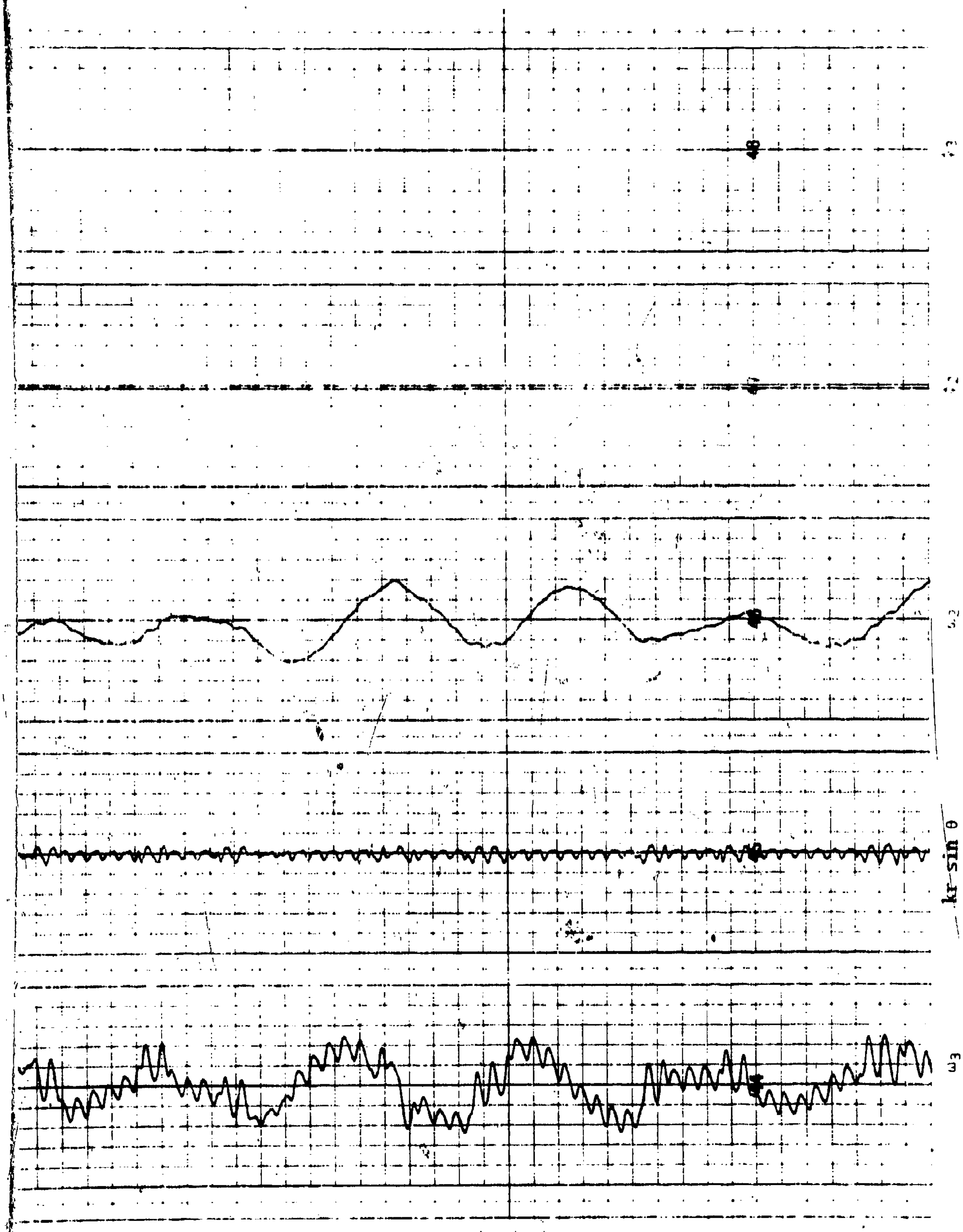
Page 23 shows the closed loop response for a given gain K_1 as a multiple of the feedback signal $r \sin \theta$. In this case the seeker was not tracking (as can be seen by ϕ_2 and ϕ_3), but was locked to a stationary target, $X = C$, $Y = 0$, $Z = 0$. Thus K_1 $r \sin \theta$ has a small amplitude and a phase which is changing rapidly to compensate for directional changes of the spin axis from the line of sight. Page 24 shows the response to a target moving at a constant rate in the Y direction. This condition results in an error signal K_1 $r \sin \theta$ which has a constant amplitude, reflecting the constant target rate, and a fixed phase dictated by the

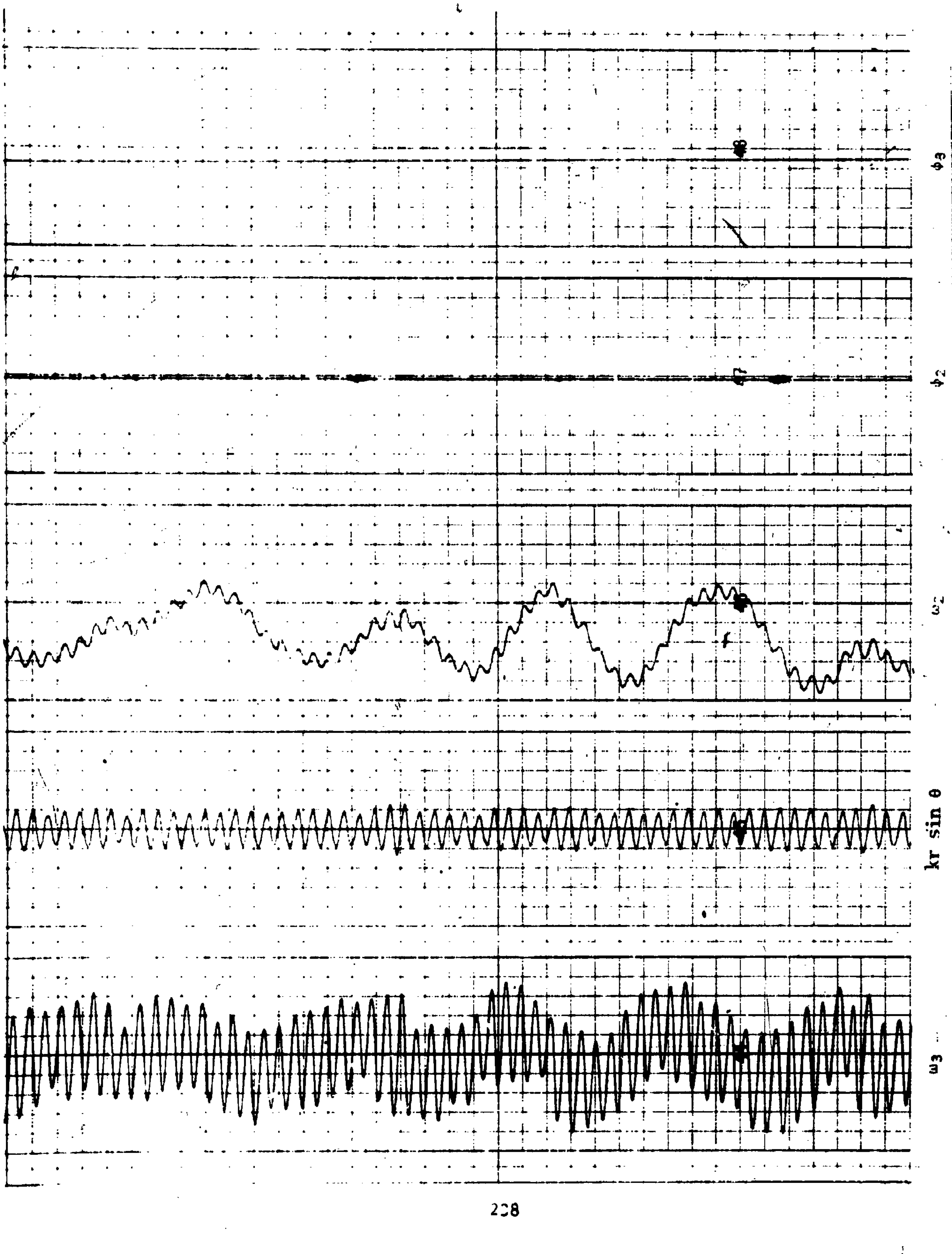
Y direction. (Note that since the torque applied to the gyro is about the $\bar{s}_3$ axis, it appears on the ω_3 channel.) The demodulated outputs of the error signal for target rates in the Y and Z direction are shown on pages 25 and 26 respectively. In this case γ was chosen as 0°, so that r sin θ was in phase with one reference signal and 90° out of phase with the other reference signal, for each tracking condition. Thus the perpendicular tracking directions result in alternate full value and zero value readings on the demodulated outputs R_1 and R_2 as shown on pages 25 and 26 . The slow rise of the R_1 and R_2 signals is due to filtering on the demodulated outputs and not to the demodulator circuits.

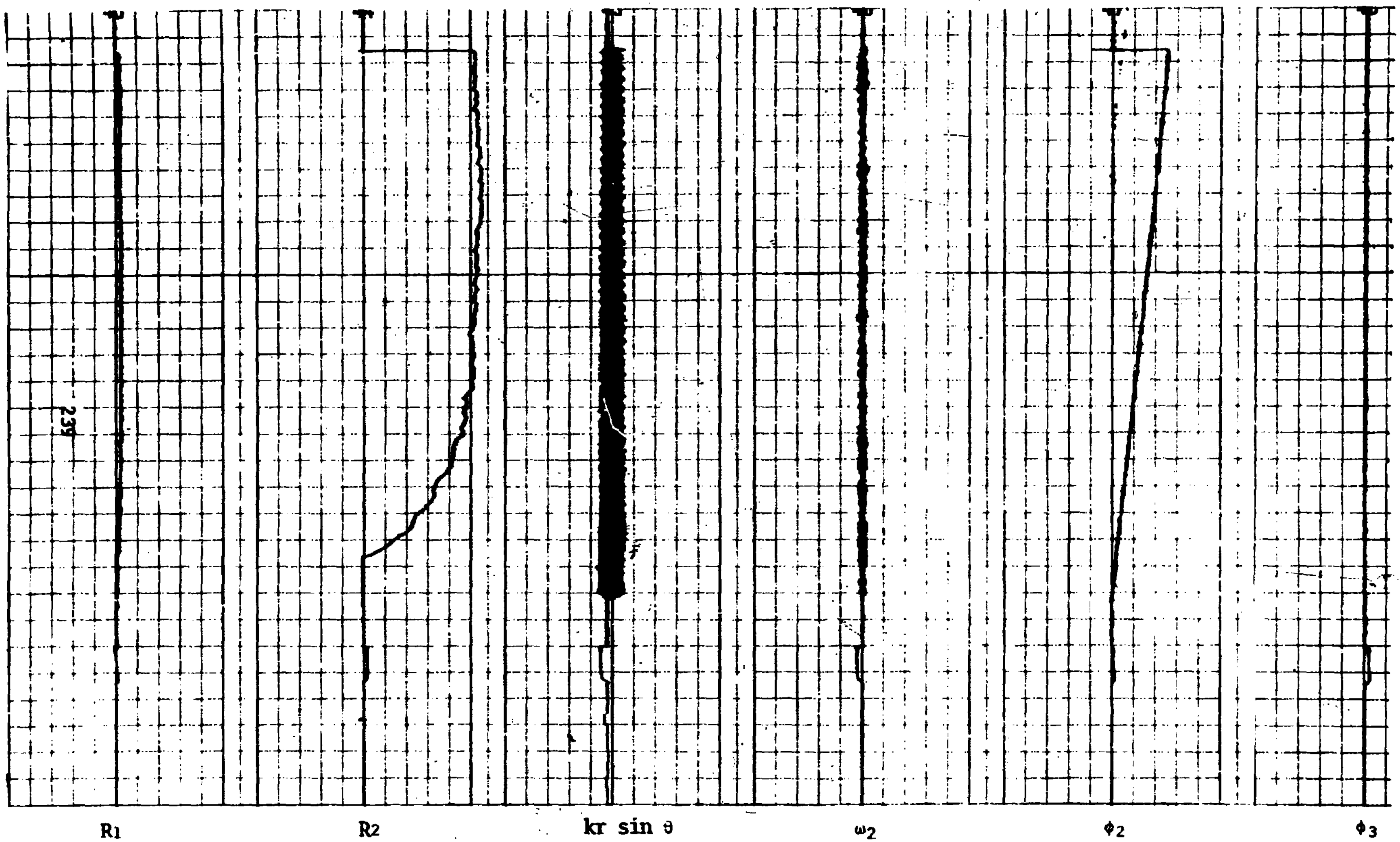
Error signals can also be produced by gyro motion such as gyro drift. If we assume a constant gyro drift rate and stationary target conditions then an error signal is produced to overcome the drift. Page 27 shows the simulation results for these conditions. In this case drift was produced by an appropriate torque in the ϕ_3 plane. The result of the error signal, Kr sin θ, generated by these conditions can best be observed on the demodulated outputs. One can think of these outputs as false tracking commands caused by gyro drift.

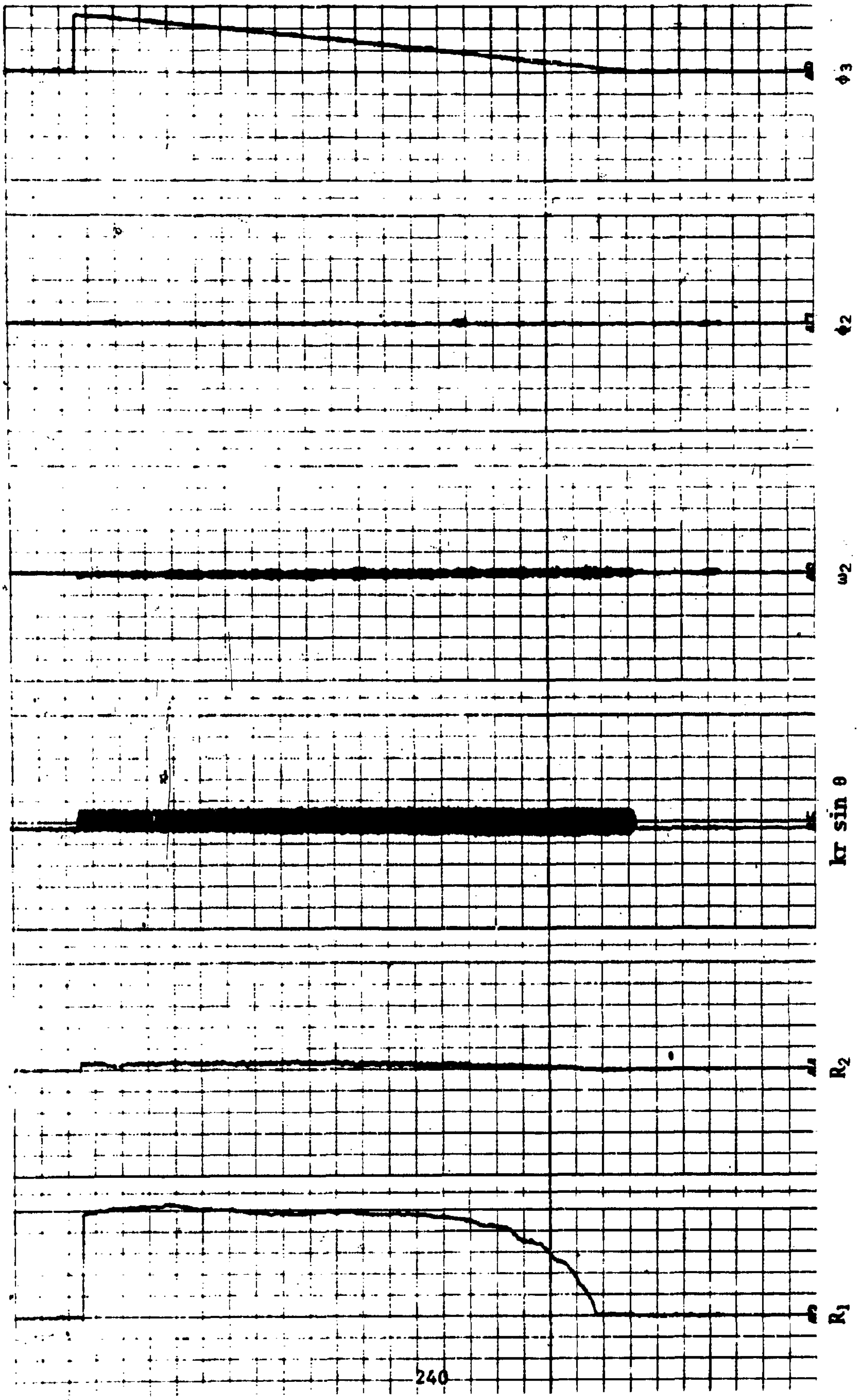

ω1
ω3



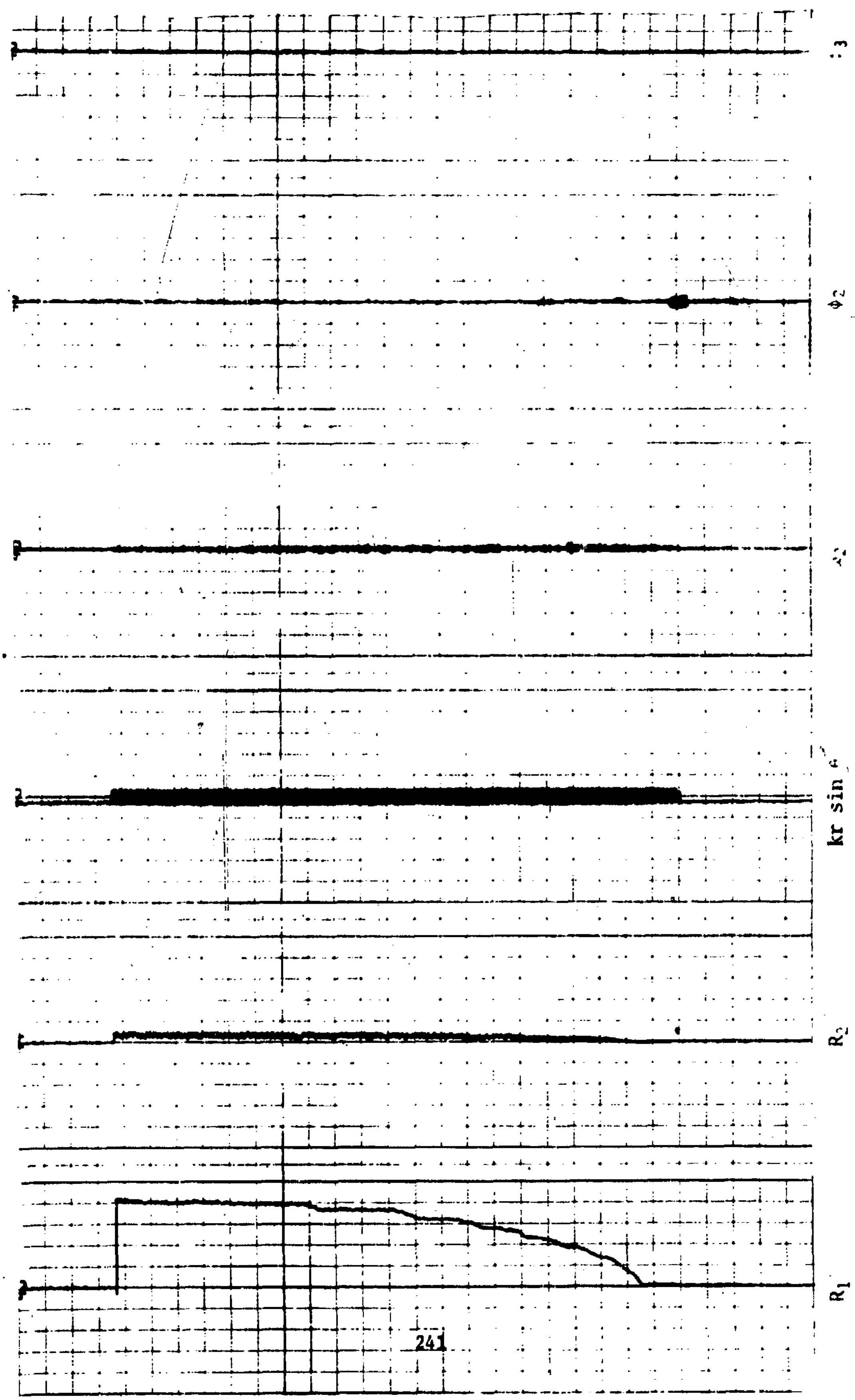

kr sin θ
ω_3


ϕ_3
ϕ_2
ω_2
kr sin θ
ω_3


R_1
R_2
$kr \sin \theta$
ω_2
ϕ_2
ϕ_3



φ3
φ2
ω2
kr sin θ
R2
R1



Φ_2
kr sin
R_2
R_1

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AN EMPIRICAL APPROACH TO ANALYSIS OF THE INTERACTION CHARACTERISTICS OF A SIX-COMPONENT ROCKET ENGINE TEST STAND

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INTRODUCTION. LANCE Developmental testing within the T&RE Laboratory has been accomplished on test stands designed to measure the six components of thrust reaction. The basic problem inherent in such stands is that of restraining the engine with a measurement system that permits the engine six degrees of freedom, without the introduction of unknown effects upon the engine. This, of course, is further compounded by the requirement to supply propellants thru a high pressure plumbing system that shunts the measurement system. It is obvious that a thorough stand calibration program must be implemented to resolve this problem.

The facility requirements stated for the present phase of LANCE are presented in Table I. Paralleling these are our estimates of our then existing capability. These estimates were derived from extensive calibration tests performed on the original test stand. Only a casual observance is required to realize that this represents a significant step forward. A less casual but limited preliminary error analysis indicated more clearly the difficulties involved and concluded that a portion of the requirement was clearly beyond the state-of-the art. Briefly this is indicated, when load cell accuracy requirements are derived from consideration of the angularity and position requirements. Considering only the load cell capacities, dictated by the thrust magnitude and the practical geometry of the stand, the position requirement means a vertical load cell resolution of 0.05% F.S. and the angularity requirement a side load cell resolution of 0.3% F.S. The thrust and side force requirements are much less severe - approximately by a factor of 3 times.

This dilemma was resolved by a joint decision to proceed on a best efforts basis. It is this effort that I will summarize this afternoon.

II. PRINCIPAL DESIGN FEATURES. The present Thrust Measurement System, (Figure 1) like its predecessor system, contains seven load cells. This is the TP series stand with component convention illustrated. The four vertical cells are parallel and symmetrically placed about the vertical centerline or Z axis. These cells react thrust as well as the moments about the two horizontal axis designated MX and MY. Side forces designated X and Y are reacted by the horizontal cells placed on or parallel to these axes. The pair of cells paralleling the Y axis also reacts the roll moment (MZ). Each load cell is assembled with flexures to permit maximum compliance with all modes of loading except those acting directly along the cell axis. Thus, the magnitudes of force interaction inherent in the system are minimized.

Figure 2 illustrates additional features. These views are of the present system. Features to be noted are:

1. The unsupported long elbow sections of pipe in the propellant line approaches to the engine.

2. Arrangement of the three lines approaching the engine symmetrically about the vertical centerline in an attempt to balance the restraints.

3. Displacement of the plumbing from the centerline to permit application of single point calibration loads.

4. Inclusion of rupture disc housings (appearing as boxes between the lines and engine) that permit the pretest installation of these discs without disturbing the propellant line connections to the measurement system.

Load cell placement and alignment of the system and especially the alignment of the calibration input devices are of critical concern. Optical tooling was used to control these factors to precisions of better than .005 inches in position and .05 mil radians in orientation.

The basic calibration scheme developed for the original stand was first employed in the calibration of this stand. The premise of this calibration attempt was that there are many sources of interaction which combine to produce the net effect on the system. These include each of the components of force input, the static pressurization of the propellant lines, the dynamics of flow through these lines and the effects of temperature over the conditioned range of -40 to 160°F. Thus, each source was tested and its effects observed. It was further presumed that these independently derived effects could be summed to express the net effect. This approach is illustrated in the slide by the multiple calibration devices.

III. REVISED CALIBRATION METHOD. Time will not permit a detailing of the extensive calibration program by which it was determined that this latter premise was invalid. Perhaps it is sufficient to say that this led to a phase that is often referred to as an agonizing reappraisal. If the basic premise that there were no synergistic characteristics was invalid, how then could the stand be calibrated. Alternate schemes were considered but the one adopted involved the use of an existing program from the Computation Center files. This program - a so called Nonsimple Stepwise Multiple Linear Regression Analysis shortened herein to MLR Analysis was used to fit the data to empirical expressions of the input components in terms of the outputs. These expressions need be limited only by the capacity of the program which permits up to 59 independent terms. The scheme was attempted on data then available to us and a very close fit was obtained. This data did not cover the full ranges of interest, however, so a recalibration plan was formulated and loading hardware was designed to permit the acquisition of data sufficient for this purpose. The plan was first implemented in January 1969 for ambient temperature condition and analysis of results were completed in March 1969. Due to a stand renovation, a repeat of ambient condition calibrations have been made as well as a calibration at each of the temperature extremes. Analysis is in progress on this data. The scheme developed for the January 1969 effort was used thru out and is the primary subject to be presented to this panel.

The loading scheme for this series of tests involved a series of input vectors whose locations, attitudes and magnitudes were closely controlled and/or measured. The input assembly, (Figure 3) starting at the "hard" point, involved a bi-directional translation device with planned displacements indexed by a series of dowel pin holes. This device was centered on the vertical centerline through the use of optical tooling. Upon this device was mounted a hydraulic jack which was linked to the input load cell through a universal flexure. A rod extended from the load cell to another flexure near the engine mounting fixture and another smaller bi-directional translation device connected this flexure to the mounting fixture. It, too, was indexed for the planned displacements. During loading operations, initial displacements were set with these translation devices to effect either vector displacement, vector angularity, or a combination of both. Then, to assure that any change in these initial conditions was known, displacement gages were used to monitor any lateral displacements above the upper flexure or below the lower flexure. The loads were then cycled under control of a servo loop to create the load sequence depicted in Figure 4. To combine the static pressure effects each of the four cycles were run at different line pressure conditions as noted. Four cycles of this type constituted a test run. The recalibration plan involved 30 runs.

Digital data acquisition and processing techniques were employed throughout the calibration process except for the displacement gages used to monitor the input rod attitude. The analog data obtained from these were manually reduced and entered into the digital analyses. Many of the features contained within the computer programs are illustrated in Figure 5. Automatic normalizing of all output data at the zero input point eliminates the negative thrust portions of our load cycles. Redundant data sections are also eliminated by an edit routine. The low range bridge of the input cell defines the sustain level sections indicated while the high range bridge defines the original boost level segments indicated by solid black lines. A range limiting feature that was subsequently added redefines the boost level to conform to the designated upper range segments.

For each of these segments, the 200 sample/sec acquisition rate was reduced by editing and averaging to approximately 1 sample/sec for boost data and 10 s/s for sustain data. Computation of input and output components were then made at these data rates and taped for input into the MLR Analysis. Inputs were based on initial and monitored change of rod attitudes while outputs were functions of the stand geometry and the load cell results. Tabulations and CRT type plots were prepared to aid in analysis.

The MLR Analysis determined the empirical equation coefficients for each component in turn. Data from 25 of the 30 calibration runs was considered collectively. The determined coefficients were then programmed to provide predictions of the input components which were then compared to the known inputs and a residual error computed. When applied to runs 26 thru 30, a validity check was made of the total process. When applied to all 30 runs in turn a standard error is obtained for each run. It is the combination of these errors that determined our estimate of measurement system capability.

Several forms of the MLR Analysis were performed as indicated in Table II. To understand this table, you must realize that these are the stand output parameters that are considered as independent variables in each of six equations involving the known input values or dependent variables. The equation form, as noted, is Input Component = $\Sigma\ A_n T_n$ with A_n the unknown coefficients and T_n the terms selected from the chart. The analysis is performed on each of these equations in turn to determine a best fit set of unknown coefficients, one for each of the independent terms. Having determined these values by calibration, it is assumed that the equation holds for unknown input conditions and thus these conditions can be predicted from any set of output values contained within the calibration ranges. The original premise was that irrelevant terms would be effectively eliminated by the analysis and a 59 term form was chosen. The highest order terms used were second degree terms and their cross products. This provided a very good fit for the 25 runs used in the analysis but prediction for runs 26 thru 30 were very poor. It was reasoned that this was due to the inclusion of too many irrelevant terms and the equations were reduced to a 16 term form. (Diagonal shading.) All cross products are eliminated. This produced significantly improved prediction results even though the standard errors from the MLR Analysis were increased. This form was adopted for the sustain range of data. An addendum note should be made at this point. In setting the program controls, no test was made on the exclusion of terms. Present analysis includes this feature but its effectiveness is not yet known.

The next significant step was taken when it was observed that even though the standard error for an over-all fit was acceptable, the fit at the upper limit of the range was at times unacceptable. This led to a revised analysis based on data contained within the interval of 25 to 40K. The 16 term form was selected for this analysis. The residual error plots seemed to suggest a correlation with the ratios of Z input with each of the components. Thus, two forms containing 23 terms in the equations were attempted next. These forms include the 16 terms and those shown in dots in Table II. Alternates shown involve the products or ratios with "Z". Results were only slightly improved in terms of the standard errors obtained, however the residual error plots pointed-up a preference for either of these over the 16 term form for most of the six components. This will be demonstrated in future figures. Attempting further improvement, two additional forms with 33 terms were used. The first attempt added the terms identified with horizontal shading. The second was broken into two sub-analyses - the 16 term form plus an analysis of the residual errors in terms of the balance of 17 terms. Significantly improved standard errors were obtained with the first of these but again the results of both produced poor predictions for the confirmation runs. A comparative charting of the five forms as they apply to runs 26 through 30 are shown in Table III. The best equation results are in shaded areas for each component. The 16 term form for Y and MX; the 33 term for MZ and the 23 term ratios for the balance. The graphs for Run 27 which are typical of the runs examined will illustrate further the reasoning for our preferences. These are residual error vs Z input plots from the 16 term analysis with data points plotted. A computer determined best fit is line plotted thru these points and appears as a broken line. Similarly determined best fit plots for each of the 23 term forms have been manually transferred to these plots for comparison. Comparison should be made between the line plots. The ideal result would appear as a zero error through out the range. Figure 6 is the X component plot. The 23 term results are obviously best with maximum error for the mean data fit of approximately 3 pounds. Figure 7 is the Y plot with a maximum error of 8 pounds. The Z plots (Figure 8) show a slight preference for the 23 term ratio form. All data is under 50 pounds of error in the presence of inputs ranging to 40K of thrust, 700 lbs. of side force and 2000 ft-lbs of moments. The MX plots (Figure 9) show a mean error of less than 30 ft-lbs. The MY plots (Figure 10) show error ranging to 100 ft-lbs for this run but this is somewhat larger than the normal MY error. The MZ plots (Figure 11) show little preference between these three forms but the standard error is less than 4 ft-lbs for all forms.

The foregoing description suggests a much more direct path than that of our actual experience. From an experiment standpoint there were several areas that could have been explored in greater depth but were ignored since this was primarily an attempt to define the specific capabilities of the stand as they related to the LANCE facility requirements. A final summary of the attainment of this goal is presented in a comparison of the facility requirements with the demonstrated capability (Table IV). Based on these results and our continued surveillance of the stand through verification testing, we are confident that our data quality must be rated equal to or superior to that obtainable at any facility of this type.

A concise summary of the major points of the calibration problem, the implemented solution and the clinical questions to be asked of this panel is now in order.

THE PROBLEM. Since it is desired to resolve a force vector in six degrees of freedom it is necessary to evaluate the interaction effects of the electromechanical measurement system in six degrees of freedom. For this purpose an experiment is required that will adequately measure these effects throughout the range of interest and determine the precision characteristics of the total measurement process.

THE IMPLEMENTED SOLUTION.

A. An experiment of 30 input vectors involving differing combinations of the six components of force input and static pressurisation levels with ranging of each parameter to near maximum expected value.

B. An evaluation of the six empirical "best fit" transfer equations relating the observed input and output data. The terms of the equations were arbitrarily selected and a coeffients derivation made by a multiple linear regression analysis.

C. A derivation of precision limits based on the combination of the overall experiment data fit precision with standard deviation values for the laboratory standard and for the transfer to the field standard. This comibnation is by the square root of the sum of variances method.

THE CLINICAL QUESTIONS.

1. Is method valid?

2. If so, are the thirty tests excessive or inadequate?

3. Are more practical methods known?

TABLE I. REQUIREMENT VERSUS CAPABILITY ESTIMATE AT START OF XRL PROGRAM

Parameter	Requirement (3 sigma)	Capability Estimate (95% confidence)
Side forces (lb)	± 50	± 105
Sustain phase thrust (lb)	± 160	± 100
Boost phase thrust (lb)	± 200	± 250
Boost phase vector location (in.)	± 0.03	± 0.125
Boost phase vector angularity (mrad)	± 0.372	± 2.6

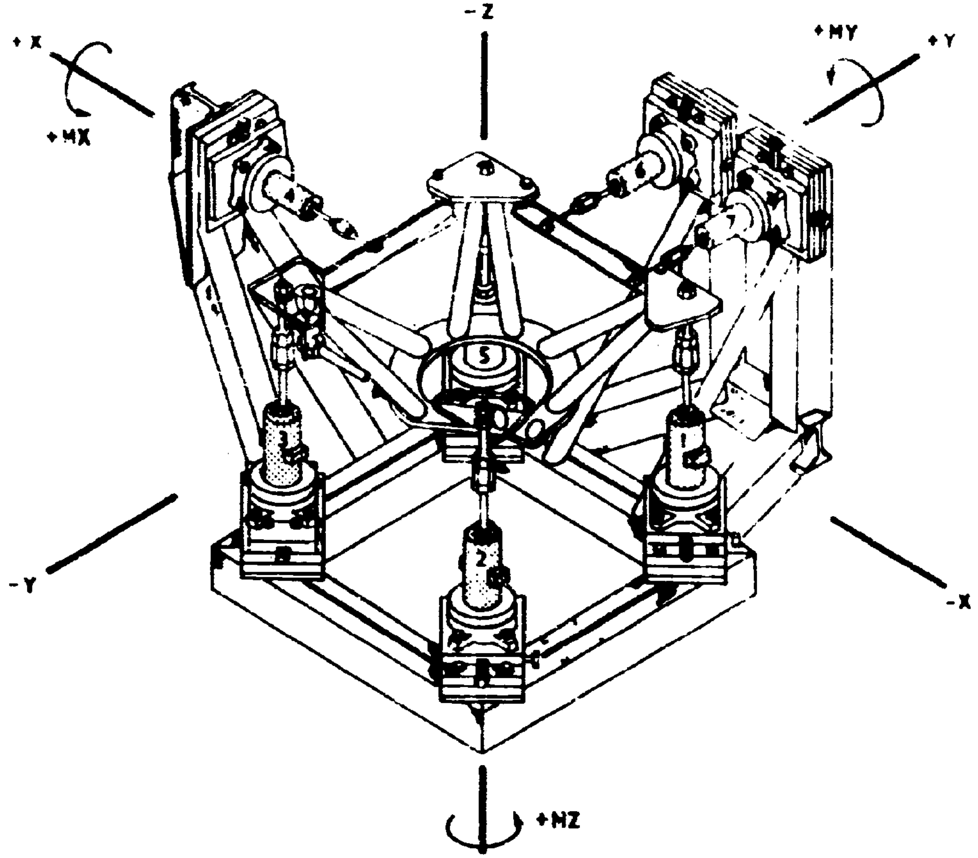

FIGURE 1. FORCE MEASUREMENT SYSTEM FOR TECHNICAL PROTOTYPE LANCE WITH DESIGNATION OF COMPONENTS CONVENTION

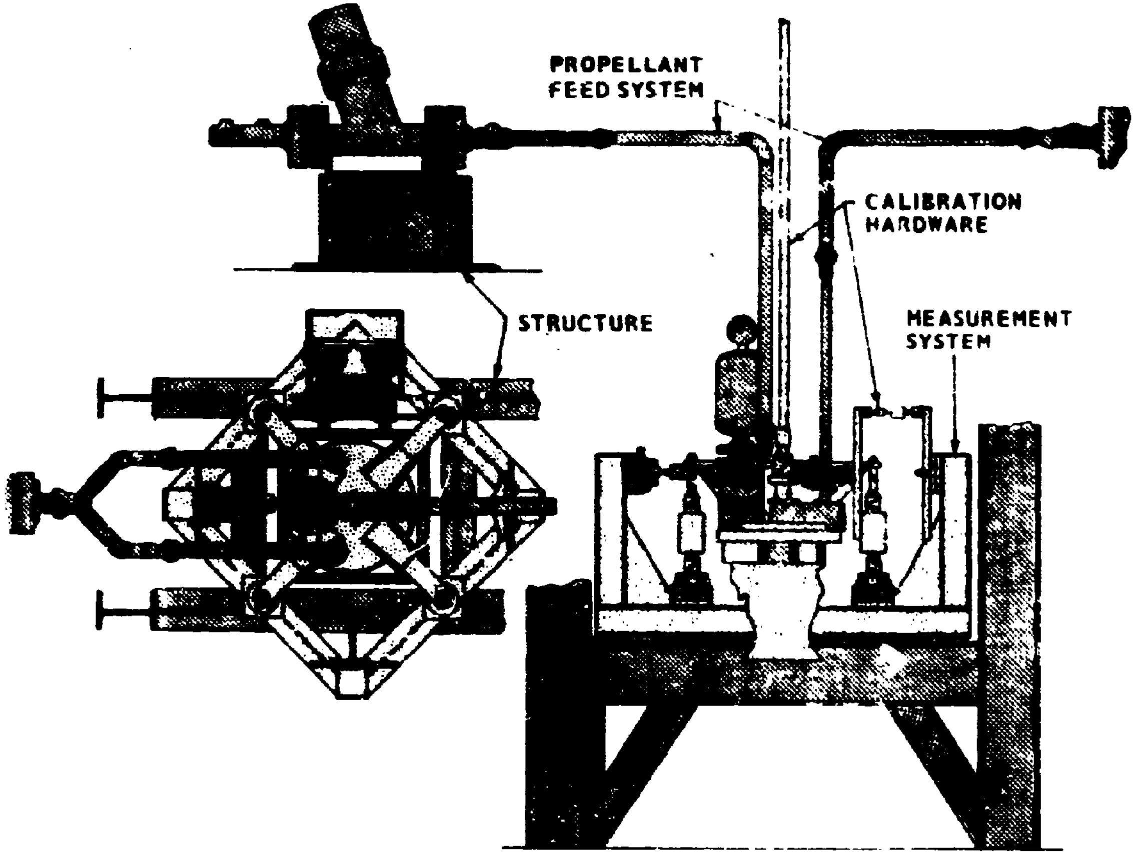

FIGURE 2. PRESENT FORCE MEASUREMENT SYSTEM

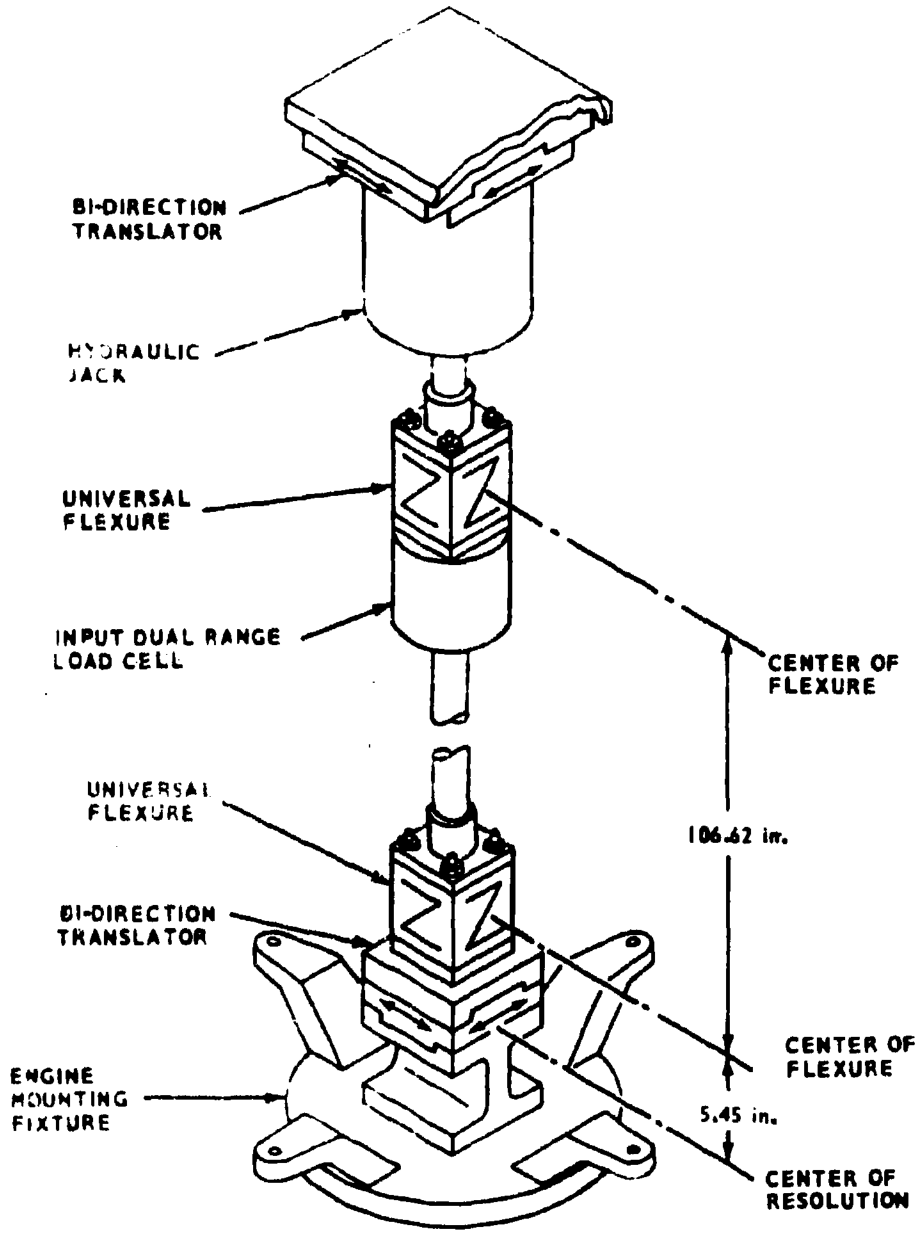

FIGURE 3. CALIBRATION FORCE INPUT ASSEMBLY

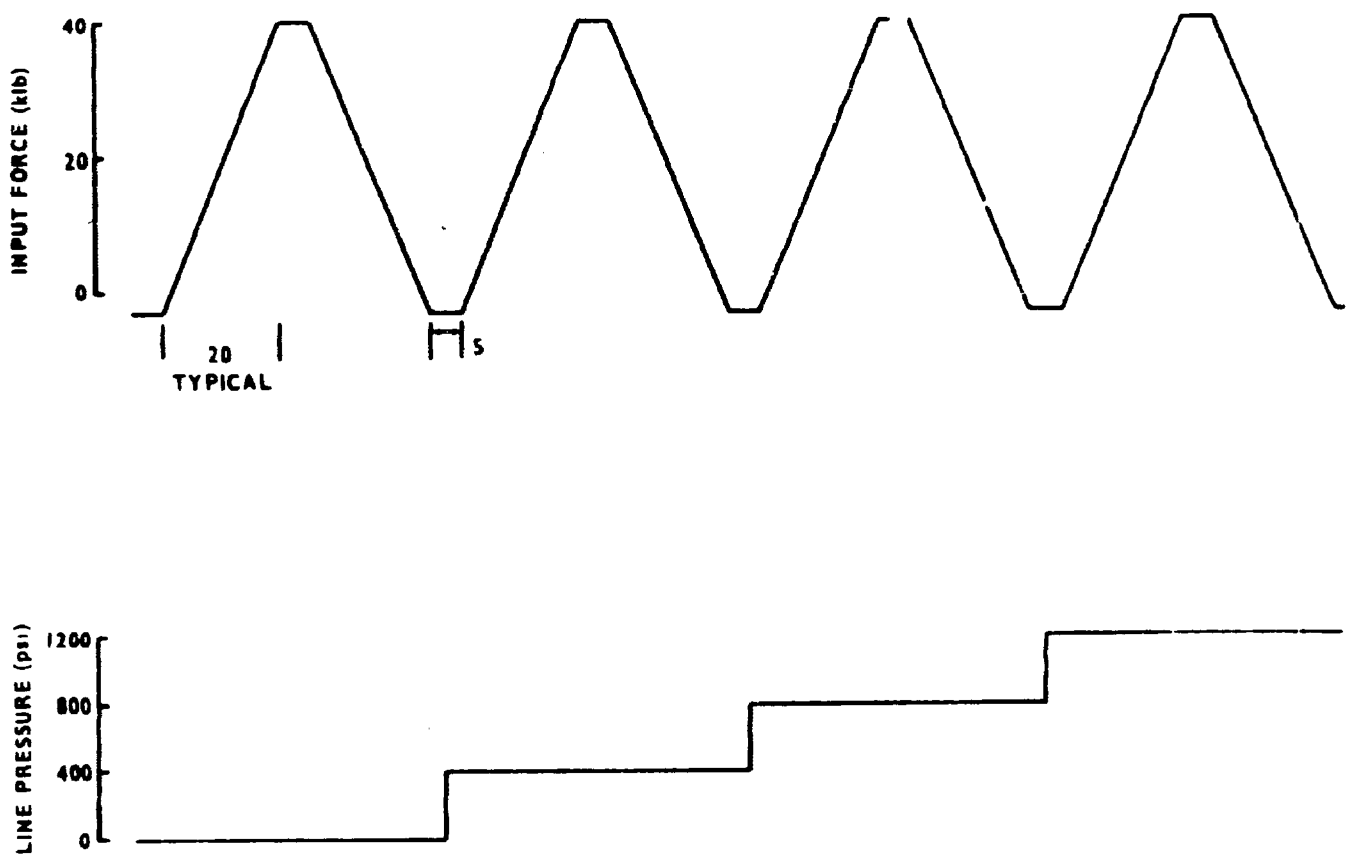

FIGURE 4. CALIBRATION TEST RUN LOADING SEQUENCE

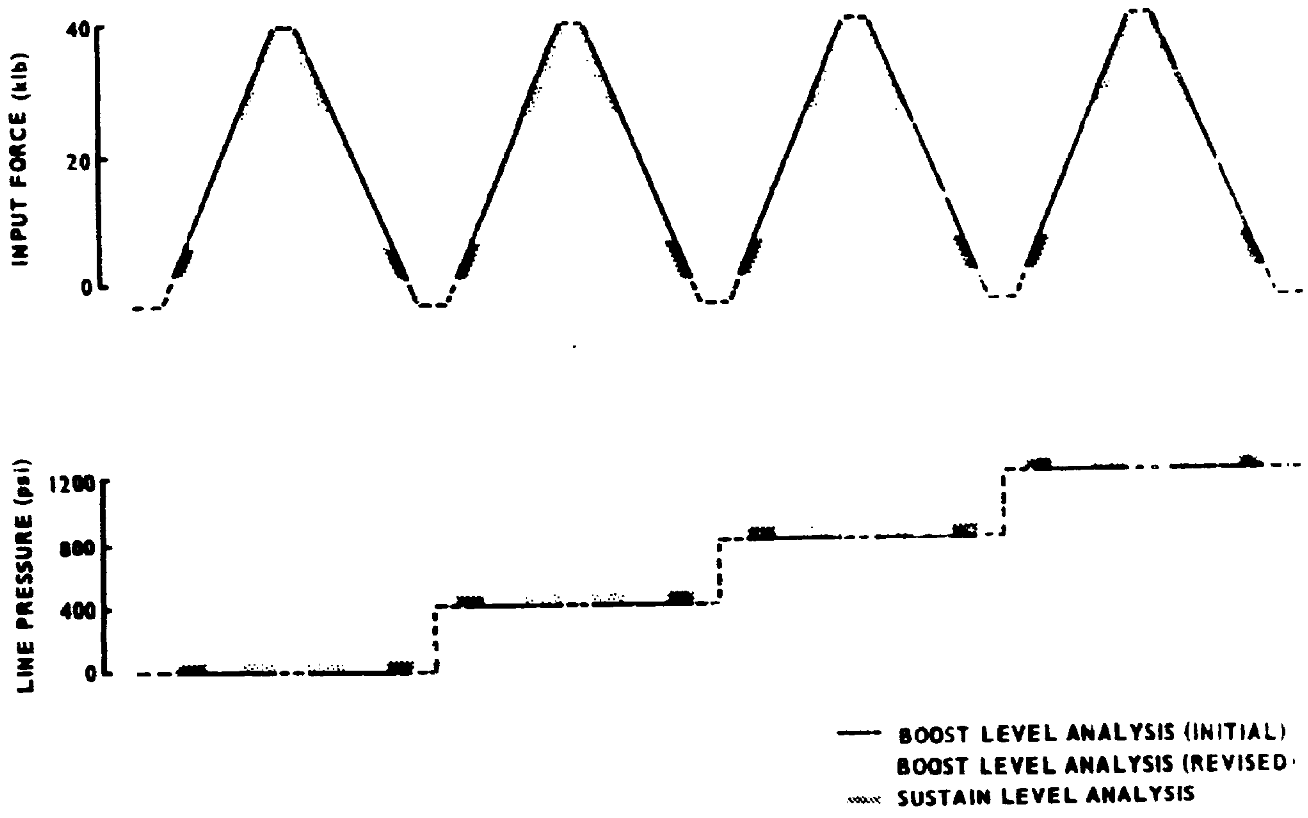

FIGURE 5. SEGMENTS OF LOADING SEQUENCE CONSIDERED BY ANALYSIS PROGRAM

X^2	Y^2	Z^2	$(MX)^2$	$(MY)^2$	$(MZ)^2$	P_F^2	P_e^2
X	Y	Z	MX	MY	MZ	P_F	P_e
XZ / X/Z	YZ / Y/Z	$(MX)Z$ / MX/Z	$(MY)Z$ / MY/Z	$(MZ)Z$ / MZ/Z	* P_F/Z	* P_e/Z	XY
$X(MX)$	$X(MY)$	$X(MZ)$	$Y(MX)$	$Y(MY)$	$Y(MZ)$	$(MX)(MY)$	$(MX)(MZ)$
$(MY)(MZ)$	X^2Y	X^2Z	$X^2(MX)$	$X^2(MY)$	$X^2(MZ)$	Y^2X	Y^2Z
$Y^2(MX)$	$Y^2(MY)$	$Y^2(MZ)$	Z^2X	Z^2Y	$Z^2(MX)$	$Z^2(MY)$	$Z^2(MZ)$
$(MX)^2X$	$(MX)^2Y$	$(MX)^2Z$	$(MX)^2(MY)$	$(MX)^2(MZ)$	$(MY)^2X$	$(MY)^2Y$	$(MY)^2Z$
$(MY)^2(MX)$	$(MY)^2(MZ)$	$(MZ)^2X$	$(MZ)^2Y$	$(MZ)^2Z$ **	$(MZ)^2(MX)$	$(MZ)^2(MY)$ **	

* NOT INCLUDED IN 59 TERM FORM
**DELETED TO PERMIT INCLUSION OF P_F^2 AND P_e^2

a) OUTP[illegible] FORCE COMPONENTS ARE X, [illegible], MY AND MZ
b) STAT[illegible] PRESSURES ARE P_e AND [illegible] AND OR LINE, RESPECTIVELY
c) TERMS CONTAINED IN 16-TERM FORM ARE BLOCKED IN [illegible]
d) TERMS CONTAINED IN 23-TERM FORMS ARE BLOCKED IN [illegible] WITH ALTERNATE TERMS SHOWN
e) TERMS CONTAINED IN 33-TERM FORMS ARE BLOCKED IN [illegible]
f) EMPIRICAL EQUATIONS ARE

INPUT COMPONENT = $\Sigma A_n T_n$

WITH A_n = UNKNOWN COEFFICIENTS
T_n = TERMS DEFINED IN CHART

g) THE MLR ANALYSIS DETERMINES VALUES OF A_n THAT PROVIDE A BEST COMPROMISE FOR A DATA CONSIDERED IN ANALYSIS

TABLE II. VARIOUS FORMS OF MLR ANALYSIS CONSIDERED

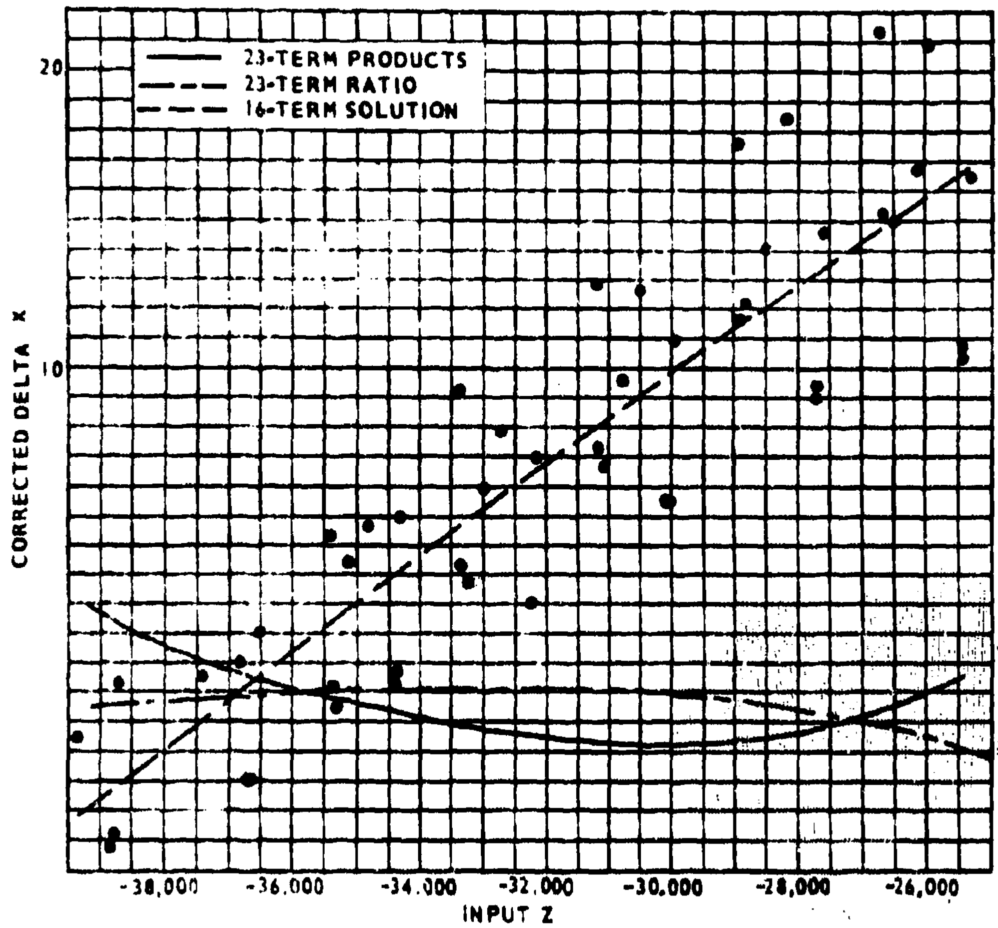

FIGURE 6. RESIDUAL ERROR PLOTS, X COMPONENT

EQUATION FORM	X COMPONENT (lb)					Y COMPONENT (lb)					Z COMPONENT (lb)				
	RUN 26	RUN 27	RUN 28	RUN 29	RUN 30	RUN 26	RUN 27	RUN 28	RUN 29	RUN 30	RUN 26	RUN 27	RUN 28	RUN 29	[illegible]
16-TERM	6	[illegible]	10	10	15*	12*	5*	9*	21	14*	18	26	14	[illegible]	[illegible]
23-TERM PRODUCTS	3*	7	9	7*	17	12*	8	9*	22	15	16*	28	11*	14	[illegible]
23-TERM RATIOS	3*	6*	9	7*	17	12*	8	9*	22	[illegible]	16*	25*	11*	14	[illegible]
33-TERM	21	16	5*	8	20	13	21	13	19*	[illegible]	[illegible]	79	20	15	44
16 AND 17	24	25	18	8	17	22	29	19	22	16	39	59	20	11*	35
	MX COMPONENT (ft-lb)					MY COMPONENT (ft-lb)					MZ COMPONENT (ft-lb)				
16-TERM	34*	27	70*	79*	100*	21	55	34	33	47	3.5	3.5	2.4	4.1	[illegible]
23-TERM PRODUCTS	35	25*	70*	79*	102	17*	82	31*	30	45	3.6	1.9	2.2	[illegible]	[illegible]
23-TERM RATIOS	35	25*	70*	79*	103	17*	83	31*	29*	45	3.5	2.1	2.2	[illegible]	
33-TERM	233	256	219	83	119	43	18*	59	31	37*	0.8	0.9*	0.4*	[illegible]	[illegible]
16 AND 17	216	237	139	84	119	20	74	43	37	40	0.2*	10.6	3.5	3.6	

*BEST RESULTS FOR TEST RUN INDICATED
"BEST EQUATION" RESULTS IN SHADED BLOCKS

TABLE III. STANDARD ERRORS RESULTING FROM MLR PREDICTIONS, BOOST RANGE 25 TO 40 KILOPOUNDS

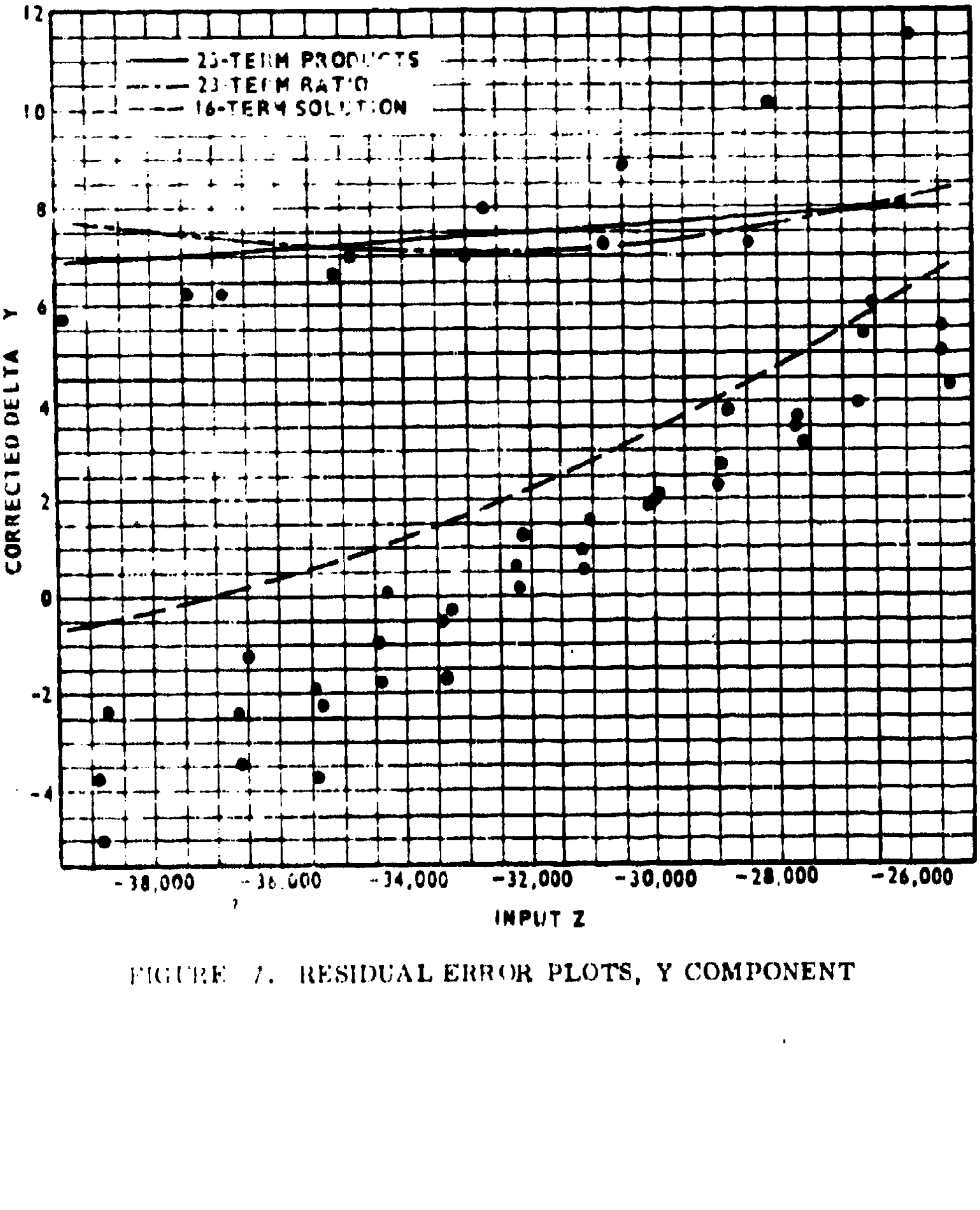

FIGURE 7. RESIDUAL ERROR PLOTS, Y COMPONENT

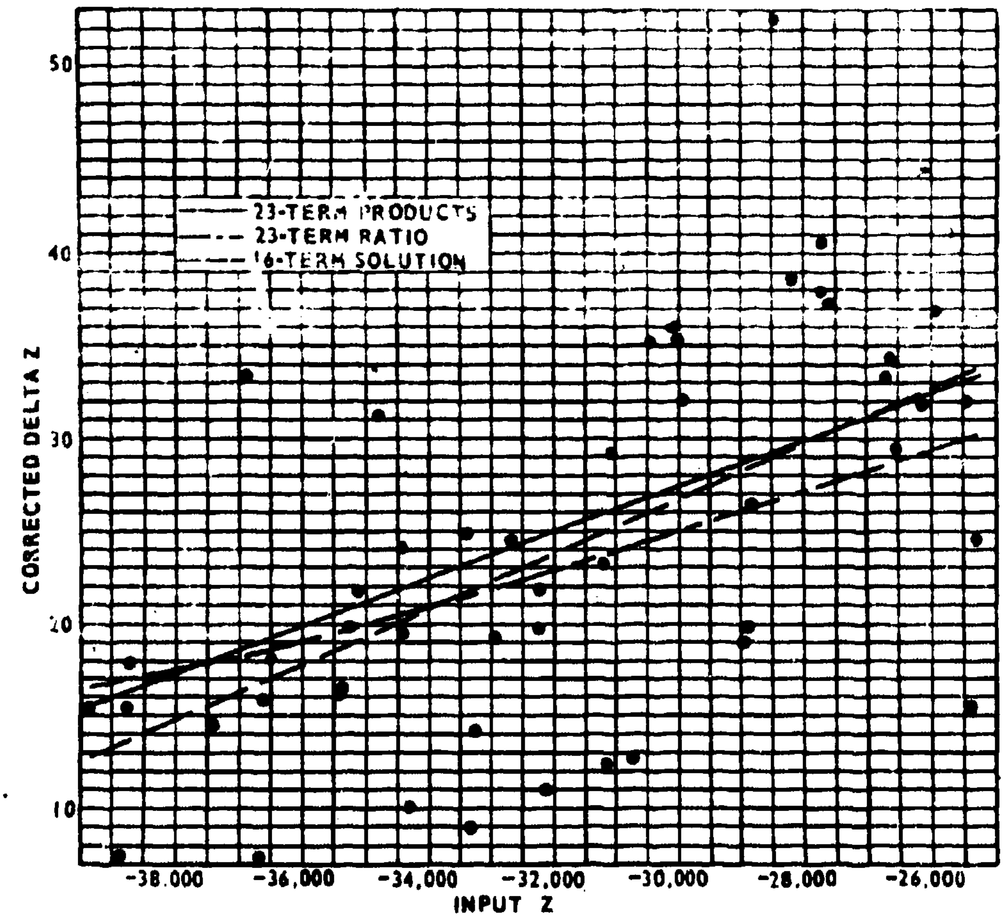

FIGURE 8. RESIDUAL ERROR PLOTS, Z COMPONENT

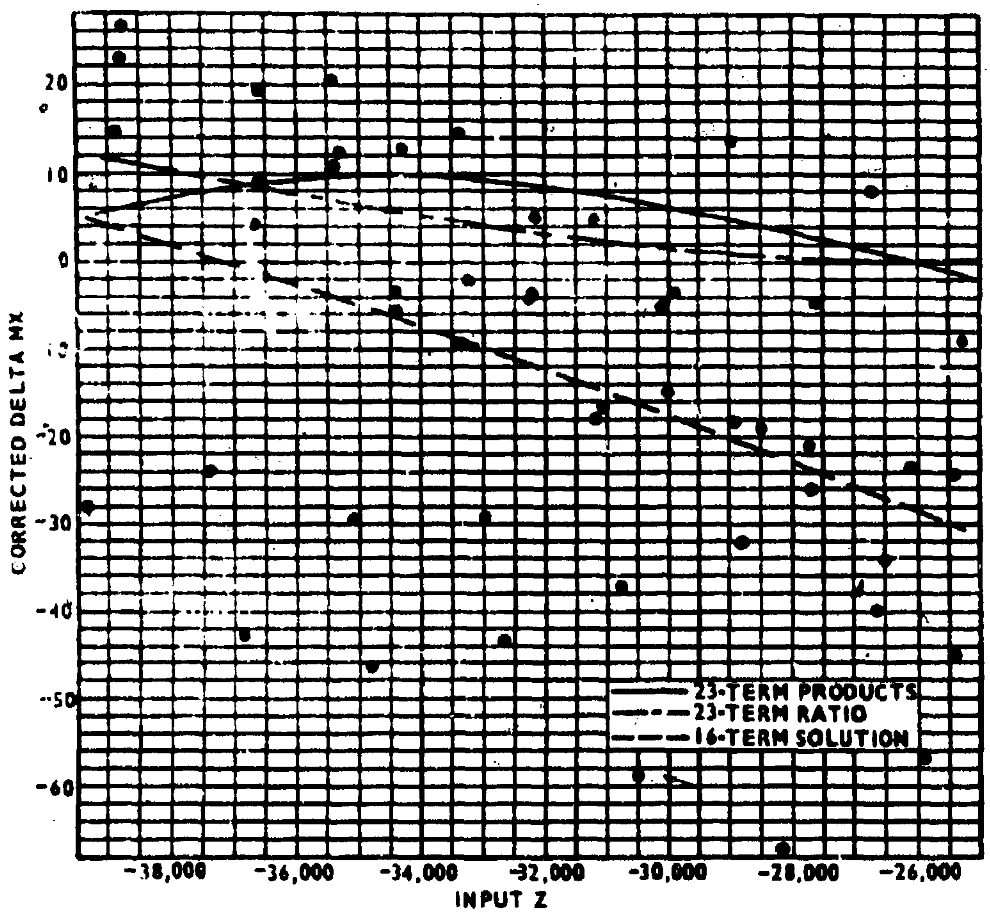

FIGURE 9. RESIDUAL ERROR PLOTS, MX COMPONENT

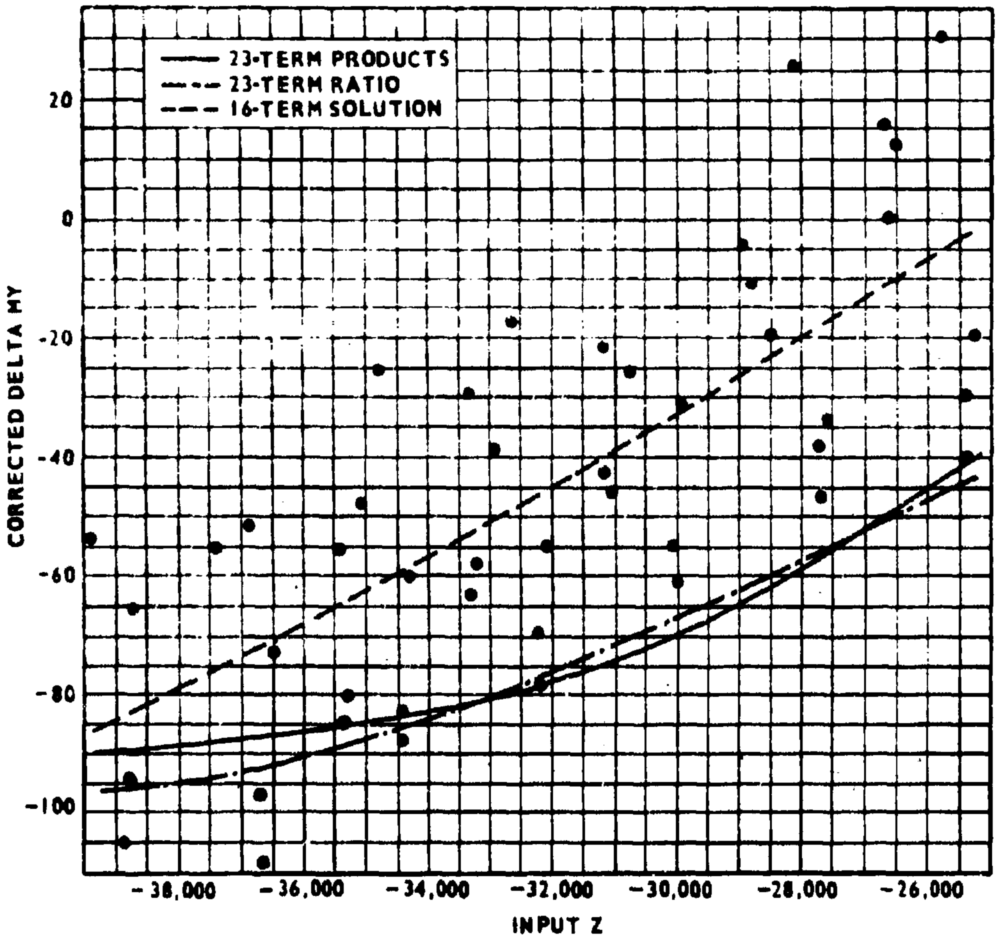

FIGURE 10. RESIDUAL ERROR PLOTS, MY COMPONENT

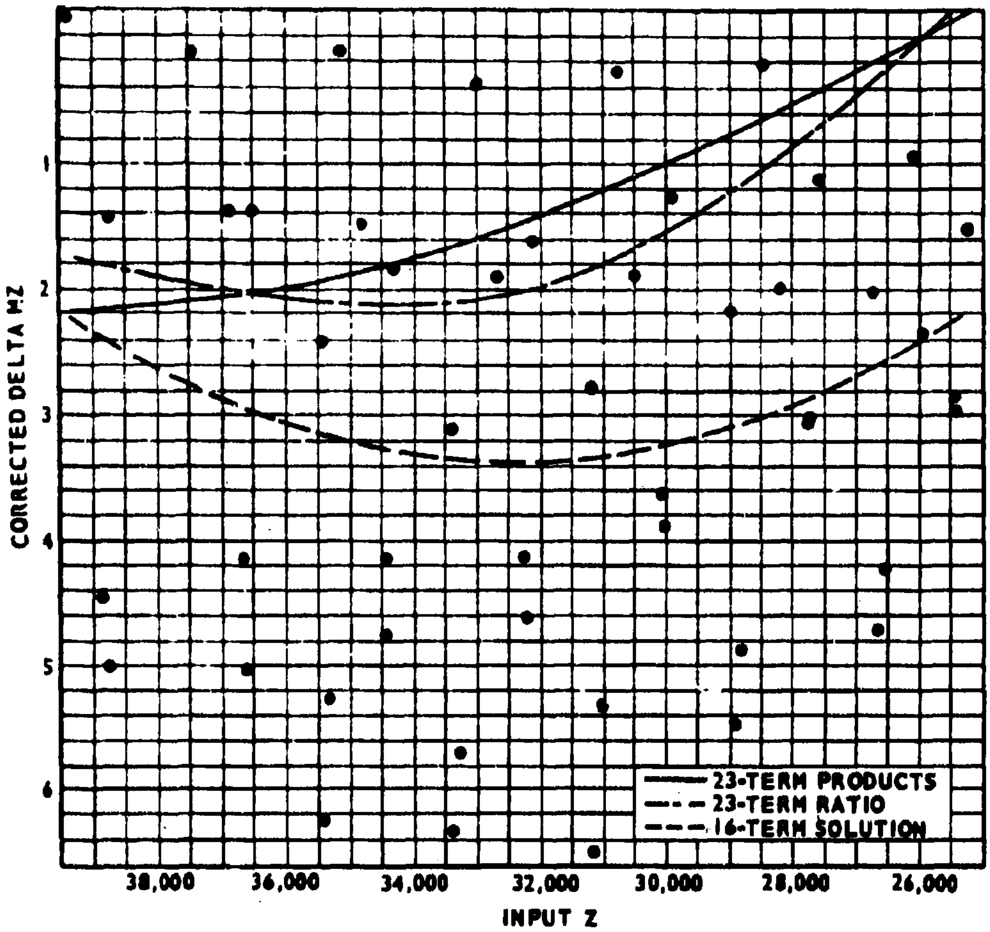

FIGURE 11. RESIDUAL ERROR PLOTS, MZ COMPONENT

TABLE IV. REQUIREMENT VERSUS DEMONSTRATED CAPABILITY AT END OF CALIBRATION ANALYSIS

Phase	Parameter	Requirement	Capability
Boost	Side force (lb)	±50	±15
	Vertical force (thrust) (lb)	±200	±110
	Vector location (in.)	±0.03	±0.06
	Vector angularity (mrad)	±0.372	±1.2
Sustain	Side force (lb)	±50	±20
	Vertical force (lb)	±160	±32
	Vector location (in.)	--	±0.18
	Vector angularity (mrad)	--	±4

INTERLABORATORY STUDY OF A METHOD FOR MEASURING AMMONIUM PERCHLORATE PARTICLE SIZE

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ABSTRACT

An interlaboratory study of a method for measuring the particle size distribution of finely ground ammonium perchlorate was conducted by the Analytical Chemistry Working Group of the Interagency Chemical Rocket Propulsion Group (ICRPG). The primary objective of the study was to determine the suitability of the method for use as a standard specification procedure by evaluating its precision. Single analyses of two different ammonium perchlorate samples, having weight median diameters in the range of 20-30μ were made by each of nine laboratories, using the same liquid sedimentation technique and equipment. The random error within laboratories and the systematic error among laboratories were resolved, and confidence intervals were placed on the determination of specific surface areas and weight mean diameters. The random error estimate was acceptably small; however, the systematic error estimate was so large that the method is not recommended for use as a standard specification procedure.

INTRODUCTION

Ammonium perchlorate (AP) is widely used as an oxidizer in composite and composite-modified double-base propellants. The particle size distribution of the AP has a pronounced effect on the propellant processing characteristics and ballistic properties, and therefore must be precisely measured and controlled. The recent use of finely ground AP in high burning rate propellants places greater demands on the precision of particle size analysis.

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Several analysis methods were evaluated and compared during a program conducted within the Army Propulsion Laboratory and Center.[1] One of these methods, based on a liquid sedimentation analysis using an apparatus manufactured by the Mine Safety Appliances (M-S-A) Company,[2,3] gave precise results and was recommended for the general analysis of fine AP.[1] The method was used with success by a number of laboratories throughout the propulsion industry.

The interlaboratory study (Round Robin) described here was subsequently conducted by the Analytical Chemistry Working Group of the Interagency Chemical Rocket Propulsion Group (ICRPG), with nine laboratories participating. The objectives were: (1) to determine the suitability of the M-S-A method for use as a standard specification procedure based on an estimation of its precision; (2) to determine the effectiveness of a simple experimental design; and (3) to evaluate and compare the performances of the participating laboratories.

EXPERIMENTAL

Each of the nine laboratories was sent three samples of nominal 20-30μ AP mixed with an inert polymer, one selected at random from each of three different batches. One of the samples was provided simply for practice prior to initiation of the Round Robin. The other two, designated materials A and B, were to be analyzed in accordance with the detailed Round Robin procedure. The instructions specified that the analysis be conducted by a skilled operator, and that the A and B samples be analyzed on different days.

Briefly, the particle analysis procedure[1] was as follows. A 15-mg sample of the material was dispersed with a surfactant and suspended in a feeding liquid composed of 60% chlorobenzene and 40% benzene by volume. The particle suspension was placed on top of chlorobenzene in a special centrifuge tube. The larger particles were allowed to fall under the influence of gravity, and the smaller particles were centrifuged. All of the particles were collected in a uniform bore capillary at the bottom of the centrifuge tube. The diameter schedule used in the Round Robin and a typical analysis are shown in Table 1.

The sediment height at each particle diameter of the schedule was measured as a function of settling times precalculated from Stokes law. The percentage by weight (volume) of particles greater than each successive diameter was calculated by dividing the corresponding sediment height by the total height at the end of the analysis. It will be noted from the table that the percentages are not independent.

Table I

Typical Ammonium Perchlorate Particle Size Analysis
with the Mine Safety Appliances (M-S-A) Analyzer

Diameter (µ)	Sedimentation Mode	Sedimentation Rate (rpm)	Sedimentation Time (min, sec)	Sediment Height (Relative)	Weight % > Diameter
200	Gravity		0, 4.1	0	0
149			0, 7.5	0.5	1.45
105			0, 15.0	1.0	2.90
74			0, 30.2	2.2	6.38
52			1, 1	5.5	15.94
37			2, 1	10.8	31.30
25			4, 25	16.5	47.83
18	Centrifuge	300	0, 27	21.0	60.87
9		600	0, 58	26.2	75.94
5		1200	0, 52.5	29.5	85.51
3		1800	1, 19	31.5	91.30
2			2, 2	32.5	94.20
1.2			5, 16	33.5	97.10
0.6		3600	6, 2	34.2	99.13
0.4			9, 55	34.5	100

RESULTS AND DISCUSSION

The particle size distribution data are given in Tables II and III. The data for laboratories 3 and 8 were omitted from the calculations of averages and the estimates of variances (S^2) and standard deviations (S) because of the outliers[4] in Table III and the abnormal shapes of their particle size distribution curves.

The average particle size distribution data are plotted on log-probability scale in Fig. 1. The shapes of the curves are typical of those obtained for finely ground unimodal ammonium perchlorate. The difference between the particle size distributions of the AP in the two materials was purposely made small so that the random errors for their respective analyses could be assumed to be equal.

Table II

Particle Size Analysis Data for Ammonium Perchlorate in Material A
(Values in wt. % > diameter)

Diameter (μ)	Laboratories 1	2	3	4	5	6	7	8	9	Average[a]	S^{2} [a]	S [a]
200												
149					0			0		0	-	-
105		0		0	0.59	0	0	3.16	0	0.08	0.0497	0.2230
74	0	0.86	0	2.12	5.14	0.31	2.19	7.91	0.88	1.64	3.071	1.752
52	5.17	6.90	3.77	9.39	15.41	3.38	10.38	19.30	8.21	8.41	15.33	3.915
37	17.24	20.26	10.46	24.85	27.47	15.38	23.77	35.44	23.75	21.82	18.92	4.350
25	34.48	35.34	18.83	42.42	46.05	32.00	38.52	51.90	41.06	38.55	24.55	4.955
18	50.29	46.55	37.24	58.79	58.89	45.85	56.01	59.81	53.37	52.82	29.54	5.435
9	68.97	68.53	68.62	83.94	82.21	72.92	78.42	77.53	75.95	75.85	37.00	6.083
5	81.03	82.76	83.68	90.30	93.48	86.15	89.89	91.77	87.39	87.29	19.19	4.381
3	89.66	91.38	90.79	93.33	98.62	92.00	94.54	96.52	92.96	93.21	8.088	2.844
2	93.39	93.53	94.14	96.06	100.00	95.69	96.72	98.10	95.89	95.89	4.912	2.216
1.2	95.69	95.69	96.23	97.88		98.46	98.36	99.37	98.24	97.76	2.447	1.564
0.6	98.56	99.57	100.00	100.00		99.38	100.00	100.00	99.71	99.60	0.2705	0.5201
0.4	100.00	100.00				100.00			100.00	100.00	-	-

[a] Laboratories 3 and 8 omitted.

Table III

Particle Size Analysis Data for Ammonium Perchlorate in Material B
(Values in wt. % > diameter)

Diameter (μ)	Laboratories 1	2	3	4	5	6	7	8	9	Average[b]	S^2 [b]	S [b]
200	0			0.35			0	0		0.05	0.0175	0.1323
149	1.45			2.48	0	0	1.41	4.69	0	0.76	1.028	1.014
105	2.90	0		4.26	2.88	0.49	4.24	17.19	2.81	2.51	2.802	1.674
74	6.38	4.65	0	9.93	7.05	4.41	10.73	33.75	6.25	7.06	5.942	2.438
52	15.94	13.95	7.07	22.34	18.27	10.29	20.62	49.37[a]	20.00	17.32	17.81	4.220
37	31.30	31.01	14.13	38.30	31.09	24.51	35.87	63.44	36.25	32.62	21.41	4.627
25	47.83	49.61	28.27[a]	53.90	52.24	43.14	53.95	76.56	53.13	50.54	15.91	3.989
18	60.87	58.14	53.00	66.67	65.38	57.84	69.49	80.94	65.00	63.34	19.87	4.458
9	75.94	76.74	81.27	86.17	83.65	78.43	86.72	86.56	83.13	81.54	19.89	4.459
5	85.51	87.98	92.93	94.33	93.31	89.22	94.35	96.87	91.25	90.85	11.66	3.415
3	91.30	94.57	96.82	97.52	98.40	94.12	97.74	99.37	95.94	95.66	6.324	2.515
2	94.20	96.12	99.29	99.29	99.04	97.55	99.47	100.00	98.13	97.69	3.735	1.933
1.2	97.10	97.67	100.00	100.00	99.68	99.02	100.00		99.38	98.98	1.330	1.153
0.6	99.13	99.22			100.00	100.00			100.00	99.76	0.1627	0.4034
0.4	100.00	100.00								100.00	-	-

[a]Outliers by Dixon's test.
[b]Laboratories 3 and 8 omitted.

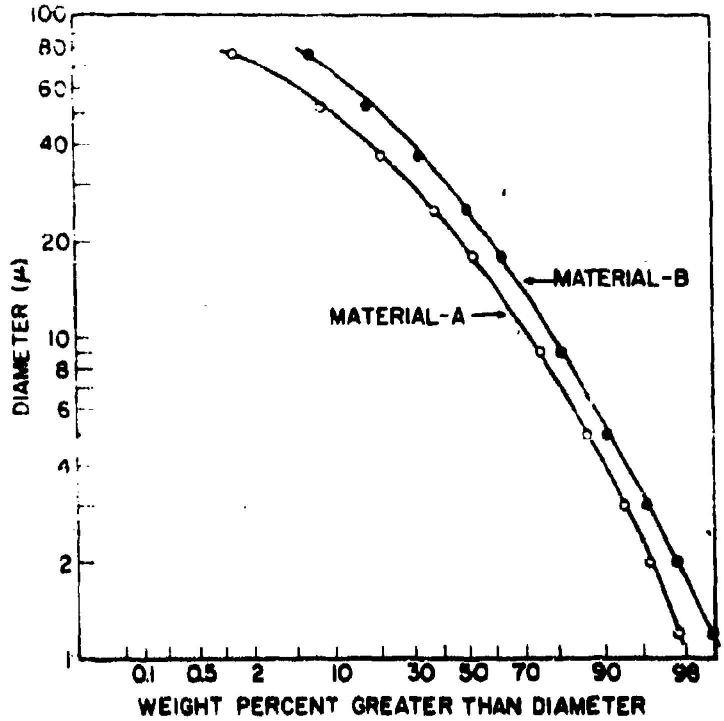

Fig. 1. Ammonium Perchlorate Particle Size Distribution Curves

Of the large number of single-valued variables that can be calculated from the particle size distribution data, the two chosen for this program were specific surface area (S_w) and weight mean diameter (d_w). The specific surface area correlates well with propellant burning rates[1] and is very sensitive to variations in the diameters of small particles; the weight mean diameter is very sensitive to variations in the diameters of large particles. The S_w and d_w values, assuming spherical particles, were calculated from the data in Tables II and III by the following formulas:

$$S_w(m^2/g) = 3.077 \sum_{i=1}^{n} \frac{W_i}{d_i} \quad (1)$$

$$d_w(\mu) = \sum_{i=1}^{n} d_i W_i \, . \quad (2)$$

where W_i is the weight fraction of particles in the i^{th} size interval, and d_i is an average diameter of the i^{th} interval.

The S_w and d_w values were initially examined using graphical procedures developed by Youden.[5,6] A two-material XY plot for S_w is shown in Fig. 2. Horizontal and vertical lines were drawn through the medians of the points, and a 45° line was drawn through their intersection. The perpendicular distances between the points and the 45° line are a measure of the random error within laboratories, and the spread of points along the 45° line is a measure of the systematic error among laboratories.

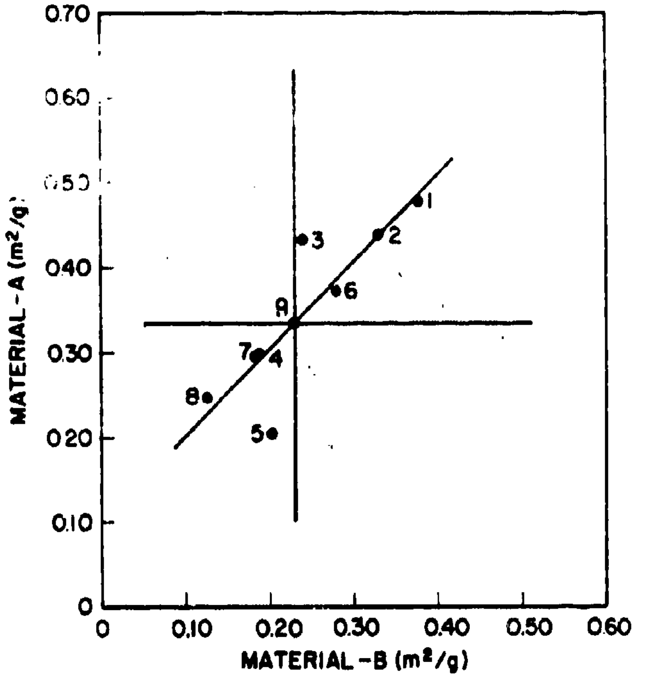

Fig. 2. Two-Material Plot of Ammonium Perchlorate Specific Surface Areas

The arrangement of points shows that the laboratories tended to get either high or low results on both materials. Moreover, the systematic error is appreciably larger than the random error. Most of the laboratories have a small random error, indicating that they did careful work. The differences between the two materials for laboratories 3 and 5 were found to be statistically significant at the 95% confidence level when compared with the average difference for all laboratories.

Figure 3 is the two-material XY plot for d_w. The laboratory 8 value differs markedly from the others, particularly for the analysis of material B, and was not considered when drawing the horizontal and vertical lines. The random error for d_w appears to be greater than that for S_w. It should be recognized, however, that the d_w values are larger and that d_w and S_w were calculated from the same particle size distribution data. The systematic error among laboratories is not noticeably larger than the random error within laboratories.

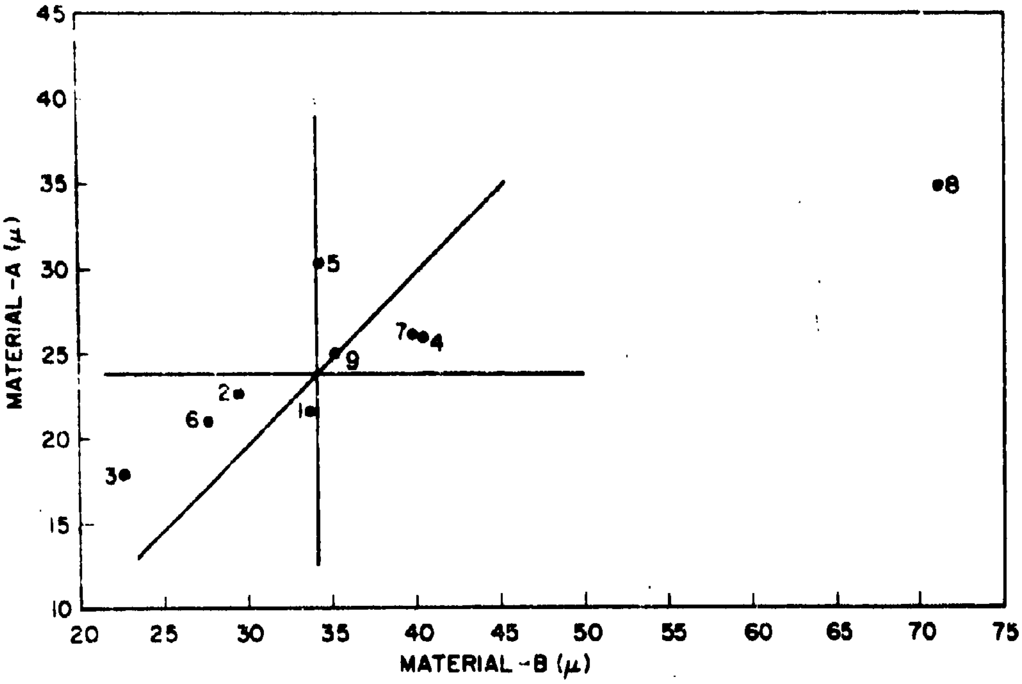

Fig. 3. Two-Material Plot of Ammonium Perchlorate Weight Mean Diameters

The extremely high d_w value for laboratory 8 clearly indicates some deviation from the recommended procedure, and this laboratory was omitted from some of the statistical analyses described later. Laboratory 3 was also omitted from some of these analyses because of its low d_w value and the outliers and abnormally shaped curves mentioned earlier.

An analysis of variance[6,7] is given in Table IV. The results, as expected, agree with the qualitative interpretations of the data in Figs. 2 and 3. The variance of S_w averages among laboratories is statistically significant when compared with the estimated random error variance (S_R^2); whereas that for d_w is not statistically significant. The laboratories x materials mean square $(MS)_{LM}$ is considered to be the best estimate of the true random error variance.

Table IV

Analysis of Variance (Laboratories 3 and 8 Omitted)

Source of Variation	DF	Specific Surface Area (m^2/g)		Weight Mean Diameter (μ)		EMS
		SS x 10^4	MS x 10^4	SS	MS	
Among Laboratories (L)	6	806.13	134.36[a]	168.60	28.10	$\sigma_L^2 + \sigma_{LM}^2 + 2\sigma_L^2$
Between Materials (M)	1	272.27	272.27[a]	338.20	338.20[a]	$\sigma_e^2 + 7\sigma_M^2 + \sigma_{LM}^2$
Laboratories x Materials (LM)	6	46.74	7.79	45.59	7.60	$\sigma_e^2 + \sigma_{LM}^2$
Total	13	1125.14		552.39		

DF - degrees of freedom　　MS - mean square
SS - sum of squares　　EMS - expected mean square

$$S_R^2 = S_e^2 + S_{LM}^2 = (MS)_{LM}$$

$$S_L^2 = \frac{(MS)_L - S_R^2}{2}$$

$$S_d^2 = S_L^2 + S_R^2/n$$

[a]Statistically significant at the 95% or higher confidence level.

The estimate of the systematic error variance among laboratories (S_L^2) is also an important component of the estimate of the overall variance (S_d^2) of the method. The formulas for calculating S_L^2 and S_d^2 are given in the table. The expected mean squares (EMS) are for a random model, which was assumed in this case.

Estimated random and systematic error variances are given in Table V for various combinations of the S_w and d_w data. The S_R^2 and S_L^2 values for specific surface area determinations are not significantly affected by omitting laboratories 3 and 8, but they are significantly reduced in the case of weight mean diameter determinations. The value of S_R^2 for the determination of specific surface area is significantly reduced when laboratories 3 and 5 are omitted, as could have been expected from Fig. 2. However, a comparison with the 0.00137 value independently obtained by replicate determinations within laboratory 1 (the originating laboratory) indicates that the value of 0.000048 is not a good estimate of the true random error variance.

Table V

Estimated Random and Systematic Error Variances

	S_R^2	S_L^2
Specific Surface Area (m^2/g)		
All Laboratories	0.001171	0.006192
Omit Laboratories 3 and 8	0.000779[b]	0.006328[b]
Omit Laboratories 3 and 5	0.000048	--
Within Laboratory 1[a]	0.001374	--
Weight Mean Diameter (μ)		
All Laboratories	48.78	62.94
Omit Laboratories 3 and 8	7.60[b]	10.25[b]
Within Laboratory 1[a]	1.73	--

[a]DF = 5.

[b]Values used for calculating confidence intervals.

The S_R^2 for the replicate determination of weight mean diameter within laboratory 1 is significantly smaller at the 95% confidence level than S_R^2 for the Round Robin data with laboratories 3 and 8 omitted.

The precision of the method is defined by the confidence limits ($X \pm L_{\frac{1}{2}}$), where X is an analysis result and $L_{\frac{1}{2}}$ is one-half the length of the confidence interval, in this case at the 95% level. These one-half values are given in Table VI. They were calculated from the formulas in the table and the estimated S_R^2 and S_L^2 variance components noted in Table V. The degrees of freedom associated with S_d were estimated by Satterthwaites approximation.[7]

Table VI

Precision of M-S-A Analysis Method
($L_{\frac{1}{2}}$ at the 95% confidence level)

Number of Analyses (n)	Specific Surface Area (m^2/g)		Weight Mean Diameter (μ)	
	Within a Laboratory[a]	Laboratory at Random[b]	Within a Laboratory[a]	Laboratory at Random[b]
1	0.0683	0.201	6.75	9.55
2	0.0483	0.196	4.77	8.47
3	0.0394	0.194	3.89	8.08
4	0.0341	0.193	3.37	7.88

[a] $L_{\frac{1}{2}} = t_f S_R / \sqrt{n}$, where t_f is Student's t with f degrees of freedom. ($f = 6$)

[b] $L_{\frac{1}{2}} = t_f S_d$ ($f = 6.69$ for specific surface area, $f = 9.02$ for weight mean diameter).

The estimated precision of analyzing AP samples within a single laboratory (random error) is given by the confidence intervals in the second and fourth columns of Table VI. These intervals apply for the analysis of AP samples having particle size distributions within the range shown in Fig. 1. Note that the precision improves with an increase in the number of replicate analyses.

The estimated precision of analyses, considering the random plus the systematic error, by any laboratory selected at random is given by the confidence intervals in the third and fifth columns. Assuming the participating laboratories are representative of the entire population of laboratories, these precision estimates determine the suitability of the method for use as a standard specification procedure. The error is larger than for analyses within a single laboratory because of the contribution of the systematic error variance (S_L^2). Nor is the precision improved much in this case by replicate analyses, because the replicates (n) reduce only the smaller S_R^2 component of S_d^2.

Perhaps in actual practice a higher degree of confidence than 95% would be desired. For higher degrees of confidence the value of t_f would be larger, and the confidence intervals would increase accordingly. The accuracy (bias) of the method could not be estimated in this Round Robin, because a standard AP sample of accurately known particle size is not available.

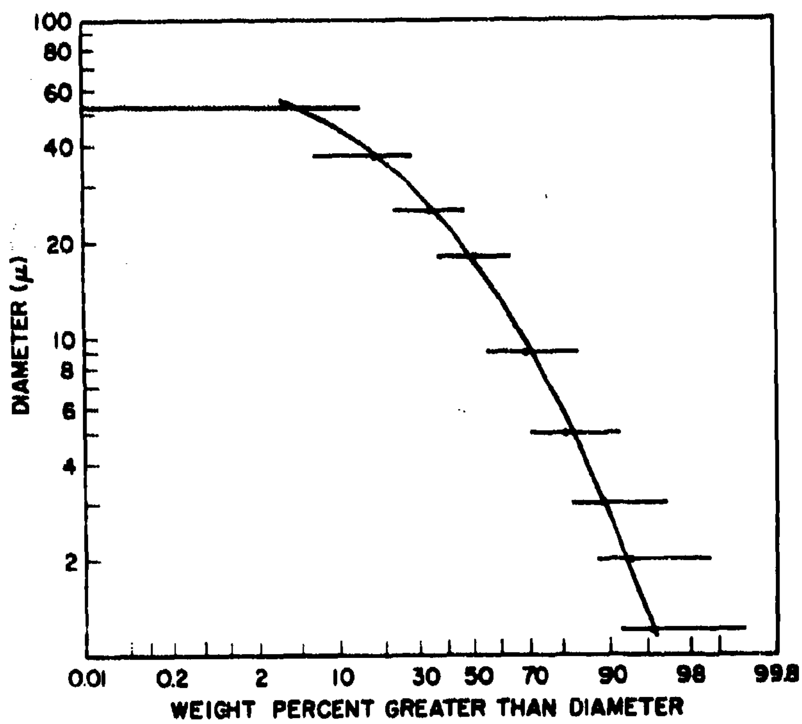

Fig. 4. 95% Confidence Intervals on the Percentage Points of the Particle Size Analysis of Ammonium Perchlorate in Material A by Laboratory 1

Confidence intervals can also be placed on the percentage points of a particle size distribution curve. This is illustrated in Fig. 4 for the analysis of a single sample of material A by laboratory 1. The confidence intervals were calculated using the variance estimates in Table II. Such a precision estimate is of value for determining whether the variations in particle size distribution are due to the analysis procedure or the grinding process.

CONCLUSIONS AND RECOMMENDATIONS

The precision of analyses within a single laboratory is considered adequate, and the fact that a number of laboratories are successfully using the procedure supports this conclusion. However, the method is not recommended as a standard specification procedure for the particle size analysis of nominal 20-30μ ammonium perchlorate, because of the large systematic error among laboratories in the determination of specific surface area. The great difference between these two errors could be due to some deficiency in the analytical procedure that permits laboratories to introduce their own variations. One likely area of inconsistency is in the dispersion of the AP particles, but there are no known alternative techniques that would not also affect the accuracy of analyses.

The simple experimental design, without replication, encouraged laboratories to participate, thus enabling a more reliable estimate of systematic error among laboratories. Past experience has shown that this systematic error is almost always significantly greater than the random error within laboratories.

The comments and suggestions of the clinical session panelists are particularly solicited with respect to the following elements of the Round Robin statistical analysis:

(1) Estimation of the degrees of freedom associated with S_d and S_L.

(2) Determination of confidence intervals or regions for particle size distribution curves.

(3) Criteria for the rejection of extreme laboratories and data.

(4) Experimental design for Round Robins and possible alternatives.

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NEW ANALYSES AND METHODS LEADING TO IMPROVED TARGET ACQUISITION REQUIREMENTS INVOLVING SYSTEMS, GEODETIC AND RE-ENTRY ERRORS, AND INCREASED WEAPONS EFFECTIVENESS FOR CONVENTIONAL WEAPONS*

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SUMMARY. After a cursory critique of currently used methodology for the study of target-accuracy requirements for artillery weapons, this research note is concerned with the development of analytical methods and two different though interrelatable and essentially additive optimization concepts. If implemented within the context of TACFIRE, these methods are conservatively estimated to provide on the average a 30% greater weapons effectiveness. Although the intra and extra weapons systems employment parameters are interdependent, variable, and changing, an integrated operational optimization is achieved. The methods outlined are also useful in weapons R&D and related systems analyses. Furthermore, the rather cogent requirement and related recommendations or conclusions arrived at may be of considerable significance for certain R&D and combat development activities.

FOREWORD. It was originally contemplated to finalize this study in 1967. The author who was also investigating more powerful methods in connection with burst and target height variabilities and the use of conventional cratering and nuclear weapons became, however, increasingly convinced that rudimentary or short-cut methods had to be considered unsatisfactory. A more rigorous and mature approach required time and concentration in view of the slow progress made in the past and also because of a satellite systems study performed during 1967. As to the word "improved" in the title, this should rather be interpreted as "less restrictive," Implementation of the methods and concepts developed would undoubtedly lead to a significant increase of Army weapons effectiveness. In addition, the new methods are expected to have some ramifications pertaining to a variety of R&D and combat development activities. The technical responsibility for this study is exclusively the author's who appreciates USAETL's continued interest in this kind of effort.

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NEW ANALYSES AND METHODS LEADING TO IMPROVED TARGET ACQUISITION REQUIREMENTS INVOLVING SYSTEMS, GEODETIC AND RE-ENTRY ERRORS, AND INCREASED WEAPONS EFFECTIVENESS FOR CONVENTIONAL WEAPONS

1. Introduction.

1.1 The essential ideas underlying this report were developed in August 1966 after an evaluation of the following material: "Target Acquisition Accuracy Requirements, 1965-1975 (U)" (1)[1]; "A Model for Determining Target Location Accuracy Requirements" (2); "Trip Report to CDC Artillery Agency" (3); and "A Technical Analysis to Support Map Accuracy Requirements" (4).

1.2 According to Ref. (3), additional contractual work, to start in July 1967 and expected to last one year, was considered necessary by USACDC in order to improve the methodology report (1). It led to the report "A Study of Target Location Accuracy Requirements for Artillery Weapons – Army 1975 (U)" (5). This study, conducted at the Combined Arms Research Office, Fort Leavenworth, Kansas, and coordinated with the USACDC Artillery Agency, Fort Sill, Oklahoma, applied the methodology of Ref. (1) to all artillery weapons of the 1966-1975 time frame.

1.3 The methodology in both Ref. (1) and Ref. (2) is essentially restricted to the 2-dimensional problem of fragmentation projectiles with impact fuzes and thus less applicable with respect to height bursts. It consists of computing a measure of effectiveness f (see Ref. (1), B-1, eq. (2)) and a fractional coverage C (C^1 in Ref. (2)) so that the fraction of casualties $F = f \cdot C$. Although it has not been spelled out explicitly, f is the probability of hitting the target which is computed by dividing the common area between target and effects pattern, a, by the target area A_T. The determination of f involves the use of a quantity A_L called the lethal area. A_L and C are calculated under the assumption of a uniform target distribution. As to multiple volleys, the assumption is made that the percentage reduction g in F will be directly proportional to the respective g_n in F_n (n volleys). Through the use of this methodology, Spears strives to arrive at the conclusion that "Changes in single-volley coverage of a target by a weapons effects pattern (a quantity relatively easy to determine) can be used as a basis for determining critical reductions in *effectiveness* of multi-volley fire (a quantity difficult to determine accurately)." Through the introduction of $\frac{a}{A_p} \sum_i A_{Li}$ as a measure of the

1. Numbers in parentheses appearing in the text refer to "LITERATURE CITED," p. 13, while numbers in parentheses on the right margin refer to equations.

average fraction of the firepower which hits the target, i.e., the total casualty potential reduced by the factor "common area between target and effects pattern divided by the effects pattern area" and use of the Poisson distribution with $-\frac{a}{A_p}\sum_i A_{Li}/A_p$ in the exponential, Spears arrives at a new non-factorized, or mixed, F. This derivation is, however, not permissible and conflicts accordingly with the former result F = f · C. At best, it is an approximation within an approximate framework. Reference (4) states that the assumption of proportionality between casualties and fractional coverage is the weakest link in the methodology employed in Ref. (1) and (2) and criticizes various other assumptions made. Under the criterion that the target acquisition does not degrade the accuracy of the weapons systems itself by more than 10% and under the assumption that the map accuracy or error is the principal contributor to the weapon site survey error and the target location error, it is concluded in Ref. (4) that present map accuracies can be relaxed or that the Class A National Map Standards have about twice the required precision. This result has been obtained by simple calculations based on the assumption that the total variance is the sum of the individual variances including the geodetic one (target location error). This assumption is wrong and is one of the basic weaknesses of all analyses so far, apart from a rather primitive methodology.

1.4 In view of the shortcomings enumerated in paragraph 1.2 and in order to provide a sound basis for decision making, this report was undertaken. Objectives of the report are as follows:

1.4.1 A rigorous mathematical-statistical analysis involving a direct, physical approach to the problem. In particular, this analysis shall be independent of assumptions regarding target distributions and simplifications involving, e.g., proportionality between casualties and fractional coverage, and casualty potential.

1.4.2 Inclusion of multiple shots and multiple volleys without loss of rigor or generality.

1.4.3 Optimization for multiple volleys as a new and most significant discovery.

1.4.4 Consideration of inhomogeneous target distributions and its change after the first volley.

1.4.5 Incorporation of meteorological-error variances.

1.4.6 Utilization of non-circular distribution parameters.

1.4.7 Application of non-isotropic fragmentation patterns.

1.4.8 A total systems optimization or marginal utility analysis involving the whole range and employment spectrum of a weapons systems, i.e., a grand optimum.

1.5 A contemplated Part II of this study will include a supplemental analysis for height bursts (time and ambient fuzes) including vertical target location errors.[2] Parts III and IV will deal with cratering conventional weapons and nuclear weapons respectively, and a partially different methodology will be required in these areas.

1.6 The optimal aiming pattern analysis together with the optimal overall weapons-systems employment concept developed in this report allow—on the average—a considerable relaxation pertaining to stringent target-acquisition requirements in general and map-accuracy requirements in particular. They tend to shorten the firing engagement time and are also advantageous in case of ammunition shortage. An exception would be hardened-point targets. According to the experience gained (as mentioned in footnote 4), target-location errors can be very large, and identification and location problems will probably exist for longer distances if direct distance and azimuth measurements are performed, though to a lesser extent. Meteorological errors are also not supposed to become negligible under many combat conditions. In view of the above, numerically fixed and extreme accuracy requirements synonymous with sophisticated and very expensive equipment which very often does not live up to expectations under realistic conditions are unnecessary. The R&D process in the areas of more accurate mapping and target location being essentially independent of that pertaining to new weapons systems should be pursued at a normal technological pace and should not overemphasize accuracy but rather concentrate on versatility, reliability, and survivability. This is also consistent with a recent directive of the Army Chief of Staff.

As exhibited by this study, the intra and extra weapons systems employment parameters are interdependent, variable, and changing but nevertheless allow a continuous integrated operational optimization. In so far, the study is also of significance for the Geographic Intelligence and Topographic Support Systems Study (GIANT), the development of the Position and Azimuth Determining System (PADS), and the development of the Long Range Position Determining System (LRPDS). Finally, the methods outlined can serve as a research and weapons-systems analysis tool for both the Combat Developments Command and the Materiel Command.

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2. In this regard, it may be worthwhile to mention that, according to Ref. (3), O. S. Spears, Scientific Advisor to CDC, Artillery Agency, has stated the following: "It is not that we don't consider the vertical component important, we simply realize that it is a difficult problem to solve. Once we get a complete handle on the horizontal accuracies, we will be able to start tackling that problem more intelligently."

2. **Individual Hit Probabilities.** The fragment-damage pattern of a particular artillery shell is not isotropic as can be inferred from Fig. 1. It depends also on range, height of burst, and impact angle. Tabulations contain, in general, isotropic data including distance from burst, total number of effective fragments, and average number of effective fragments per area unit.

An example from Ref. (6) is given below:[3]

Fragment Damage of Shell, HE, 155-mm, M107;
Initial Fragment Velocity 3,500 f/s
Source: Army TM 9-1907, Table XXXV

Dist from Burst in Feet	Total Number of Effective Fragments	Average Number of Effective Fragments Per Sq Ft	For the Lightest Effective Fragment: Weight ozs	For the Lightest Effective Fragment: Vel f/s
r	N	B	m	v
20	1880	.374	.0108	2340
30	1740	.154	.0148	2000
40	1640	.0816	.0195	1740
60	1450	.0321	.0310	1380
80	1300	.0162	.0440	1160
100	1220	.00971	.0562	1030
150	1040	.00368	.0832	845
200	940	.00187	.109	738
300	770	.00068	.166	598
400	640	.00032	.235	503
700	420	.00007	.515	340

From individual, i.e., unaveraged, fragment patterns, it is possible to determine through the use of sampling techniques individual hit probabilities. Thus, p_1 (F, ρ, ϕ) would be the average probability that a person or item with cross section F which is located at a distance ρ and azimuth ϕ from the burst suffers exactly one hit. In this respect, the azimuth is to be counted counterclockwise from the line of fire. By p_I $(F, \rho, \phi) = p_1 + p_2 + \ldots$ we designate the probability of at least one hit. With reference to human beings, it would be possible to drop the letter F. For identification purposes, we denote a semi-fixed pattern of human beings by superscripts and have

3. An excellent introduction into kinds and characteristics of explosives (munitions) is given in Ref. (7).

Source: Army TM 9-1907

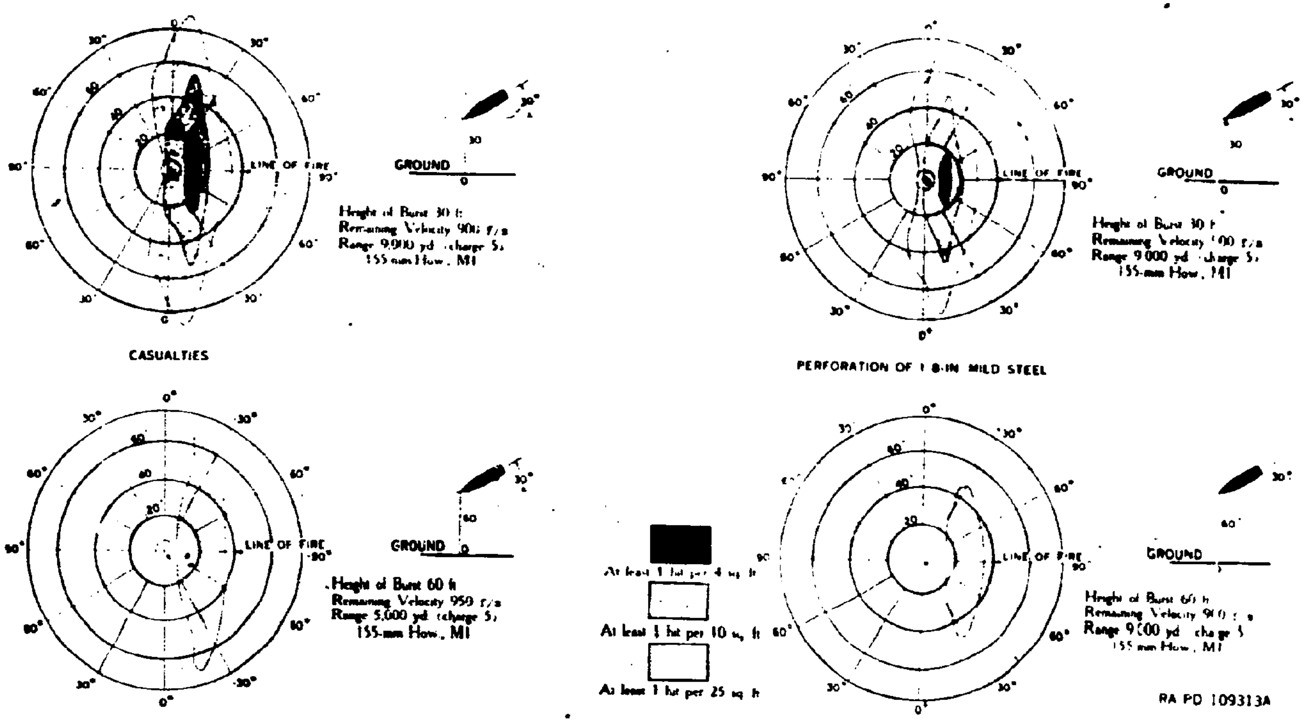

Fig. 1. Damage pattern — 155-mm HE shell, M107.

thus for example, $p_1^{14}(\rho, \phi) = p_1(\rho_{14}, \phi_{14})$. It is to be kept in mind that these probabilities refer to constant range, height of burst, and impact angle.

3. **Distributions and Distribution Parameters.** Weapon distribution parameters for a specific range are the line of fire and lateral standard deviations S_r and S_ℓ. As already mentioned, height uncertainties are considered negligible in this investigation (Part I). In addition, we have target-location errors depending, e.g., on map accuracy, target identification, and location,[4] and meteorological errors. The corresponding distributions may for simplicity be described by the two parameters σ_T and σ_M. Since we restrict ourselves to normal distributions, we may establish the relation

$$\text{var } r_T + \text{var } r_M = \text{var } \ell_T + \text{var } \ell_M = \sigma_T^2 + \sigma_M^2 = \sigma^2 \tag{1}$$

It is important to remember that σ_T^2 and σ_M^2 need not be considered constant for a certain range. Hence, σ^2 may allow a few classes of variability depending on circumstances.

4. **Formulation of Multiple Volley Optimization Problem for Stationary Personnel Distribution.** In Fig. 2, the general target coordinate system for aiming purposes is denoted by x, y. At the origin, the combined target location distribution

$$f(\bar{\xi}, \bar{\eta})\, d\bar{\xi}\, d\bar{\eta} = \frac{1}{2\pi\sigma^2}\, e^{-\frac{1}{2}\frac{\bar{\xi}^2 + \bar{\eta}^2}{\sigma^2}}\, d\bar{\xi}\, d\bar{\eta} \tag{2}$$

has a maximum. The aiming point for the first volley is represented by 0_2, with coordinates a_1, b_1; and the respective gun-aiming points separated by the distance c are

4. The accuracy of a class A map of 1/50,000 scale within a single sheet can be expressed approximately by a standard deviation of 25m. Though this is not a negligible parameter and accuracies decrease with reference to lower quality maps, additional errors enter in case of target identification (which includes determination of a reference point for the whole target configuration) and target location on the map. The latter type of error can be very sizable, and standard deviations of the order 500m have been found according to Ref. (8), (9), and (10). For simplicity, we lump map, target identification and target location variances together into var r_T. Smaller σ_T's are, of course, to be expected in case of a direct link including distance and azimuth measurements between observer and a suitable target reference point. Directional σ_T's might also be generated by moving targets and target configurations.

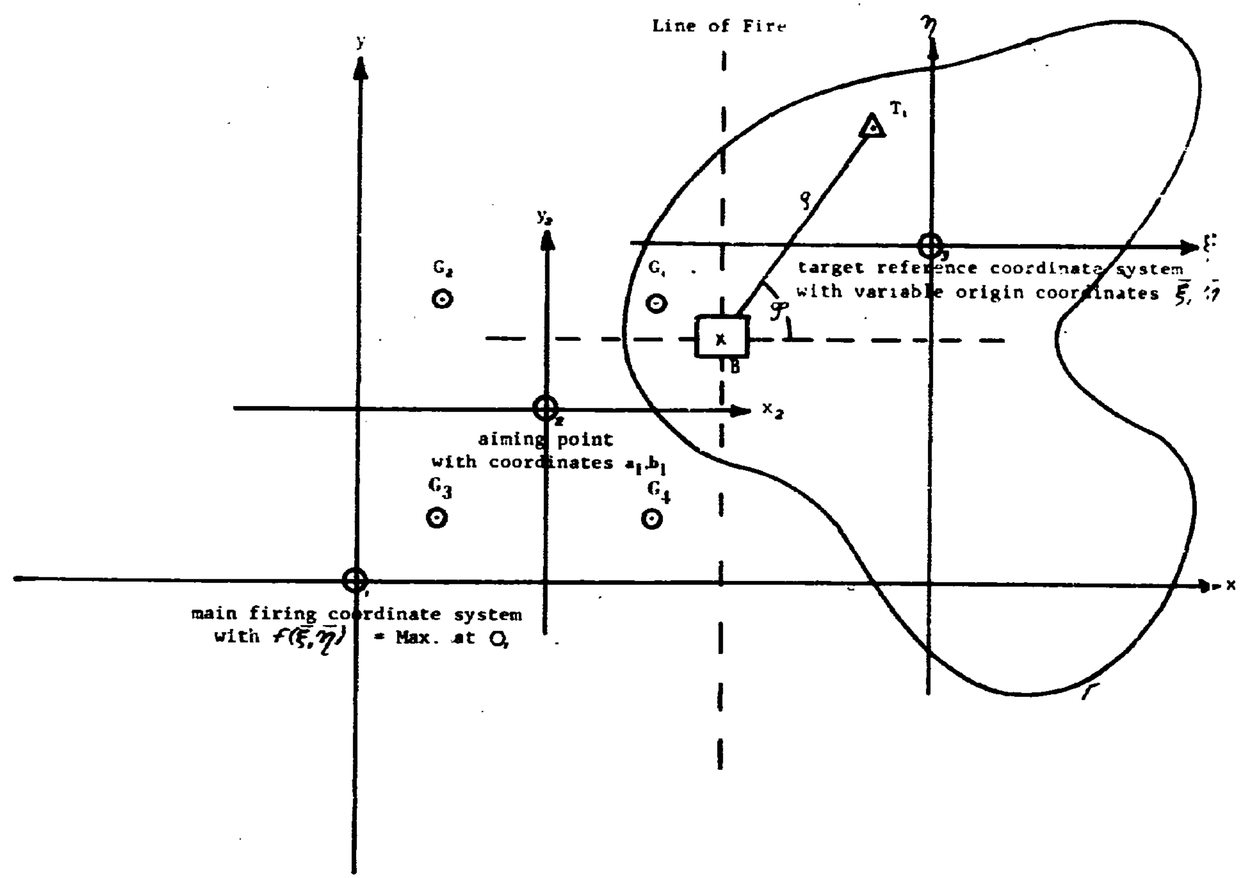

Fig. 2. The general target coordinate system.

G_1, G_2, G_3, G_4. The burst point for which a total impact probability is to be computed and which lies in a finite area element (for numerical purposes) is B. Only one individual target, T_1, is indicated within the target area with boundary Γ. The distance from B to T_1 is ρ and the azimuth is ϕ commensurable with the denotations of para. 2.

For the first volley, we have four burst distributions designated by

$$\lambda_1(x,y;a_1+\frac{c}{2},b_1+\frac{c}{2};S_r,S_\varrho)dxdy,\ \lambda_2(x,y;a_1-\frac{c}{2},b_1+\frac{c}{2};S_r,S_\varrho),\ \text{etc.} \tag{3}$$

For sufficiently small area elements $\Delta x\Delta y$, we arrive then in integral form at an intermediate average probability of hitting T_1 at least once,

$$P^1_{i1} = \iint_{-\infty}^{+\infty} \sum_1^4 \lambda_v \cdot p^1_1(\rho,\phi)\,dxdy = \iint_{-\infty}^{+\infty} \sum_1^4 \lambda_v\, p_1(x,y,\bar{\xi},\bar{\eta},\xi_1,\eta_1)\,dxdy \tag{4}$$

and, since 0_3 obeys a distribution law, at

$$P^1_1 = \iint_{-\infty}^{+\infty}\iint_{-\infty}^{+\infty} \sum_1^4 \lambda_v\, p_1(x,y,\bar{\xi},\bar{\eta},\xi_1,\eta_1)\, f(\bar{\xi},\bar{\eta})\,dxdyd\bar{\xi}d\bar{\eta}\ . \tag{5}$$

For k volleys and μ individual targets, we obtain the total expected casualty result

$$n_{1\sum_1^k} = \sum_{\tau=1}^{\tau=k}\sum_{\mu=1}^{\mu=N} \iint_{-\infty}^{+\infty}\iint_{-\infty}^{+\infty} \sum_{v=1}^{v=4} \lambda_v(x,y;a_\tau,b_\tau)p_1(x,y;\bar{\xi},\bar{\eta};\xi_\mu,\eta_\mu)f(\bar{\xi},\bar{\eta})dxdyd\bar{\xi}d\bar{\eta} \tag{6}$$

The optimization conditions can be formulated as

$$\frac{\partial}{\partial a_\tau}\, n_{1\sum_1^k} = 0\ ; \qquad \frac{\partial}{\partial b_\tau}\, n_{1\sum_1^k} = 0 \tag{7}$$

For a circular and homogeneous (uniform) target distribution conditions (7) reduce to fewer equations, i.e., the respective aiming pattern, consisting of a set of k origins 0_2 would be invariant under a rotation about 0_1.

After the first volley which has in many cases a surprise effect, a degradation with respect to P_1 can be expected which can be expressed as a transition from ground to air burst[5] and by an empirical reduction factor $R < 1.00$. Taking this into consideration, a more general result corresponding to eq. (6) reads as

$$n_1\sum_1^k = \sum_{\mu^{(1)}=1}^{\mu^{(1)}=N} \int\!\!\int_{-\infty}^{+\infty}\int\!\!\int_{-\infty}^{+\infty} \sum_{\nu=1}^{\nu=4} \lambda_\nu \cdot p_I(x,y;\bar{\xi},\bar{\eta};\xi_{\mu(1)},\eta_{\mu(1)})f(\bar{\xi},\bar{\eta})\,dx\,dy\,d\bar{\xi}\,d\bar{\eta}$$

$$+\sum_{\tau=2}^{\tau=k}\sum_{\mu^{(2)}=1}^{\mu^{(2)}=N} R \int\!\!\int_{-\infty}^{+\infty}\int\!\!\int_{-\infty}^{+\infty} \sum_{\nu=1}^{\nu=4} \lambda_\nu \cdot \pi_{\mu(2)} p_I^{c}(x,y;\bar{\xi},\bar{\eta};\xi_{\mu(2)},\eta_{\mu(2)})f(\bar{\xi},\bar{\eta})\,dx\,dy\,d\bar{\xi}\,d\bar{\eta} \qquad (8)$$

In eq. (8), the first term refers to the first volley. The second term containing the reduction factor R reflects a changed hit probability function and includes the factor $\pi_{\mu(2)}$.

A particular $\pi_{\mu(2)}$, say $\pi_{3(2)}$, requires the computation of the individual P_I^3 from eq. (5). It is then

$$\pi_{3(2)} = 1 - \kappa\, P_I^3 \qquad (9)$$

where κ denotes the (average) probability that an individual, hit at least once, remains at the initial position. The index (2) in $\pi_{3(2)}$ indicates the transition from the initial target configuration to a second, more protective one.

It should be mentioned that, in connection with an evaluation of eq. (8), an average $n_1\sum_1^k$ for typical target distributions under consideration of protective obstacles can be determined. It is also possible to classify targets by size and concentration indices (cf. para. 7). Furthermore, it is possible to split P_I up into probabilities for exactly 1 hit, 2 hits, etc.

5. **Probability Distributions.** The probability distribution associated with $n_1\sum_1^k$ of eq. (8) can easily (though approximately) be found by setting $\bar{p} = \frac{n_1\sum_1^k}{N}$, where N

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5. According to Ref. (6), p. 181, ground bursts generally are more effective against material and personnel in case of no shielding by revetments, but personnel in foxholes or trenches should be attacked by air-burst fire.

is the total number of individual targets. According to Kendall (11), we have then the moments about the mean $N\bar{p}$

$$\mu_2 = N\bar{p}\bar{q} \qquad \bar{q} = 1 - \bar{p}$$

$$\mu_3 = N\bar{p}\bar{q}(\bar{q} - \bar{p})$$

$$\mu_4 = 3\,N^2\,\bar{p}^2\bar{q}^2 + \bar{p}\,\bar{q}\,N\,(1 - 6\,\bar{p}\,\bar{q})$$

from which a Pearson Type I curve can be calculated which is evaluable in terms of the incomplete B-function. Alternatively, the binomial distribution and associated tables might be used.

6. **Symmetric Aiming Patterns.** As an illustration, some symmetric aiming patterns applicable for circular and completely homogeneous conditions are shown in Fig. 3. Equivalent solutions would result through an arbitrary rotation.

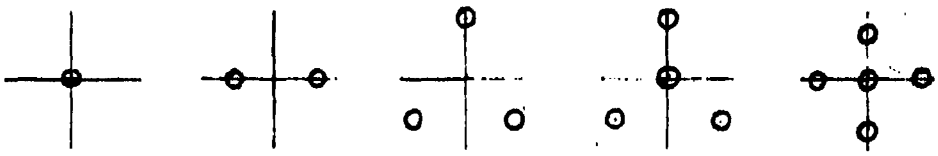

Fig. 3. Symmetric aiming patterns.

7. **Computation and Utilization Considerations.** A particular kind of quasi-circular target could profitably though not exclusively be characterized by r (range to origin O_1) S (target size, 3 indices), D (quadrant density, 3 indices), σ (combined target location error, 3 or 4 indices), k (number of volleys). From these data, k azimuth and angular height corrections for the optimal aiming points would be immediately available. Of course, the optimization computations should be conducted by a large-scale digital computer, i.e., not in the field. The corrections · functions of variable input data would be available as stored digital information and incorporated in TAC-FIRE procedures. Probability statements depending on k could be added. Too many indices are to be avoided. As to D, there are 3^4 variations (with three elements of the fourth class and repetitions). Some of these variations can be omitted because of practical reasons. Nine quadrants leading to 3^9 complexions would be prohibitive. From a

scientific standpoint, it also appears that an effective employment of a single artillery weapons system (battery) represents a rather formidable problem.

8. Views on Optimal Weapons Systems Employment.

8.1 With reference to optimization considerations, we assume the existence of the following scheme:

$$\left.\begin{array}{lll} r_1 & h_{11}\, U_{11}\,(k_{11}) \ldots\ldots\ldots\ldots\ldots & h_{1n}\, U_{1n}\,(k_{1n}) \\ r_2 & h_{21}\, U_{21}\,(k_{21}) \ldots\ldots\ldots\ldots\ldots & h_{2n}\, U_{2n}\,(k_{2n}) \\ \vdots & ------------------- & \\ r_n & h_{m1}\, U_{m1}\,(k_{m1}) \ldots\ldots\ldots\ldots\ldots & h_{mn}\, U_{mn}\,(k_{mn}) \end{array}\right\} \qquad (10)$$

In this discrete scheme, valid for a particular weapons system (e.g., artillery battery), the symbols r, h, U, and k denote range, relative frequency of employment, mean weapons effect for a particular type of target, and number of volleys respectively. Strictly speaking, there has to be a greater number of discrete schemes with associated scheme frequencies in order to account for variations in target size and target location. This involves an additional frequency matrix with elements $j_{a,b}$. The total mean weapons systems effect can then be formulated as

$$\bar{U} = \sum_{\alpha,\beta} h_{\alpha,\beta} \sum_{a,b} j_{a,b}\, U_{\alpha,\beta;a,b}\,(k_{\alpha,\beta;a,b}) \qquad (11)$$

We shall distinguish between $\bar{U}$ and $\hat{U}$ with $\hat{U}$ considered optimized by the analysis outlined in para. 4. In other words, $\bar{U}$ does not imply the utilization of optimal aiming patterns.

If we apply the rather usual criterion of 30% damage or casualties with a 90% assurance, we arrive at

$$\bar{U}_{30\%/90\%} \cdot \quad K = \sum_{\alpha,\beta} \sum_{a,b} k_{\alpha,\beta;a,b} \qquad (12)$$

and

$$\bar{U}_{30\%/90\%} \cdot \quad \hat{K} = \sum_{\alpha,\beta} \sum_{a,b} \hat{k}_{\alpha,\beta;a,b} \qquad (13)$$

with $\bar{U} \approx \hat{U}$ and $\hat{K} < K$.

A reasonable measure for the effectiveness increase expressed in percent is evidently

$$\eta = 100 \; \frac{K - \hat{K}}{\hat{K}} \tag{14}$$

8.2 A different approach would consist of stipulating a constraint, say

$$\hat{K}^{(2)} = \sum_{\alpha,\beta} \; \sum_{a,b} \; k^{(2)}_{\alpha,\beta:a,b} \leq A \tag{15}$$

and to compute the $k^{(2)}$'s in such a way that

$$\hat{U}^{(2)} = \text{Max.} \tag{16}$$

For the purpose of comparison, we may assume

$$\hat{K}^{(2)} = \hat{K} \tag{17}$$

The optimization expressed by eq. (15) and (16) implies *a fortriori*

$$\hat{U}^{(2)} > \hat{U} \tag{18}$$

and a relatively greater expansion of volleys with respect to closer range targets and those involving smaller σ's. On the other hand, for some targets with less favorable characteristics, the 30%/90% criterion might not be fulfilled.[6] What can be said with certainty is that the utilization of optimal aiming patterns makes the ground optimization described in para. 8.2 quite attractive. It is conservatively estimated that optimal, aiming-pattern utilization incorporated in TACFIRE would result in a 15% increase in systems' effectiveness. The systems' overall optimization would yield an additional 15% increase and thus lead to a combined improvement of 30%.

6. This is, however, not a serious limitation since it can be partially or completely overcome by a greater A in eq. (15). This would particularly apply to defensive positions with a large ammunition supply.

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An Air Defense Comparative Model

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ABSTRACT

This paper proposes a new and unique approach for conducting comparative experiments or evaluations between existing or proposed air defense weapon systems. It is based upon the game theory "minimax" philosophy and provides several distinct advantages over the use of computer simulation methods. Submodels for objectively determining the optimal deployment of the defense and the optimal attack routes to be used by the attacking aircraft are discussed.

INTRODUCTION

Development and deployment of air defense systems having a large degree of effectiveness against high altitude aircraft, has resulted in increased interest in the operation of tactical aircraft at low altitudes [1]. As a result many weapons systems analysts have become deeply involved in analytical and experimental studies evaluating the effectiveness of existing or proposed defense systems for defeating the low altitude threat. Historically, war games have been extensively used to "model" military situations for such experimental and evaluative purposes. The different types or classes of war games that have been used are: (1) field exercises, (2) board games, and (3) computer simulations.

When considering air to ground conflict situations, experimentation using the computer simulation technique has proven to be the most feasible and efficient. Consequently, computer simulation models have evolved from very simple and basic models into those which are now very large, complex and time consuming. This increase in size and complexity has arisen due to the desire to approach, as near as practically possible, an exact model of the real life situation. Unfortunately, as realism has increased, so too has the computer time required to run the experiments.

This paper proposes a new approach for conducting comparative experiments or evaluations between existing or proposed air defense weapon systems. It is based upon the game theory "minimax" philosophy and provides several distinct advantages over existing computer simulation models. These include:

1. Less computation time required.
2. Fewer necessary assumptions and simplifications, hence greater realism.
3. Additional useful information is generated such as optimal defense system deployment, optimal attack routes, etc.
4. Real world scenarios, (actual situations) may be used.
5. Only one computer run per defense system is required.

MODEL PHILOSOPHY

The scenario used for the Air Defense Comparative Model should be a real world situation i.e. a specific piece of terrain which is to be defended. The concept of the low altitude attack is to utilize the masking effects of the terrain (hills, valleys, etc.) and of the earth's curvature, to prevent the defense from being able to detect and engage the attackers until the targets are reached [1]. Hence, any experimental evaluation of the defense system must take this into account.

The defensive problem in our scenario can be stated as follows:

1. Given a specific sector of terrain (with hills, valleys, etc.) which is to be defended by n or less defensive units.
2. Given the characteristics of the defense system (i.e. range, maximum and minimum elevation angles, azimuth scan angle, etc.).
3. Given the feasible locations and pointing angles for the placement of defensive units. (i.e. cannot be located in middle of lakes, bottom of ravines, etc.).
4. Find those n locations and pointing angles which (a) minimize the range from any attacker to a systems radar and (b) maximizes the visibility of the combined radar systems. This must take into account the masking effects of terrain features and earth's curvature.

Likewise, we can state the problem faced by the offense or attackers in our scenario. This can be done as follows:

1. Given a set of targets to be destroyed which are contained within a specific piece of terrain which is defended by a set of ground based air defense missile systems which are optimally deployed.

2. Given that the attacker or penetrator has complete knowledge from intelligence operations of these allocations and of the defensive capabilities.

3. Determine the best location to enter the defended sector, and then the least risk route to follow in order to reach a designated target. The least risk route is that which minimizes the visibility time and maximizes the survival probability.

In the proposed Air Defense Comparative Model, these problems are solved objectively and optimally by sub-models. The objective, optimal solution to both problems, is a unique feature of the proposed model. The tactics are not determined by educated guess as in other war game models. It should be pointed out however, that the two sub-models (optimal allocation of defense units and optimal attack route analyzer) can be used to set the scenario and tactics for other computer simulation models. It is a firm conviction of the authors, that where tactics are determined by educated guess, the experimenter may inadvertently penalize a system by his choices. Allocating or placing the defensive units by the use of the optimal allocation model on the other hand, allows each different system to capitalize on its strengths and minimize its weaknesses.

The concept of the proposed comparative model can now be stated. The philosophy followed is:

1. Determine the optimal defense system deployment for each system to be considered, based upon its own characteristics and the terrain features of the sector to be defended.
2. Determine the optimal attack routes against each defense system which minimizes the risk to the attacking aircraft.
3. Determine the risk incurred by the attacker for each defense system to be compared.
4. The defense system that maximizes the enemy's risk is the preferred system.

Maximization of risk to the enemy has been chosen as the measure of effectiveness for a very straightforward reason. The purpose of the air defense system is to protect field army value units such as supply depots, vehicle concentrations, artillery positions, troop concentrations, etc. The purpose of any offensive weapons system is to destroy a given set of targets (value units) with the least possible cost. It is a generally accepted fact that a defensive system cannot prevent a determined and powerful offense from destroying a given number of these targets if the offense is willing to pay the price. The defense objective then is to try to extract a high cost from the offense. In gaming theory terms then, the goal of the defense is to maximize the offensive cost while minimizing the defensive cost. Both offensive and defensive costs are direct functions of the risk incurred by the offense in carrying out its attack.

The proposed Air Defense Comparative Model is a Game Theoretic Model utilizing the maxi-min principle of optimality. [2] Stated simply the defense chooses that strategy which maximizes minimum risk while the offense chooses that strategy which minimizes maximum risk. The value of the game is then calculated for each defensive system to be compared and the one which extructs the highest risk to the attackers, is the preferrable system.

An overall schematic of the model is shown in Figure 1. Due to space limitations, it will not be possible to give detailed descriptions of the sub-models in this paper. However detailed descriptions of the component sub-models, including computer programs, may be found in references 3, 9, 10, and 11. Short descriptions of the sub-models are given in the following sections.

MAVD MODEL

Basic to the proposed Air Defense Comparative Model is the visibility subroutine called MAVD (Minimum Altitude Visibility Diagram). MAVD is a new concept and subroutine for calculating the visibility of targets to the defensive system sensor units [3].

The input to the MAVD Model is an array of digitized topographic data which is stored on magnetic tape. The Army Map Service has expended a considerable amount of time and effort in the digitization, and storing on magnetic tape, of topographic data. An $m \times n$ grid of horizontal ($m = 1,2,\ldots,i$) and vertical ($n = 1,2,\ldots,j$) lines is overlaid over the topographical map of the piece of terrain of interest. The spacing or grid interval between the lines is equal. The standard army battle map is a transverse Mercator projection of the Gauss-Kruger type [4]. The primary coordinate system for the map is a square grid system called the Universal Transverse Mercator grid [5]. Points of interest can be located on the map by their UTM grid coordinates. The UTM grid will appear on any map as a square grid system where the numerical values of the coordinates of a point are positive, and increase as one moves the point east and north. For good terrain definition, it has been found that the grid spacing should not exceed about 1,500 feet or 300 meters. The local altitudes above sea level for the grid points thus defined are read off the topographical map and entered along with their grid point designation (i,j) as inputs.

MAVD (Minimum Altitude Visibility Diagram) is a geographic representation of the minimum local altitude at which a target may fly above the local terrain and still be visible to the given air defense sensor. Thus a MAVD value of 150 feet at point 5, 45 (the i,j grid representation of a specific point on the terrain) means that any aircraft at 150 feet altitude or above is visible to the sensor, or conversely any aircraft below 150 feet altitude is not visible (either masked by terrain irregularities or the curvature of the earth) to the sensor.

The MAVD routine is used to compute all the MAVD values for every designated point (i.e., a point defined by the intersection of two grid lines) on a grid for all given sensor locations. Figure 2 represents an example of a MAVD display. The top figure (2a) is the original terrain map and the bottom figure (2b) is the MAVD display where each MAVD value is given for the corresponding point on the original terrain map. In three dimensions a surface through all the MAVD values could be represented and any aircraft on or above this surface is visible to the given sensor(s).

The values on the MAVD represent, as mentioned, the minimum altitude values at which a penetrator is visible to a sensor at any given grid point (intersection point represented by the intersection of an "i" and "j" line). The effect of the curvature of the earth's surface and all terrain irregularities are considered in the computation of these values. The calculation procedure is straightforward and uses basic plane and analytical geometry techniques. The detailed computer program for this subroutine (written in Fortran IV) may be found in reference 3. The present program is capable of handling a 211x211 grid size.

DEFENSE ALLOCATION MODEL

The second sub-routine utilized is the Allocation program which provides a systematic, objective method for computing the optimal deployment of any air defense system. Allocation is defined, with respect to this paper, as the assignment (or placement) of air defense system sensors at specific points on the given piece of terrain. The optimal allocation is that deployment which maximizes the attacker's risk. It may be also thought of as that deployment which minimizes the probability that an attacking aircraft or missile penetrates the defensive system undetected.

A survey of the literature uncovered an almost negligible amount of effort towards devising any systematic, objective, assignment of sensor units to terrain. The majority of models surveyed assigned sensor locations at random or at best, use an educated guess based on an "analysis" of the terrain involved. This analysis consists of little more than looking at the terrain map and attempting to visualize the effect of placing a sensor at a certain point. Such methods of choosing sensor locations are far from optimum. It is highly subjective and consequently it is doubtful that any two people would choose the same locations.

The mathematical formulation of optimal deployment problems falls into a sub-class of non-linear, zero-one programming problems. Although this has been previously recognized, [6] it has not been possible to apply the existing methods of zero-one programming to any practical size problem due to the severe limitations of these mathematical methods. A new procedure called Complementary Programming, was therefore developed as a part of this research [7] and is applicable to very large problem types. For example, it was used to compute the optimal deployment of radars within the continental United States. The results were then compared with those previously proposed by Smallwood [8] utilizing a much more involved and time consuming procedure. The Complementary Programming method achieved an improved deployment over Smallwood's "optimal method."

Tests conducted during the evaluation of an early version of the Allocation Program showed that optimal deployment was sensitive to both range and visibility. The tests showed that, for a large terrain area, the primary factor in deployment was range, and visibility was only secondary. This observation led us to divide the original allocation program into two separate programs. We have designated the first program as the Coarse Allocation Program and the second as the Fine Allocation Program. The main concern of the Coarse Allocation Program is the minimization of range distance while the purpose of the Fine Allocation Program is the maximization of radar system visibility. The two programs are then used sequentially (see Figure 3). A good analogy to this method is the process used in turning to a station on the radio. One first turns the selector to the vicinity of the station in one rapid motion. When the station vicinity on the dial is reached, you then fine tune the selector until the station is optimally received. The Coarse Allocation Program achieves an initial, coarse deployment based primarily on range considerations. This coarse deployment is then used in conjunction with the Fine Allocation Program to achieve a new final deployment based primarily on visibility considerations. The sequential operation of the two programs thus provides a final deployment that has both minimized the range from any attacker to a systems radar and maximized the combined radar system visibility.

It now becomes necessary to define a measure of the "goodness" of the coverage or visibility of the sensor for the points within its defined sector. Visibility was previously defined as a measure of the ability of a sensor or sensors to detect a target (or targets) within the air space over a given terrain area. This measure can be represented by a range of numerical values from zero to one and will be called the Visibility Value. A value of zero will be defined as there being no visibility over a given grid point for a specified altitude range. For example, if a grid point is not within the sector or range of a certain sensor, a zero is given to the Visibility Value for that sensor for the grid point. Another example of zero Visibility Value would be if the MAVD value for a grid point (for a given sensor) was 10,000 feet and the probability of an attack at that altitude or above was zero. We then would assign a zero to the Visibility Value. A value of one would require that, for the grid point, there exists visibility for all possible altitudes of attack.

The method used to convert MAVD values to visibility values is simple. First a limit is set on the altitude values of interest. Since the emphasis for this paper is on low altitude attacks and since an attack at high altitudes is visible to almost any sensor allocation, it is unnecessary to consider any altitudes above a specified ALT MAX. ALT MAX will be assigned a value for which there is (a) essentially zero probability of attack at altitudes $\geqq$ ALT MAX or (b) considering terrain altitudes and irregularities, there is an almost certain probability of detection of any targets above ALT MAX.

The Visibility Value would then be calculated as:

$$\text{Visibility Value} = \frac{\text{ALT MAX} - \text{MAVD}}{\text{ALT MAX}}$$

where if the MAVD exceeds ALT MAX we assign a Visibility Value of zero i.e., we do not allow negative Visibility Values. Thus the Visibility Value is proportional to the per cent of air space that the sensor can see between the point on the surface of the local terrain and ALT MAX.

The Visibility Values as computed are then written on the computer drum in the order shown below:

Visibility Values on Drum

B Grid Point \ Sensor	1	2	3	4	5	......	240
1,1	.000	.900	.905	1.000	.500	......	.000
1,2	.100	.000	.810	.150	.400	......	.200
1,3	.000	.800	.000	.300	.200	......	.800
.	.	.	.	.	.	......	.
.	.	.	.	.	.	......	.
.	.	.	.	.	.	......	.
50,50	.900	.900	.000	.600	.750	......	1.000

Thus each column represents the Visibility Values for a possible sensor location for all points on the grid. Our objective is then to combine a specified number of the above possible sensors so that the resulting "sum" of their coverage is maximum.

As perviously mentioned a new heuristic programming method called Complementary

Programming was derived for accomplishing this. The method is based upon the basic principle of the union of sets from set theory, where each of the columns in the above table of Visibility Values is considered an ordered set. The development, justification and complete computer programs for accomplishment can be found in references 9 and 10. The present computer program will handle a situation with 199 x 199 grid size over the terrain, 38,601 possible candidate radar locations and/or 463,212 possible candidate location - pointing angle combinations (if radar has less than 360° azmith capability).

ATTACK ROUTE MODEL

Having determined the optimal deployment of the defense, the next step is to turn our attention to the offense. As stated earlier, the problem of the offense may be stated as, "given an airspace over a specific piece of terrain that is defended by a ground based air defense system, find the least risk route that may be taken over this terrain to reach an assigned target. Based on the mini-max principle, it is assumed that the air defense system is optimally deployed over the terrain and that the offense has complete knowledge of both the defense system deployment and capabilities. The least risk route solution would then specify at which point(s) to enter the defended air space, the path to follow through the air space to the target, and the probability of survival. Such a computer model has been developed [11] and will now be briefly described.

A survey of the literature showed that very little had been accomplished in the area of the systematic, objective determination of optimal attack routes. Furthermore, of the few methods proposed, none was capable of handling anything except very small problems. It was therefore, decided to provide a relatively new approach rather than try to build onto or refine an older approach. The optimal attack route problem was formulated in terms of a classical network problem. This was a natural approach in view of the grid overlay used for the terrain description in the MAVD and Defense Allocation models. With regards to the network description, we can state our problem as: "Determine the least risk route through the network, where we may enter the network at any outer node (intersection of grid lines) and travel on any branch (grid line between nodes) in either a forward or lateral direction.

The risk in traveling from one node to another in the network is then expressed as follows:

$$R = f(V, Ra, t)$$

where:

$R =$ Risk

$V =$ Visibility factor (i.e. is the target visible or not)

$R_a =$ Range from target to defense system

$t =$ The time in which the target is visible to the defense system.

Consequently, a least risk path would in general minimize the time the target is visible, minimize the number of times the target is visible, and maximize the range to the defense system (for the times in which the target is visible). Under this description, each node of the network may be assigned a value of risk. The "cost" of going from one node to the other is then the difference in risk from one node to the next, or the probability of survival from one node to the next.

Under the network formulation, several methods for solving the classical "shortest route through a network" are available. The most efficient methods are linear programming and dynamic programming. While either of these methods can solve a small problem, it was found necessary to utilize dynamic programming for the larger real world problem because of computer storage requirements. For example, the solution of a problem by linear programming (Hungarian algorithm) would require storage of at least N^2 points (where N is the number of grid points). The storage requirements of dynamic programming are more on the order of 4N. Since determination of a flight path is a multi-stage decision process, dynamic programming was particularly well suited to the problem.

The method of dynamic programming is discussed in detail in the literature [12] and thus will be touched on only briefly. Generally speaking, dynamic programming is a method of solving multi-stage decision decision problems. Unlike linear programming, there is no standard mathematical formulation of the problem. It is a general approach and the particular equations used must be developed to fit each separate case at hand.

As stated, we use the same grid overlay as used in the MAVD and Allocation Models. But under our dynamic programming formulation we let each row (i.e. nodes i,j with i constant) represent a decision stage (see Figure 4). Each stage in turn, has a number of states associated with it. In our case, the states of each stage are simply the nodes of each row. In general, the states are simply the various possible positions in which the aircraft might be at any stage of the problem. In a multi-stage problem with discrete stages, (as in this problem) decisions are to be made at the beginning of the stages. The policy decision to be made at each stage is the destination for the next stage i.e. which state in the next stage. It is dependent upon the situation at the time of decision, upon the decision itself and upon the stage of the system. Each decision affects not only the next stage but all subsequent stages. The solution of the problem is a sequence of decisions that yields the least risk route. This is essentially Bellman's principle of optimality, "An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

The particular version of dynamic programming used in this problem computes the flight path in a "backward" manner. That is, one first starts at the target and then determines the optimal paths from each state in the previous stage to the target. Once this is done, the optimal paths from each state in the M-2 stage to the M-1 stage is computed. At each stage only the values of optimal paths need to be stored. This procedure is repeated until we are at the initial stage (i.e. row one). At this point all of the optimal paths from any of the entry points to the target are available.

As with the allocation model discussed in the previous section a two phase sequence is used. The first phase or calculation of the course attack route is primarily predicated on minimizing visibility (or risk) and the second phase or calculation of the fine course route is primarily to minimize exposure time. The solution procedure requires data in the form of two matrices. These matrices are (a) visibility matrix and (b) missile flight time matrix.

The visibility matrix provides the probability of detection for each node of the terrain sector. The aircraft altitude, h, and the MAVD values for each node are compared. If the MAVD value is greater than h, the aircraft is invisible and the risk is zero. It should be noted that the MAVD value to be compared with the aircraft altitude is always the minimum value of all the radar sites within range of the node. If the MAVD value is less than h, then the visibility value for that node has some probability value associated with it. True aircraft visibility is not just a line of sight yes or no variable. Range from the target, reflective surface of the target, transmitter power, size of the antenna, etc. are all variables which may affect the visibility of a target to a radar. A review of the variables affecting visibility indicates that the effect of each is dependent to a large degree, upon the range of the aircraft from the radar. For this reason, the range of the target from the radar having the best MAVD value was selected as the best single variable to measure visibility. The relationship of range to probability of detection can be expressed graphically and is obtained from an analysis of the performance specifications of the missile system under consideration. The range value is thus converted to a visibility probability value based on each missile systems specifications. The risk is set equal to the probability value for the specified range. The detailed development of the model with the procedure coded in FORTRAN V language is given in reference 11.

SUMMARY

This paper has proposed a new and unique approach for conducting comparative experiments or evaluations between air defense weapon systems. The sub-models which were briefly discussed were developed as means of improving existing digital, computer stimulation experiments. It is believed, however, that these submodels and developed methodologies can be utilized as the basis for a completely independent, "unified air defense system comparison model." Such a model could be used for realistically analyzing and evaluating air defense systems in what we believe would be a far more economical and accurate manner than is presently available from simulation models.

ACKNOWLEDGEMENT

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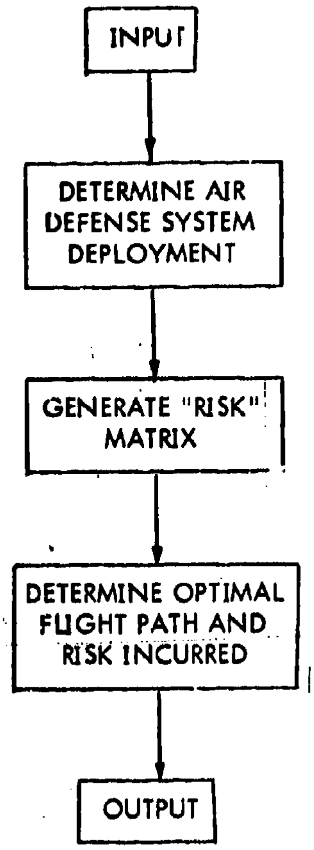
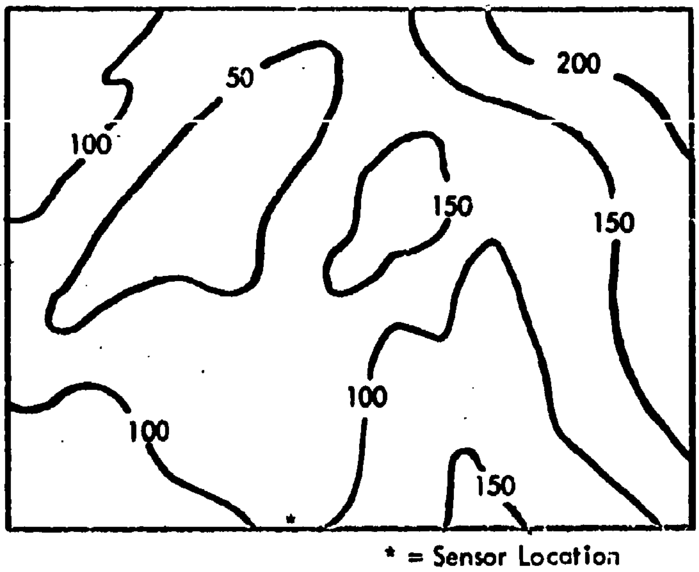
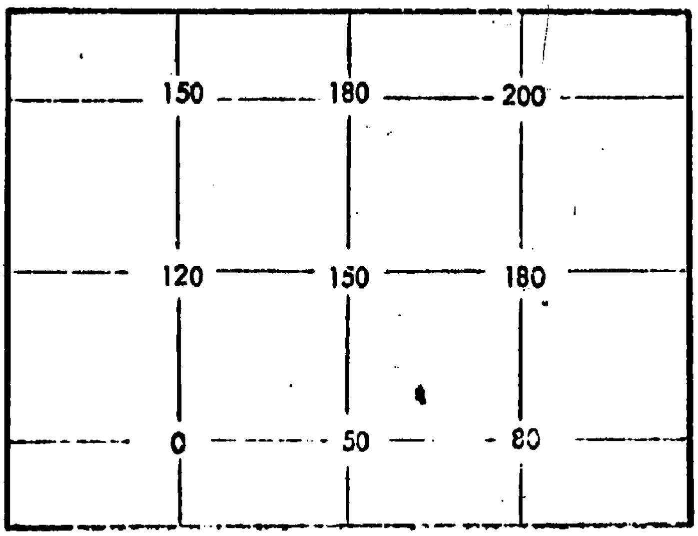
FIGURE 1


INPUT
DETERMINE AIR DEFENSE SYSTEM DEPLOYMENT
GENERATE "RISK" MATRIX
DETERMINE OPTIMAL FLIGHT PATH AND RISK INCURRED
OUTPUT

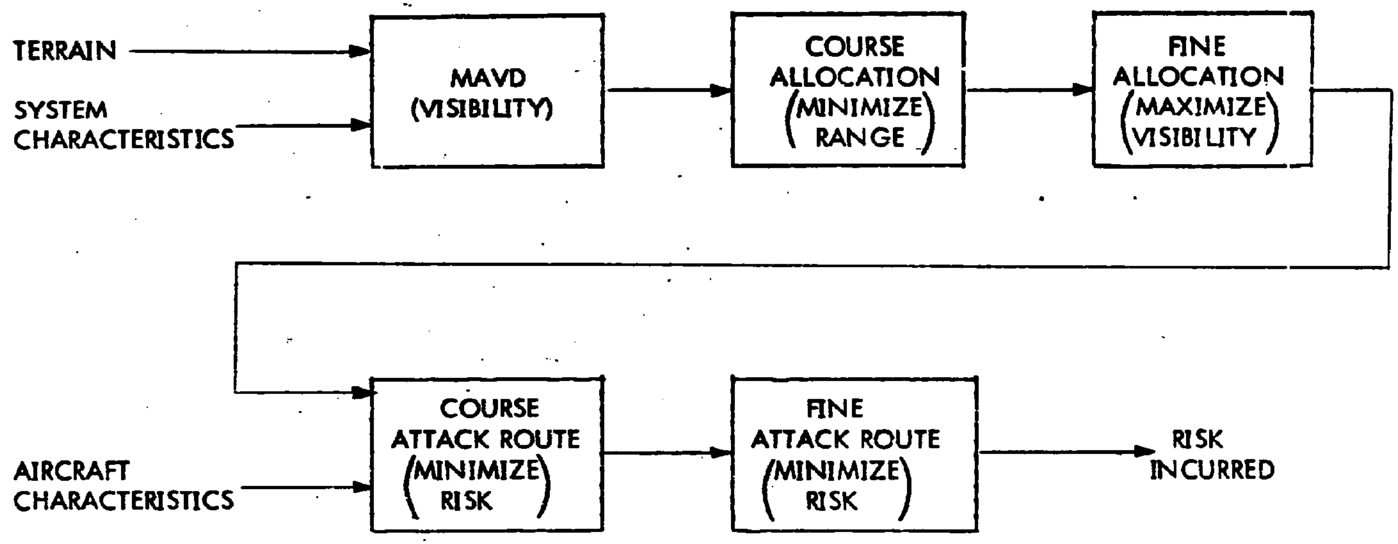


2a: TERRAIN MAP



2b: MAVD

FIGURE 2: MAVD



SCHEMATIC - GAME THEORY

COMPARATIVE MODEL

FIGURE 3

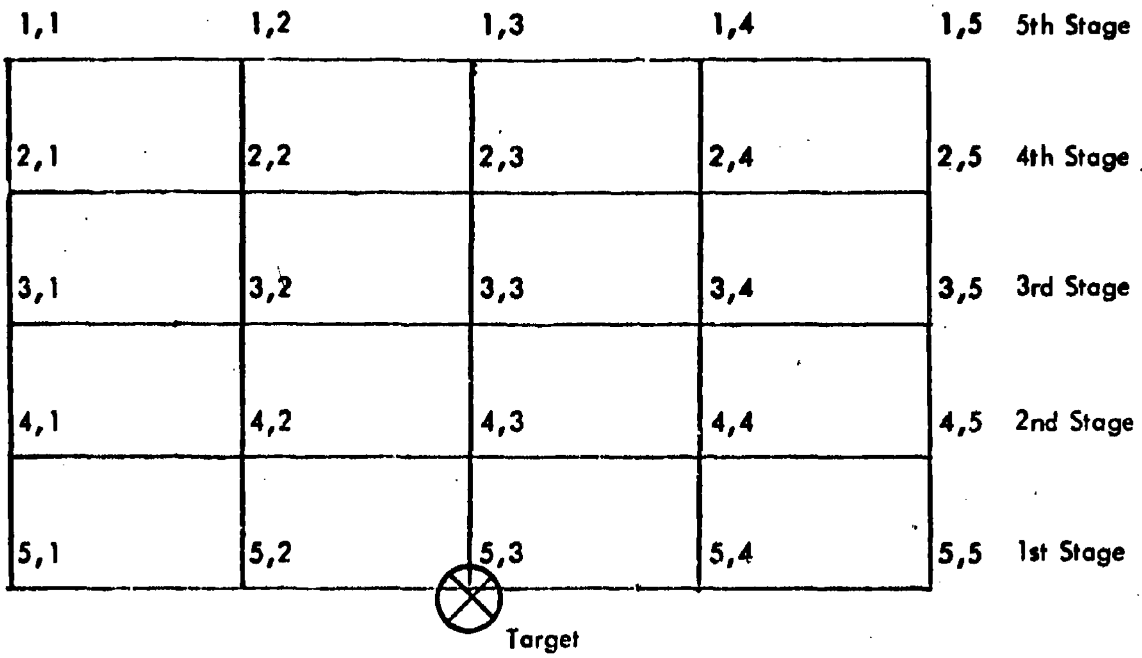

1,1
1,2
1,3
1,4
1,5
5th Stage
2,1
2,2
2,3
2,4
2,5
4th Stage
3,1
3,2
3,3
3,4
3,5
3rd Stage
4,1
4,2
4,3
4,4
4,5
2nd Stage
5,1
5,2
5,3
5,4
5,5
1st Stage
Target

FIGURE 4

PROBABILISTIC MANPOWER PLANNING FOR THE RESEARCH AND DEVELOPMENT ORGANIZATION

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INTRODUCTION

This paper proposes a statistical approach to one of the problems confronting all Army research, development and testing organizations. That is, manpower planning.

Though the problem addressed involves manpower, it should be noted that the mathematical techniques are applicable to all types of inventory by redefining the paramaters involved.

The question for long-range planning is not what should be done tomorrow, but rather what can be done today to cope best with the uncertain tomorrow. Management must understand the alternatives available to them, the risk associated with each, and choose rationally among the alternatives rather than plunge into uncertainty only on the basis of intuition or previous experience.

DEFINITION OF PROBLEM

If a given organization has a large number of programs planned for the future, it is usually reasonable to assume that the manpower requirement is somewhat normally distributed. However, most R & D organizations do not have a large number of outstanding programs and, therefore, the gain or loss of a single program can have gross affects on the required manpower. This problem requires that the "exact" probability distribution be known and solutions for this problem are not available in the literature.

The current need for management planning techniques with relatively few outstanding programs motivated the study described herein.

ASSUMPTIONS

This study makes four basic assumptions:

1. First it is assumed that the organization will not be required to perform every program for which current planning exists and that a subjective probability can be associated with the gain or loss of each program.

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2. Second, it is assumed that each program is statistically independent. That is, the initiation or cancellation of one program does not affect the probability of any other program.

3. Third, it is assumed that personnel of common disciplines are reasonably interchangeable. For example, any mechanical engineer or electronic technician could be utilized on any program where such skill is required.

4. Fourth, each program being considered has a proposed initiation date, and if this date is uncertain, it is assumed that a conditional probability for starting on various dates can be estimated.

DATA INPUTS

First, for each outstanding program, there must be a program plan which delineates each task to be performed, the time phasing, and the sequence in which tasks must be conducted. From this plan, the organization generates a time adjusted manpower array for performing all tasks under the development plan.

PROBLEM SOLUTION

In effort to promote appreciation of the problem at hand, an example problem will be utilized.

From the development plan, a time adjusted manpower array for performing all tasks has been generated and is presented in Figure 1. It is assumed certain that this project will be conducted, but the start date is uncertain. Utilizing the theory of Expected Values:

$$E(X_t) \quad (P_i)(P_{1/i})(X_{1/i}) + (P_i)(P_{2/i})(X_{2/i}) = \sum_{j=1}^{k} (P_i)(P_j)(X_j)$$

where $j = t/i$

and when "n" projects are considered:

$$E(X_t) = \sum_{i=1}^{n} \sum_{j=1}^{k} (P_i)(P_j)(X_j)$$

where: X_t = the units of manpower required during period t

P_i = capture probability

P_j = probability that project "i" starts in period "t"

X_j = number of manpower units required for project "i" during the period "t"

This process is illustrated graphically in Figures 2, 3 and 4.

Figure 1 - Time Adjusted Manpower Array and Probable Start Date

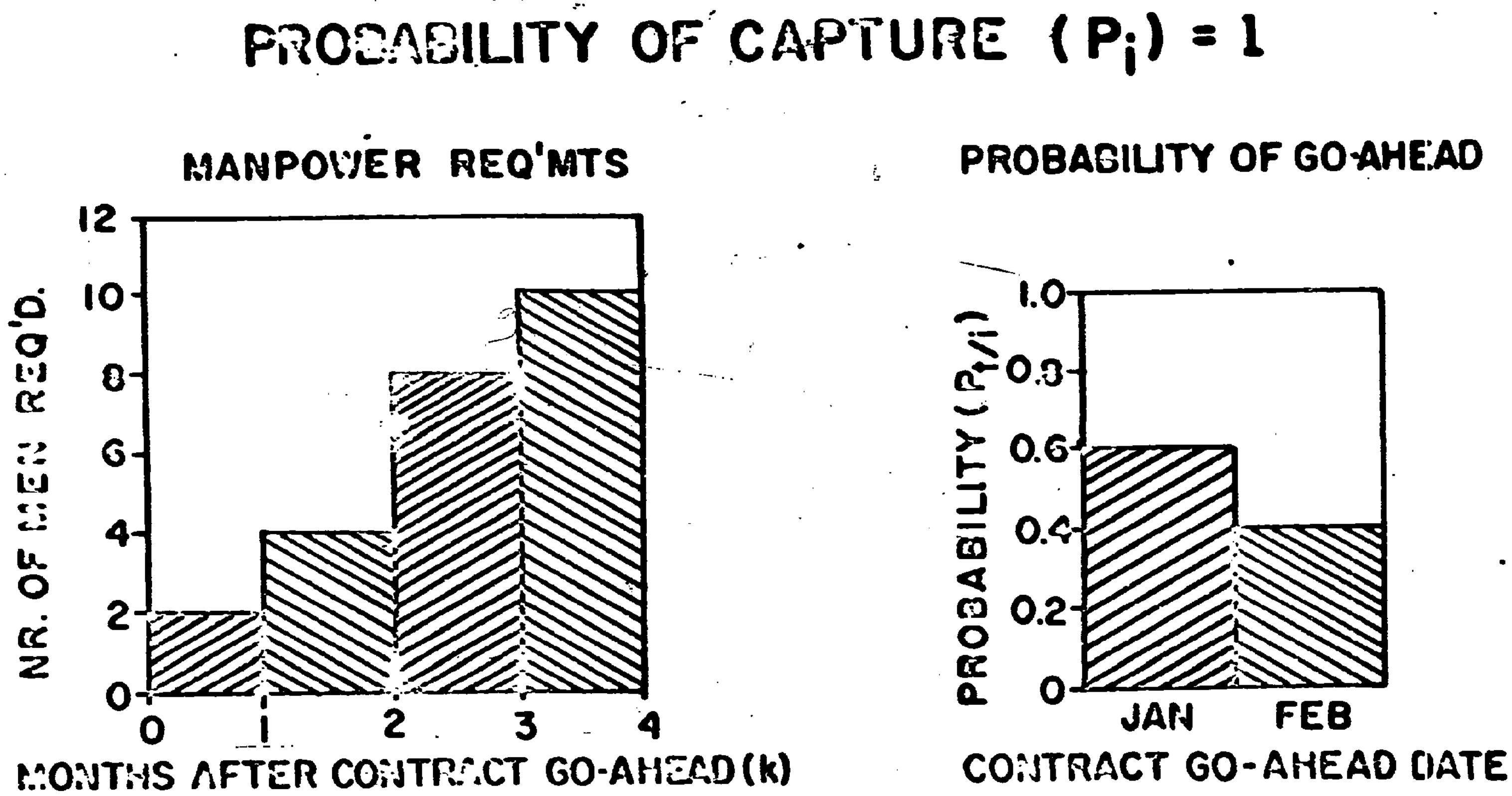


Figure 2 - Actual and Expected Manpower Requirements for January Start Date

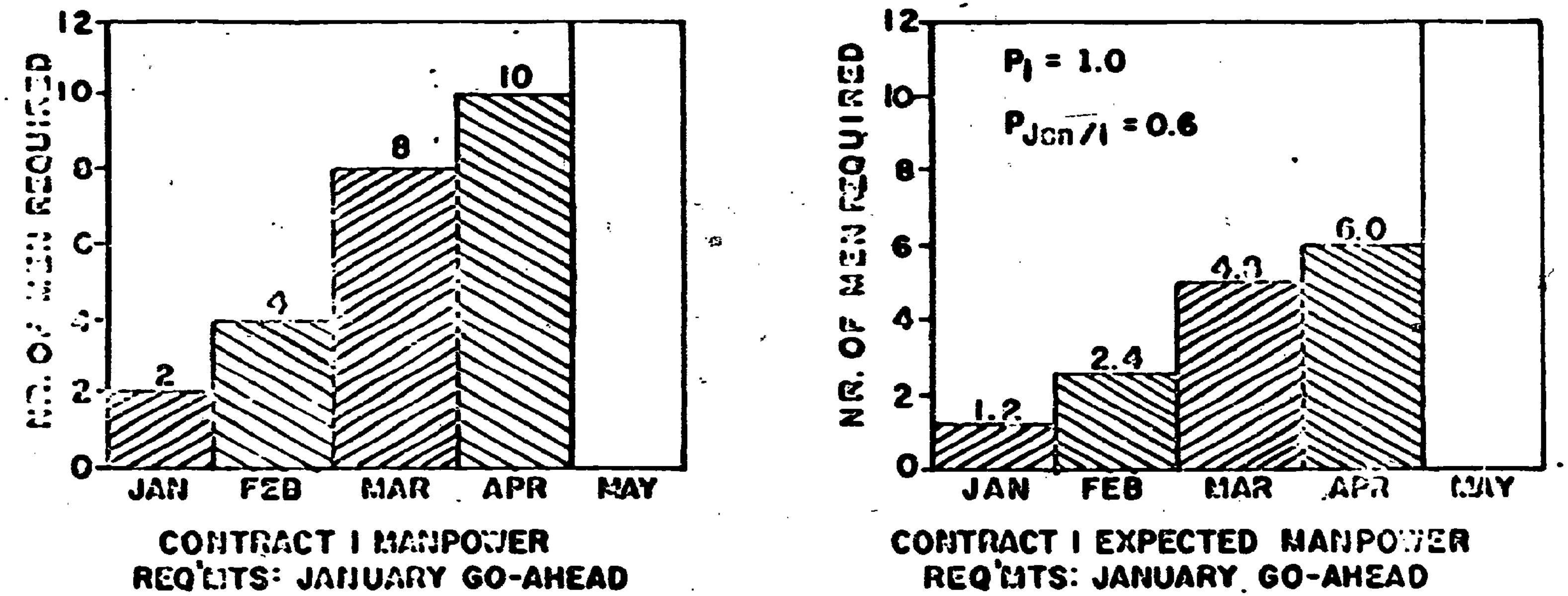
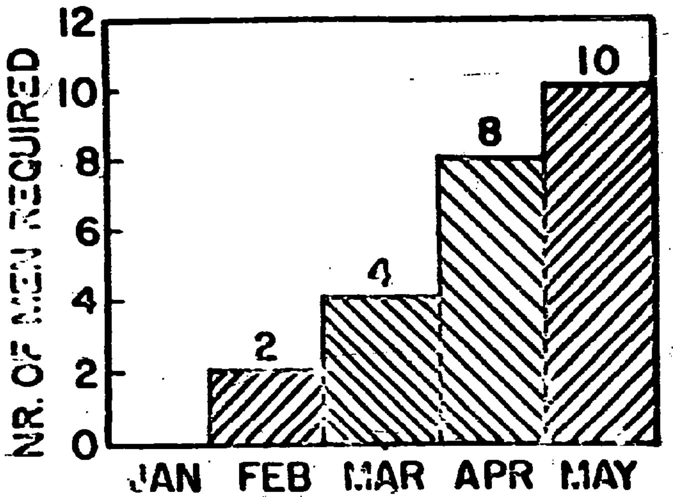
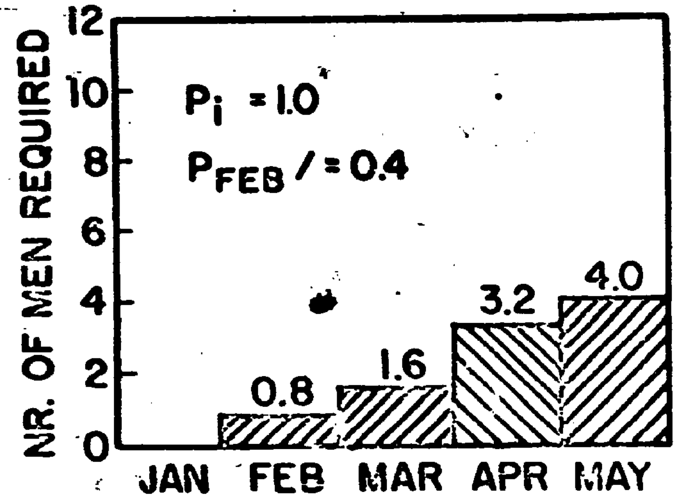


Figure 3 - Actual and Expected Manpower Requirements for February Start Date

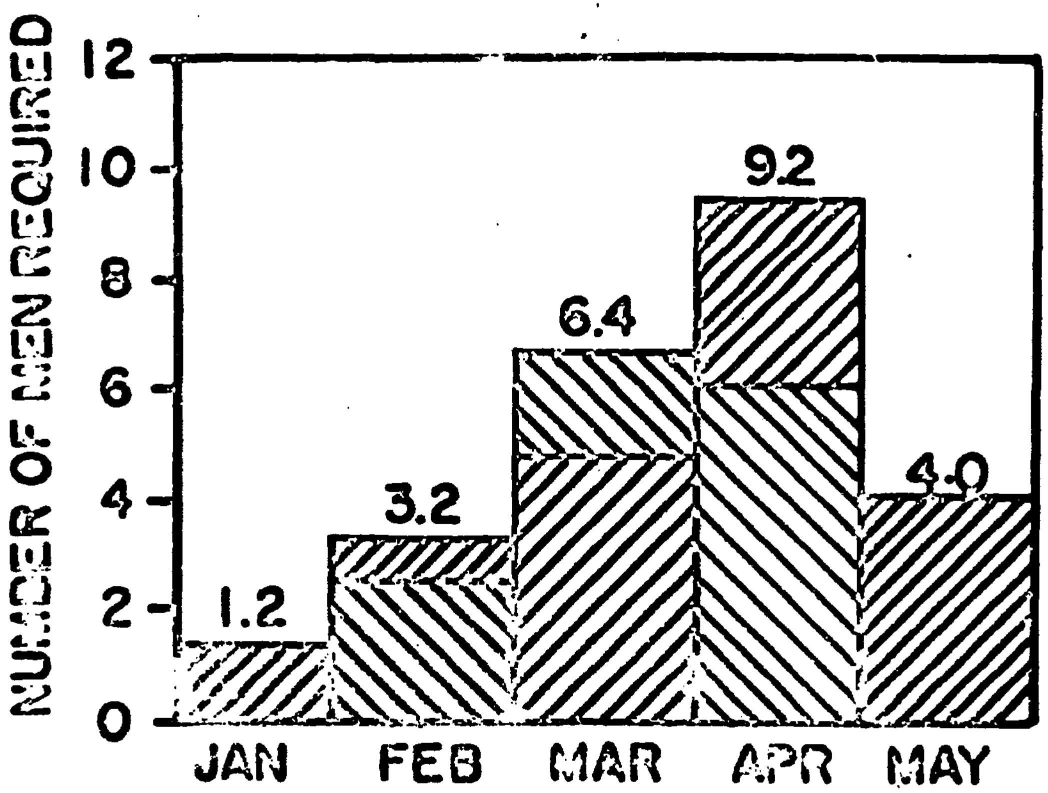


CONTRACT I MANPOWER REQ'MTS: FEBRUARY GO-AHEAD



CONTRACT I EXPECTED MANPOWER REQ'MTS: FEBRUARY GO-AHEAD

Figure 4 - Total Expected Manpower Requirement for Contract I



At this point the problem solution seems quite simple; however, because we used expected values, the "risk" associated with the solution cannot be evaluated unless there is a large number of outstanding projects (which usually is not the case for an R&D Organization). We must therefore develop a model which utilizes the exact statistical distribution. For this situation an enumeration process has been developed and is illustrated in Figure 5. Every possible workload is identified and the probability of occurrence for each is evaluated. Considering that the go-ahead date for the programs may not be "fixed" but rather can be expressed as a probabilistic function, we expand the enumeration procedure as illustrated in Figure 6 where the additional uncertainty is accounted for by t_1, t_2, etc.

A technique has thus been developed for enumerating the total array of possible workloads for an organization and the probability associated with each. This concept can readily be adapted to fit any particular problem that one may have.

MINIMUM COST METHOD

Given the total array of possible manpower requirements developed above, the corporation is now faced with the problem of determining the most economical method of performing any given workload. That is, if management were to assume that they knew specifically which one of the workloads will occur, how can they most economically perform the task realizing that if the workload exceeds capacity, they may choose to:

1. hire additional employees
2. work on an overtime basis or
3. subcontract a portion of the work.

If capacity exceeds the workload, management may choose to:

1. continue on an over-staffed basis or
2. lay-off some employees

Management must recognize that with each alternative, the total cost for conducting the workload will vary. For example, new employees must be trained, overtime costs premium wage, employee efficiency decreases with overtime and subcontract personnel may not be as affective as regular employees.

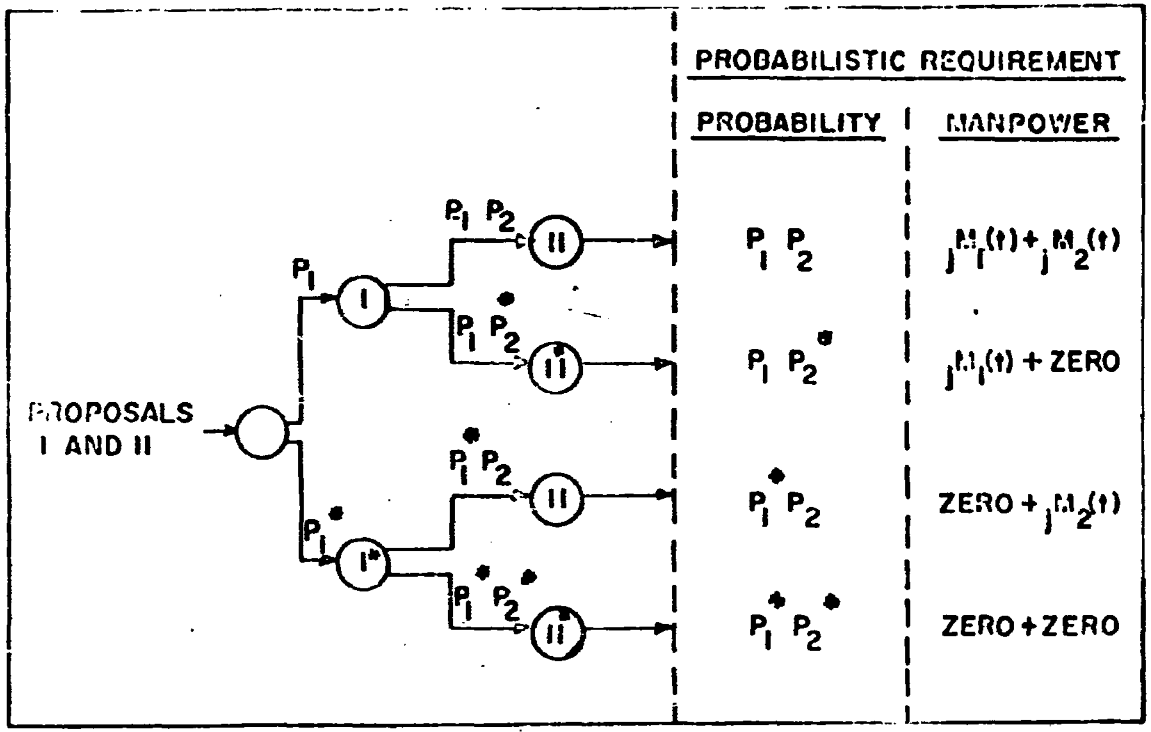
Total capacity for the organization during any period is therefore given by:

$$Z_t = \lambda_1 X_{1,t} + \lambda_2 X_{2,t} + \lambda_4 X_{4,t} + \lambda_5 X_{5,t} + \lambda_6 X_{6,t}$$

where:

λ_m is the efficiency factor for each type of manpower (overtime, subcontract, etc.)

Figure 5 - Enumeration Process When Initiation Date is Fixed
(Two Proposals)



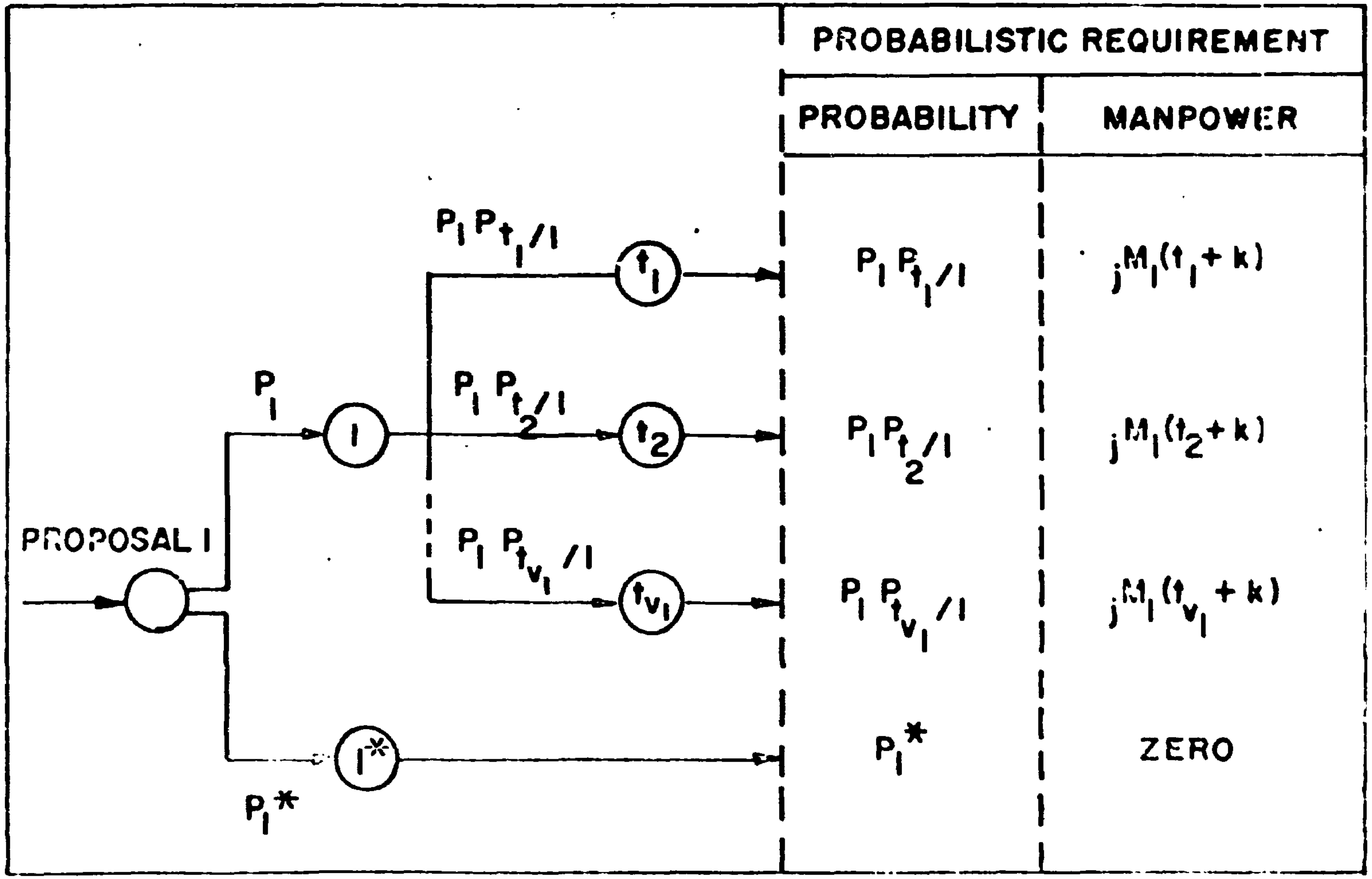
NOTE: P_1 = PROBABILITY OF GETTING PROGRAM

P_1^* = PROBABILITY OF NOT GETTING PROGRAM

$P_1 + P_1^* = 1$

${}_jM_1(t)$ = NR. OF TYPE j PERSONNEL FOR JOB 1 DURING PERIOD t

Figure 6 - Enumeration Process for Variable Initiation Date
(One Proposal)



Z_t is the effective capacity of manpower available during period t.

$X_{1,t}$ is the number of experienced employees available during period t.

$X_{2,t}$ is the number of new employees available during period t.

$X_{4,t}$ is the number of overtime units which can be worked by experienced employees during period t.

$X_{5,t}$ is the number of overtime units which new employees can work during period t.

$X_{6,t}$ is the number of subcontract personnel available during period t.

Note that $X_{3,t}$, which is the number of full time employees to be terminated during period t, is not included in the equation. Unit costs for each type of manpower must also be available.

The problem is one of determining the optimum manpower schedule for a given workload which permits the organization to operate for the duration of the planning period with minimum labor costs.

Requirements and data inputs for the minimization problem are ideal for solution by the Simplex Linear Programming Technique where the constraints, due to management policies, labor agreements, etc., limit the range of values for X_i. The Simplex not only provides a manpower plan for each workload but also the total cost for each plan.

Note that in some cases the Simplex may indicate to hire employees one month and terminate them the next. This is not fault with the mathematics but rather fault with management policies. The Minimum Cost Technique is therefore a good indicator for restraining management labor policies.

MINIMUM RISK METHOD

In the previous discussion it may be noted that the probability associated with each workload was ignored. How then can the element of risk be considered in the management plan?

Given the probabilistic manpower requirements, management needs a decision-making policy which allows them to plan for a theoretical workload and adjust with minimum consequence to the actual workload when it occurs. The Minimum Risk Method will provide such a plan.

Considering all of the possible workloads, the notion of risk is introduced where risk for a given workload is defined as the cost of making a transition, in later intervals, from some planned manpower level to the appropriate Minimum Cost Plan. In short, risk is the cost of adjusting to the workload that actually occurs. Once the workload becomes known, an adjustment or transition is made and the appropriate Minimum Cost schedule is followed.

The problem now is to identify a manpower planning level which minimizes total risk for the enumerated range of workloads.

Letting C_k^* represent the minimum cost for the k^{th} workload and C_k' represent the cost of adjusting from the planned level to the appropriate Minimum Cost Plan and completing the job, then the risk R_k is given by

$$R_k = C_k' - C_k^*$$

The expected risk (R_k') for each of the k workloads is given by

$$R_k' = P_k R_k = (P_k)(C_k' - C_k^*)$$

where P_k is the probability of occurrence determined by enumeration. The total risk $R(\cdot)$ for any manpower plan is therefore given by

$$R(\cdot) = \sum_k R_k' = \sum_k P_k(C_k' - C_k^*)$$

Our problem is now to identify a manpower plan which minimizes $R(\cdot)$. A dynamic programming technique for minimizing $R(\cdot)$ has not been developed; however, we can iterate a solution as illustrated in the following problem.

Example Problem

Suppose the scheduling period is 4 months and the initial number of experienced employees is 60. The organization has two outstanding project with estimated capture probabilities of 0.6 and 0.3 respectively and it will not be known until 1 January if the projects will be funded. The type A manpower requirements for each project are presented in Table 1. The projected manpower requirement without consideration of the two new contracts is also shown.

Table 1. Type of Manpower Requirements; Minimum Risk Problem

Identification of Manpower Requirements	Probability	January	February	March	April
Workload without new contracts	1.0	50	70	90	60
Contract I	0.6	10	10	20	30
Contract II	0.3	30	20	20	10

Due to company policies and labor agreements, constraints exist such that

$$0 \le X_{1,t} \le 100$$
$$0 \le X_{2,t} \le 20$$
$$0 \le X_{3,t} \le 20$$
$$0 \le X_{4,t} \le 30$$
$$0 \le X_{5,t} \le 30$$
$$0 \le X_{6,t} \le 20$$

Unit costs and efficiency factors for the various types of manpower are presented in Table 2. It is also assumed that new employees can be trained within one time period.

Table 2. Unit Costs and Efficiency Data for Minimum Risk Problem

Unit Costs ($)	Identification	Efficiency Factors
$C_1 = 100$	Experienced employees	$\lambda_1 = 1.00$
$C_2 = 130$	New employees	$\lambda_2 = 0.50$
$C_3 = 50$	Mandatory terminations	
$C_4 = 150$	Experienced employee overtime	$\lambda_4 = 0.70$
$C_5 = 150$	New employee overtime	$\lambda_5 = 0.35$
$C_6 = 170$	Subcontracts	$\lambda_6 = 0.80$

The problem is to determine theoretical values for $X_{1,1}, X_{2,1}$, etc., which minimize the total risk for this planning situation.

Referring to the complete enumeration technique, there are four possible workloads with probabilities of occurrence as calculated and shown in Table 3. The Minimum Cost plan for each of the four possible workloads, determined by the Simplex Technique, is presented in Table 4.

Since the primary interest for planning is full time employees, and the Minimum Cost plans of Table 4 indicate that portions of the workload should be subcontracted, the workload and Minimum Cost arrays are modified to reflect only the in-house efforts. The in-house efforts are determined by subtracting the subcontracts from Tables 3 and 4, and results of this operation are presented in Tables 5 and 6.

Table 3. Enumerated Workload Requirements for Minimum Risk Problem

Possible Workloads	Probability of Occurrence	Manpower Requirements			
		January	February	March	April
$W_1 = M_1^* M_2^*$	$p_1^* p_2^* = 0.28$	50	70	90	60
$W_2 = M_1 M_2^*$	$p_1 p_2^* = 0.42$	60	80	110	90
$W_3 = M_1^* M_2$	$p_1^* p_2 = 0.12$	80	90	110	70
$W_4 = M_1 M_2$	$p_1 p_2 = 0.18$	90	100	130	100

Table . Minimum Cost Manpower Plans for Minimum Risk Problem

Minimum Cost Plan for	Probability	January	February	March	April	Total Cost ($)
W_1	0.28	$X_{1,1} = 60$	$X_{1,2} = 60$ $X_{2,2} = 20$	$X_{1,3} = 80$ $X_{7,3} = 12.5$	$X_{1,4} = 60$ $X_{3,4} = 20$	31,725
W_2	0.42	$X_{1,1} = 60$ $X_{2,1} = 10$	$X_{1,2} = 70$ $X_{2,2} = 20$	$X_{1,3} = 90$ $X_{4,3} = 5.71$ $X_{6,3} = 20$	$X_{1,4} = 90$	39,157
W_3	0.12	$X_{1,1} = 60$ $X_{2,1} = 20$ $X_{6,1} = 12.5$	$X_{1,2} = 80$ $X_{2,2} = 20$ $X_{6,2} = 6.25$	$X_{1,3} = 90$ $X_{4,3} = 5.71$ $X_{6,3} = 20$	$X_{1,4} = 70$ $X_{3,4} = 20$	42,344
W_4	0.18	$X_{1,1} = 60$ $X_{2,1} = 20$ $X_{4,1} = 5.71$ $X_{6,1} = 20$	$X_{1,2} = 80$ $X_{2,2} = 20$ $X_{6,2} = 12.5$	$X_{1,3} = 100$ $X_{4,3} = 20$ $X_{6,3} = 20$	$X_{1,4} = 100$	51,982

Table 5. In-House Workload Requirements for Minimum Risk Problem

Possible In-House Workloads	Probability of Occurrence	January	February	March	April
W'_1	0.28	50	70	80	60
W'_2	0.42	60	80	94	90
W'_3	0.12	70	85	94	70
W'_4	0.18	74	90	114	100

Table 6. Minimum Cost Plans for In-House Workloads

Minimum Cost Plan for	Probability	January	February	March	April	Total Cost ($)
W_1'	0.28	$X_{1,1} = 60$	$X_{1,2} = 60$ $X_{2,2} = 20$	$X_{1,3} = 80$	$X_{1,4} = 60$ $X_{3,4} = 20$	29,600
W_2'	0.42	$X_{1,1} = 60$ $X_{2,1} = 10$	$X_{1,2} = 70$ $X_{2,2} = 20$	$X_{1,3} = 90$ $X_{4,3} = 5.71$	$X_{1,4} = 90$	35,757
W_3'	0.12	$X_{1,1} = 60$ $X_{2,1} = 20$	$X_{1,2} = 80$ $X_{2,2} = 10$	$X_{1,3} = 90$ $X_{4,3} = 5.71$	$X_{1,4} = 70$ $X_{3,4} = 20$	35,757
W_4'	0.18	$X_{1,1} = 60$ $X_{2,1} = 20$ $X_{4,1} = 5.71$	$X_{1,2} = 80$ $X_{2,2} = 20$	$X_{1,3} = 100$ $X_{4,3} = 20$	$X_{1,4} = 100$	43,057

The next procedure is to iterate costs and risks fo all feasible values of $X_{1,1}$ and $X_{2,1}$ so that the minimum R(·) can be i ntified. It may be noted in Table 6 that $X_{1,1}$ is 60 for all four work 's; therefore, there is only one feasible solution for $X_{1,1}$. However, $X_{2,1}$ varies from 0 to 20. The problem then is to determine a value ($X_{2,1}'$) which minimizes R(·) keeping in mind that the objective is to adjust to t Minimum Cost plan in the most expedient and economic manner consistent with the manpower constraints.

The results of the iteration process for each of the four possible workloads is presented in Tables 7 through 10 where $X_{2,1}$ was varied from 0 to 20 in increments of 5 units. Increments of five units each were arbitrarily selected for simplification of calculations in this illustrative problem. Note the heavy line in each of the tables. This line indicates when the level of full time employment reaches the Minimum Cost plan for that particular workload. Total cost for each of the trial solutions is also presented in Tables 7 through 10.

Summary of the expected risk R_k' calculations with the cost data from Tables 7 through 10 is presented in Table 11. The term R(·) may then be calculated by the equation $R(\cdot) = \sum_k R_k'$ for each of the $X_{2,1}'$ solutions. The R(·) data are tabulated in Table 12 where it is shown that R(·) is minimum when $X_{2,1}'$ is 10 units.

Table [illegible]. [illegible] Risk Plan for Workload Without New Contracts

Solution	Period W: $X'_{1,1}$	$X'_{2,1}$	January 50	February 70	March 80	April 60	Cost C'_1 ($)
$S_{1,1}$	60	0	$X_{1,1}$ = [illegible]	$X_{1,2}$ = [illegible] $X_{3,2}$ = 20	$X_{1,3}$ = 80	$X_{1,4}$ = 60 $X_{3,4}$ = 20	29,600
$S_{1,2}$	60	5	$X_{1,1}$ = 60 $X_{2,1}$ = 5	$X_{1,2}$ = 65 $X_{3,2}$ = 15	$X_{1,3}$ = 80	$X_{1,4}$ = 60 $X_{3,4}$ = 20	30,100
$S_{1,3}$	60	10	$X_{1,1}$ = 60 $X_{2,1}$ = 10	$X_{1,2}$ = 70 $X_{3,2}$ = 10	$X_{1,3}$ = 80	$X_{1,4}$ = 60 $X_{3,4}$ = 20	30,600
$S_{1,4}$	60	15	$X_{1,1}$ = [illegible] $X_{2,1}$ = 15	$X_{1,2}$ = 75 $X_{3,2}$ = 5	$X_{1,3}$ = 80	$X_{1,4}$ = 60 $X_{3,4}$ = 20	31,100
$S_{1,5}$	60	20	$X_{1,1}$ = 60 $X_{2,1}$ = 20	$X_{1,2}$ = 80	$X_{1,3}$ = 80	$X_{1,4}$ = 60 $X_{3,4}$ = 20	31,600

Table 8. Enumerated Risk Plan if Contract I is Received

Solution	Period W: $X'_{1,1}$	$X'_{2,1}$	January 60	February 80	March 94	April 90	Cost C'_2 ($)
$S_{2,1}$	60	0	$X_{1,1}$ = 60	$X_{1,2}$ = 60 $X_{2,2}$ = 20 $X_{4,2}$ = 14.3	$X_{1,3}$ = 80 $X_{2,3}$ = 10 $X_{4,3}$ = 12.9	$X_{1,4}$ = 90 —	36,980
$S_{2,2}$	60	5	$X_{1,1}$ = 60 $X_{2,1}$ = 5	$X_{1,2}$ = 65 $X_{2,2}$ = 20 $X_{4,2}$ = 7.1	$X_{1,3}$ = 85 $X_{2,3}$ = 5 $X_{4,3}$ = 9.3	$X_{1,4}$ = 90	36,360
$S_{2,3}$	60	10	$X_{1,1}$ = 60 $X_{2,1}$ = 10	$X_{1,2}$ = 70 $X_{2,2}$ = 20	$X_{1,3}$ = 90 $X_{4,3}$ = 5.7	$X_{1,4}$ = 90	35,757
$S_{2,4}$	60	15	$X_{1,1}$ = 60 $X_{2,1}$ = 15	$X_{1,2}$ = 75 $X_{2,2}$ = 15	$X_{1,3}$ = 90 $X_{4,3}$ = 5.7	$X_{1,4}$ = 90	36,255
$S_{2,5}$	60	20	$X_{1,1}$ = 60 $X_{2,1}$ = 20	$X_{1,2}$ = 80 $X_{2,2}$ = 10	$X_{1,3}$ = 90 $X_{4,3}$ = 5.7	$X_{1,4}$ = 90	36,755

Table 9. Enumerated Risk Plan if Contract II is Received

	Period		January	February	March	April	Cost
	W'_3		70	85	94	70	C'_3
Solution	$X'_{1,1}$	$X'_{2,1}$					($)
$S_{3,1}$	60	0	$X_{1,1} = 60$ $X_{4,1} = 14.3$	$X_{1,2} = 60$ $X_{2,2} = 20$ $X_{4,2} = 21.4$	$X_{1,3} = 80$ $X_{4,3} = 20$	$X_{1,4} = 70$ $X_{3,4} = 10$	38,455
$S_{3,2}$	60	5	$X_{1,1} = 60$ $X_{2,1} = 5$ $X_{4,1} = 10.7$	$X_{2,1} = 65$ $X_{2,2} = 20$ $X_{4,2} = 14.3$	$X_{1,3} = 85$ $X_{4,3} = 12.9$	$X_{1,4} = 70$ $X_{3,4} = 15$	37,685
$S_{3,3}$	60	10	$X_{1,1} = 60$ $X_{2,1} = 10$ $X_{4,1} = 7.1$	$X_{1,2} = 70$ $X_{2,2} = 20$ $X_{4,2} = 7.1$	$X_{1,3} = 90$ $X_{4,3} = 5.7$	$X_{1,4} = 70$ $X_{3,4} = 20$	36,917
$S_{3,4}$	60	15	$X_{1,1} = 60$ $X_{2,1} = 15$ $X_{4,1} = 3.6$	$X_{1,2} = 75$ $X_{2,2} = 15$ $X_{4,2} = 3.6$	$X_{1,3} = 90$ $X_{4,3} = 5.7$	$X_{1,4} = 70$ $X_{3,4} = 20$	36,335
$S_{3,5}$	60	20	$X_{1,1} = 60$ $X_{2,1} = 20$	$X_{1,2} = 80$ $X_{2,2} = 10$	$X_{1,3} = 90$ $X_{4,3} = 5.7$	$X_{1,4} = 70$ $X_{3,4} = 20$	35,757

Table 10. Enumerated Risk Plan if Contracts I and II are Received

Solution	Period		January	February	March	April	Cost C'_4 (\$)
	W'_i		74	90	114	100	
	$X'_{1,1}$	$X'_{2,1}$					
$S_{4,1}$	60	0	$X_{1,1} = 60$ $X_{4,1} = 20$	$X_{1,2} = 60$ $X_{2,2} = 20$ $X_{4,2} = 28.6$	$X_{1,3} = 80$ $X_{2,3} = 20$ $X_{4,3} = 30$ $X_{5,3} = 8.6$	$X_{1,4} = 100$	48,280
$S_{4,2}$	60	5	$X_{1,1} = 60$ $X_{2,1} = 5$ $X_{4,1} = 16.4$	$X_{1,2} = 65$ $X_{2,2} = 20$ $X_{4,2} = 21.4$	$X_{1,3} = 85$ $X_{2,3} = 15$ $X_{1,3} = 30$ $X_{5,3} = 1.4$	$X_{1,4} = 100$	46,580
$S_{4,3}$	60	10	$X_{1,1} = 60$ $X_{2,1} = 10$ $X_{4,1} = 12.9$	$X_{1,2} = 70$ $X_{2,2} = 20$ $X_{4,2} = 14.3$	$X_{1,3} = 90$ $X_{2,3} = 10$ $X_{4,3} = 27.1$	$X_{1,4} = 100$	45,345
$S_{4,4}$	60	15	$X_{1,1} = 60$ $X_{2,1} = 15$ $X_{4,1} = 9.3$	$X_{1,2} = 75$ $X_{2,2} = 20$ $X_{4,2} = 7.1$	$X_{1,3} = 95$ $X_{2,3} = 5$ $X_{4,3} = 23.6$	$X_{1,4} = 100$	44,200
$S_{4,5}$	60	20	$X_{1,1} = 60$ $X_{2,1} = 20$ $X_{4,1} = 5.7$	$X_{1,2} = 80$ $X_{2,2} = 20$	$X_{1,3} = 100$ $X_{4,3} = 20$	$X_{1,4} = 100$	43,057

Table 11. Risk Summary for Example Problem

Solution	p_k	C'_k	C^*_k	R_k	R_k
$S_{1,4}$	0.28	31,000	29,600	1500	420
$S_{1,5}$	0.28	31,600	29,600	2000	560
$S_{2,1}$	0.42	36,980	35.757	1223	514
$S_{2,2}$	0.42	36,360	35,757	603	253
$S_{2,3}$	0.42	35,757	35,757	0	0
$S_{2,4}$	0.42	36,255	35,757	498	209
$S_{2,5}$	0.42	36,755	35,757	998	419
$S_{3,1}$	0.12	38,455	35,757	2698	324
$S_{3,2}$	0.12	37,685	35,757	1928	231
$S_{3,3}$	0.12	36,917	35,757	1160	139
$S_{3,4}$	0.12	36,335	35,757	578	69
$S_{3,5}$	0.12	35,757	35,757	0	0
$S_{4,1}$	0.18	48,280	43,057	5223	940
$S_{4,2}$	0.18	46,580	43,057	3523	634
$S_{4,3}$	0.18	45,345	43,057	2288	412
$S_{4,4}$	0.18	44,200	43,057	1143	206
$S_{4,5}$	0.18	43,057	43,057	0	0

Table 12. Risk Analysis for Example Problem

$X'_{1,1}$	$X'_{2,1}$	ΣS	ΣR_k	$R(\cdot)$
60	0	$S_{1,1}+S_{2,1}+S_{3,1}+S_{4,1}$	0 + 514 + 324 + 940	1778
60	5	$S_{1,2}+S_{2,2}+S_{3,2}+S_{4,2}$	140 + 253 + 231 + 634	1258
60	10	$S_{1,3}+S_{2,3}+S_{3,3}+S_{4,3}$	280 + 0 + 139 + 412	831
60	15	$S_{1,4}+S_{2,4}+S_{3,4}+S_{4,4}$	420 + 209 + 69 + 206	904
60	20	$S_{1,5}+S_{2,5}+S_{3,5}+S_{4,5}$	560 + 419 + 0 + 0	979

What does this solution mean? The Minimum Risk plan for this illustrative problem is to retain the 60 experienced employees who will be available for January and, in addition, hire 10 new employees before January so that their services will be available during the first month. If workload J_1 occurs, the corporate plan will be as shown in Table 13. If workloads J_2, J_3, or J_4 occur, the corporate plan will be as shown in Tables 14 through 16 respectively.

Table 13. Minimum Risk Plan Without New Contracts

$X'_{1,1}$	$X'_{2,1}$	January	February	March	April
60	10	$X_{1,1} = 60$	$X_{1,2} = 70$	$X_{1,3} = 80$	$X_{1,4} = 60$
		$X_{2,1} = 10$	$X_{2,2} = 10$	$X_{6,3} = 12.5$	$X_{3,4} = 20$

Table 14. Minimum Risk Plan if Contract I is Received

$X'_{1,1}$	$X'_{2,1}$	January	February	March	April
60	10	$X_{1,1} = 60$	$X_{1,2} = 70$	$X_{1,3} = 90$	$X_{1,4} = 90$
		$X_{2,1} = 10$	$X_{2,2} = 20$	$X_{4,3} = 5.7$	
				$X_{6,3} = 20$	

Table 15. Minimum Risk Plan if Contract II is Received

$X'_{1,1}$	$X'_{2,1}$	January	February	March	April
60	10	$X_{1,1} = 60$	$X_{1,2} = 70$	$X_{1,3} = 90$	$X_{1,4} = 70$
		$X_{2,1} = 10$	$X_{2,2} = 20$	$X_{4,3} = 5.7$	$X_{3,4} = 20$
		$X_{4,1} = 7.1$	$X_{4,2} = 7.1$	$X_{6,3} = 20$	
		$X_{6,1} = 12.5$	$X_{6,2} = 6.25$		

Table 16. Minimum Risk Plan if Contracts I and II are Received

$X'_{1,1}$	$X'_{2,1}$	January	February	March	April
60	10	$X_{1,1} = 60$	$X_{1,2} = 70$	$X_{1,3} = 90$	$X_{1,4} = 100$
		$X_{2,1} = 10$	$X_{2,2} = 20$	$X_{2,3} = 10$	
		$X_{4,1} = 12.9$	$X_{4,2} = 14.3$	$X_{4,3} = 27.1$	
		$X_{6,1} = 20$	$X_{6,2} = 12.5$	$X_{6,3} = 20$	

The organization is hereby presented with a strategy for planning the future manpower requirements in the face of uncertainty. A mathematical simulation has been developed which can assist management in understanding the problem and the effects of various plans available to them. It is felt that this approach can be computerized and provide management with a rapid assessment of the situation at any given time.

ANALYSIS OF FACTORIAL EXPERIMENTS IN NON-CONNECTED BLOCK DESIGNS*

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1. Introduction

The object of the analysis is to estimate the effect of v treatments on the response of a device in the presence of b extraneous side effects (blocks), whose effects are removed from the treatment effects. The treatments may be simple treatments involving only one factor, or they may be treatment-combinations involving several factors applied simultaneously to the device. In the latter case, the effect of each factor and the effects of interaction between factors are also estimated. The results presented here require no restrictions on the experimental design. That is, the design may be unbalanced, treatments may be missing, there may be an unequal number of observations per cell, etc.

2. Model

The model is the fixed effects model:

$$y_{ijk} = \mu + t_i + b_j + \varepsilon_{ijk} \tag{1}$$

where μ - overall constant

t_i - i^{th} treatment effect, $i = 1, \ldots, v$

b_j - j^{th} block effect, $j = 1, \ldots, b$

y_{ijk} - k^{th} observation of treatment i in block j

$k = 1, \ldots, \ell_{ij}$

ε_{ijk} - experimental error in y_{ijk} assumed to be $N(0,\sigma^2)$

3. Incidence Matrix

The design of the experiment is characterized by the incidence matrix, N, which is a vxb matrix whose elements n_{ij} are one if treatment i is applied in block j, and are zero otherwise. For example, if v = 3 and b = 3, the incidence matrix might be:

Treat-ments	Blocks 1	2	3	
1	1	1	0	t_1 is applied in blocks 1 and 2
2	0	1	1	t_2 is applied in blocks 2 and 3
3	1	0	1	t_3 is applied in blocks 1 and 3

Let ℓ_{ij} be the number of observations on treatment i in block j. Then

$$L = ((\ell_{ij})) = ((n_{ij}\ \ell_{ij}))$$

*This is a condensed version of a paper which is to appear in a national journal. This article has been reproduced photographically from the author's manuscript.

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4. Definition of Connected Designs and Sets of Connected Blocks

A connected design is one in which all the blocks are connected by a chain of treatments. In the example, pg. 1, blocks 1 and 2 are connected by treatment 1 and blocks 2 and 3 are connected by treatment 2 - hence all blocks are connected and the design is said to be a connected design. The number of sets of connected blocks is one, and the set consists of $\{b_1, b_2, b_3\}$.

In the example below, blocks 1 and 2 are connected by treatment 1, but there is no treatment which connects either block 1 or block 2 to block 3 - hence the design is said to be non-connected. There are two sets of connected blocks $\{b_1, b_2\}$ and $\{b_3\}$.

$$N = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Review of Solution for Connected Designs with No Missing Treatments

The least-squares solution obtained by minimizing

$$S = \sum_{i=1}^{v} \sum_{j=1}^{b} \sum_{k=1}^{\ell_{ij}} n_{ij} (y_{ijk} - \mu - t_i - b_j)^2$$

with respect to μ, t_i and b_j, and eliminating μ and b_j, gives the reduced normal equations: $C\,\hat{t} = Q$, where

$$\hat{t}' = (\hat{t}_1, \hat{t}_2, \ldots \hat{t}_v), \text{ the vector of treatment effects}$$

$$C = R - L K^{-1} L' \qquad (2)$$

$$Q = T - L K^{-1} B$$

and: R is a diagonal matrix with diagonal elements equal to the number of observations on t_i, i.e. $r_i = \sum_{j=1}^{b} \ell_{ij}$, $i = 1, \ldots, v$

All $r_i > 0$ in the no missing treatments case

K is a diagonal matrix with diagonal elements equal to the number of observations in block j, i.e. $k_j = \sum_{i=1}^{v} \ell_{ij} > 0$, $j = 1, \ldots, b$

L is the matrix: $L = ((\ell_{ij} n_{ij}))$

$T' = (T_1, T_2, \ldots, T_v)$ is a vector of treatment totals, i.e.

$$T_i = \sum_{j=1}^{b} \sum_{k=1}^{\ell_{ij}} n_{ij} y_{ijk}, \quad i = 1, \ldots, v$$

$B' = (B_1, B_2, \ldots, B_b)$ is a vector of block totals, i.e.

$$B_j = \sum_{i=1}^{v} \sum_{k=1}^{\ell_{ij}} n_{ij}\, y_{ijk}, \quad j = 1, \ldots, b$$

The solution $\hat{t} = C^+ Q$, where C^+ is the generalized inverse of C, gives the minimum-variance, unbiased estimates of the t_i, $i = 1, \ldots, v$, subject to the constraint $\sum_{i=1}^{v} t_i = 0$. The rank of the matrix C gives the number of linearly independent treatment effects which can be estimated, hence the degrees of freedom associated with treatments. Thus for the connected design, no missing treatments case, we have the following:

Treatment effects $\hat{t} = C^+ Q$

Variance-covariance matrix $V(\hat{t}) = C^+ \sigma^2$

Sums of squares due to treatments $SS(t) = \hat{t}' Q = \hat{t}' C \hat{t}$

Degrees of freedom d.f. (t) = Rank $C = v - 1$

Let: w be the total number of observations in the experiment,

Y, the vector of observations, $((n_{ij} y_{ijk}))_{w \times 1}$

$G = \sum_{i=1}^{v} \sum_{j=1}^{b} \sum_{k=1}^{\ell_{ij}} n_{ij} y_{ijk}$, the grand total of the observations.

Then the analysis of variance table is given by:

Source of Variation	Sums of Squares	Degrees of Freedom
Treatments (adjusted for blocks)	$\hat{t}' Q$	Rank $C = v - 1$
Blocks (unadjusted)	$B' K^{-1} B - G^2/w$	$b - 1$
Error	$Y' Y - \hat{t}' Q - B' K^{-1} B$	w - Rank $C - b$
Total	$Y' Y - G^2/w$	$w - 1$

(3)

For the factorial case, the notation developed by Kurkjian and Zelen in "A Calculus for Factorial Arrangements", Annals of Math Stat, Vol. 33, No. 2, June 1962, will be used. Two operators, $\otimes$, symbolic direct product (SDP), and x, direct product (DP) are needed.

The SDP is used to order the combinations of levels of the various factors, illustrated by the following example: Assume three factors in the experiments, two at two levels and one at three levels. Let $\Theta_s' = (1, 2, \ldots, m_s)$ be a vector designating the levels of the s^{th} factor, which has m_s levels. Then:

$$\theta_1 \otimes \theta_2 \otimes \theta_3 = \begin{bmatrix}1\\2\end{bmatrix} \otimes \begin{bmatrix}1\\2\end{bmatrix} \otimes \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}111\\112\\113\\121\\122\\123\\211\\212\\213\\221\\222\\223\end{bmatrix}.$$

The final vector gives a particular order of the combinations of the levels of the three factors, and is obtained by setting the first two factors at level 1 and running through all levels of the third factor; setting the second factor at level 2, running through all levels of the third factor again; and finally setting the first factor at its second level and repeating the sequence on the second and third factors again. The procedure can easily be generalized to any number of factors at various levels.

The DP is the matrix multiplication defined as follows:

$$A_{m \times n} \times B_{p \times q} = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & & a_{mn}B \end{bmatrix}_{mp \times nq}$$

That is, each element of A is multiplied times the entire matrix B, by ordinary multiplication of a scalar times a matrix.

Now let $A_1, A_2, \ldots, A_n$ be n factors in the experiment

at $m_1, m_2, \ldots, m_n$ levels, respectively.

The number of treatments (or treatment-combinations) resulting from applying all the factors simultaneously at all combinations of their levels is $v = \prod_{i=1} m_i$.

Let $(i_1, i_2, \ldots, i_n)$ be the i^{th} treatment-combination where factor A_1 is at level i_1, factor A_2 is at level i_2, etc., and order the treatments by the SDP of the levels of the factors.

For example, if $n = 2$, $m_1 = 2$, $m_2 = 3$, then:

$$\theta_1 \otimes \theta_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 13 \\ 21 \\ 22 \\ 23 \end{bmatrix}; \quad \text{Thus} \quad t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix} = \begin{bmatrix} t_{11} \\ t_{12} \\ t_{13} \\ t_{21} \\ t_{22} \\ t_{23} \end{bmatrix}$$

Thus treatment 1 is the combination with both factors at level 1, treatment 2 is the combination with the first factor at level 1 and the second factor at level 2, etc.

Let $a_s(i_s)$ be the main effect of factor A_s at the i_s level,

$a_{rs}(i_s, i_r)$ be the second-order interaction effect between factors A_r and A_s at levels i_r and i_s, respectively,

$\vdots$

$a_{12 \ldots n}(i_1, i_2, \ldots, i_n)$ be the n^{th}-order interaction effect between the n factors at levels $i_1, i_2, \ldots, i_n$, respectively.

Then the i^{th} treatment expressed in terms of the main and interaction effects of the factors is:

$$t_i = t(i_1, i_2, \ldots, i_n) =$$

$$\sum_{s=1}^{n} a_s(i_s) + \underset{1 \le s < r < n}{\sum_s \sum_r} a_{rs}(i_r, i_s) + \ldots + a_{1,2 \ldots n}(i_1, i_2, \ldots i_n) \qquad (4)$$

For the previous example, we get the relationships:

$$\begin{aligned} t_1 &= t_{11} = a_1(1) + a_2(1) + a_{12}(11) \\ t_2 &= t_{12} = a_1(1) + a_2(2) + a_{12}(12) \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad (5) \\ t_6 &= t_{23} = a_1(2) + a_2(3) + a_{12}(23) \end{aligned}$$

Let a_X represent a general interaction term vector of effects where $X = (x_1, x_2, \ldots, x_n)$ such that

$x_i = 1$ if factor A_i is present in the interaction term and

$x_i = 0$ if not.

The number of elements in the vector is $\prod_{i=1}^{n} m_i^{x_i}$, again the elements are assumed to be in the order defined by the SDP of the levels of the factors involved, and the order of the interaction is given by $p = \sum_{i=1}^{n} x_i$.

The number of interaction terms for an n-factor experiment is $2^n - 1$, which is the number of combinations of zeroes and ones in the vector X, excluding all zeroes.

Continuing with the example, there are $2^n - 1 = 3$ interaction terms as follows:

$X = (10)$ denotes the main effect (first order) term for A_1 and

$$a_X = a_1 = \begin{bmatrix} a_1(1) \\ a_1(2) \end{bmatrix}$$

$X = (01)$ denotes the main effect term for A_2 and

$$a_X = a_2 = \begin{bmatrix} a_2(1) \\ a_2(2) \\ a_2(3) \end{bmatrix}$$

$X = (11)$ denotes the second-order interaction term A_1A_2 and

$$a_X = a_{12} = \begin{bmatrix} a_{12}(11) \\ a_{12}(12) \\ a_{12}(13) \\ a_{12}(21) \\ a_{12}(22) \\ a_{12}(23) \end{bmatrix}$$

Adding the constraints that the sum of the effects in an interaction term over all levels of any one factor is zero, i.e.

$$\sum_{i_s=1}^{m_s} a_s(i_s) = 0 \qquad s = 1, \ldots, n \tag{6}$$

$$\sum_{i_r=1}^{m_r} a_{rs}(i_r, i_s) = \sum_{i_s=1}^{m_s} a_{rs}(i_r, i_s) = 0 \qquad \begin{matrix} r = 1, \ldots, n \\ s = 1, \ldots, n \end{matrix} \; r \neq s$$

etc.,

the relationships in (4) can be solved uniquely for the interaction effects in terms of the treatment effects, giving:

$$\hat{a}_X = \frac{1}{v} M_X \hat{t} \qquad \text{where} \tag{7}$$

$$M_X = M_1^{x_1} \times M_2^{x_2} \times \ldots M_n^{x_n} \quad \text{(Using DP multiplication)}$$

$$\text{and } M_i^{x_i} = M_i = m_i I_i - J_i, \text{ if } x_i = 1$$

$$= 1_i' = (1, 1, \ldots, 1)_{1 \times m_i}, \text{ if } x_i = 0$$

where I_i is the identity matrix of order m_i, and

J_i is a matrix of all ones of order m_i.

Thus for the example, the constraints are:

$$\sum_{i=1}^{2} a_1(i) = \sum_{i=1}^{3} a_2(i) = \sum_{i=1}^{2} a_{12}(i1) = \sum_{i=1}^{2} a_{12}(i2) = \sum_{i=1}^{2} a_{12}(i3)$$

$$= \sum_{i=1}^{3} a_{12}(1i) = \sum_{i=1}^{3} a_{12}(2i) = 0$$

And the interaction effects expressed in terms of the treatment effects are given by:

$$X = (10):\ \hat{a}_X = \begin{bmatrix} \hat{a}_1(1) \\ \hat{a}_1(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix} \hat{t}$$

$$X = (01):\ \hat{a}_X = \begin{bmatrix} \hat{a}_2(1) \\ \hat{a}_2(2) \\ \hat{a}_2(3) \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & -1 & -1 & 2 \end{bmatrix} \hat{t}$$

$$X = (11):\ \hat{a}_X = \begin{bmatrix} \hat{a}_{12}(11) \\ \hat{a}_{12}(12) \\ \hat{a}_{12}(13) \\ \hat{a}_{12}(21) \\ \hat{a}_{12}(22) \\ \hat{a}_{12}(23) \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & -2 & 1 & 1 \\ -1 & 2 & -1 & 1 & -2 & 1 \\ -1 & -1 & 2 & 1 & 1 & -2 \\ -2 & 1 & 1 & 2 & -1 & -1 \\ 1 & -2 & 1 & -1 & 2 & -1 \\ 1 & 1 & -2 & -1 & -1 & 2 \end{bmatrix} \hat{t}$$

Having the interaction effects expressed in terms of the treatment effects, then the following equations hold for the connected block, no missing treatments case:

$$\hat{a}_X = \frac{1}{v} M_X \hat{t} = \frac{1}{v} M_X C^+ Q$$

$$Var(\hat{a}_X) = \frac{\sigma^2}{v^2} M_X C^+ M_X' = \sigma^2 \Sigma_X$$

$$Cov(\hat{a}_{X_i}, \hat{a}_{X_j}) = \frac{\sigma^2}{v^2} M_{X_i} C^+ M_{X_j}', \quad i \neq j \qquad (8)$$

Sums of Squares due to $a_X = SS(a_X) = \hat{a}_X' \Sigma_X^+ \hat{a}_X$

$SS(a_X)/\sigma^2$ is chi-square distributed with

$$f_X = \text{Rank}(\Sigma_X) = \prod_{i=1}^{n} (m_i - 1)^{x_i} \text{ degrees of freedom.}$$

If the design is connected, and orthogonal (i.e. $cov(\hat{a}_{X_i}, \hat{a}_{X_j}) = 0$ for all i,j, $i \neq j$)

then:

$$\sum_{i=1}^{2^n - 1} SS(a_{X_i}) = \hat{t}' Q = SS(\text{treatments}).$$

If the design is not orthogonal, the $SS(a_X)$ are not additive, but each $SS(a_X)/\sigma^2$ is statistically independent of the error term, so that F-tests are valid.

The analysis of variance table is given by:

Source of Variation	Sums of Squares	Degrees of Freedom
X_1	$SS(a_{X_1})$	Rank Σ_{X_1}
X_2 (Adjusted for Blocks)	$SS(a_{X_2})$	Rank Σ_{X_2}
$\vdots$	$\vdots$	$\vdots$
X_{2^n-1}	$SS(a_{X_{2^n-1}})$	Rank $\Sigma_{X_{2^n-1}}$
Blocks (Unadjusted)	$B' K^{-1} B - G^2/w$	$b - 1$
Error	$Y'Y - \hat{t}' Q - B' K^{-1} B$	w - Rank $C - b$
Total	$Y'Y - G^2/w$	$w - 1$

(9)

From a computational viewpoint, the calculations involved in (8) can be greatly reduced by eliminating the elements of each a_X which are linearly dependent because of the constraints in (6). The total number of elements in all the a_X vectors is $\prod_{i=1}^{n} (m_i + 1) - 1$, while only (v-1) are linearly independent in the connected design case. If all elements involving any factor at its highest level (each of which can always be expressed in terms of other elements in that term using the relationships in (6)) are eliminated, there will result (v-1) independent elements.

Then let $\bar{\hat{a}}_X$ be a vector containing the linearly independent elements of $\hat{a}_X$ (selected as above) and let $\overline{M}_X$ be the corresponding rows of M_X. Then the equations corresponding to those in (8) become:

$$
\begin{aligned}
\bar{\hat{a}}_X &= \frac{1}{v} \overline{M}_X \hat{t} = \frac{1}{v} \overline{M}_X C^+ Q \\
\text{Var } (\bar{\hat{a}}_X) &= \frac{\sigma^2}{v^2} \overline{M}_X C^+ \overline{M}_X' = \sigma^2 \overline{\Sigma}_X \\
\text{Cov } (\bar{\hat{a}}_{X_i}, \bar{\hat{a}}_{X_j}) &= \frac{\sigma^2}{v^2} \overline{M}_{X_i} C^+ \overline{M}_{X_j}' \quad i \neq j \\
SS (\bar{\hat{a}}_X) &= SS(a_X) = \bar{\hat{a}}_X' \overline{\Sigma}_X^{-1} \bar{\hat{a}}_X \\
\overline{f}_X &= f_X = \text{Rank } \overline{\Sigma}_X = \prod_{i=1}^{n} (m_i - 1)^{x_i}
\end{aligned}
\qquad (10)
$$

The dimensions of the var and cov matrices are reduced from $\prod_{i=1}^{n} m_i^{x_i}$ to $\prod_{i=1}^{n} (m_i - 1)^{x_i}$, and the matrix $\overline{\sum}_X$ is non-singular, so SS can be computed using the regular inverse.

The analysis of variance table (9) remains the same, since the sums of squares and degrees of freedom are equal.

6. Non-connected Designs and Missing Treatment Solutions

If the design is not connected, then additional constraints are needed to find a unique solution to the reduced normal equations $C\hat{t} = Q$.

Let z_1 be the number of sets of connected blocks and let S_i be the i^{th} set, $i = 1, 2, \ldots, z_1$. Let z_2 be the number of missing treatments (i.e. the number of $r_i = 0$). Then there must be $z_1 + z_2$ constraints to find a unique solution to the reduced normal equations.

If the constraints are taken to be:

$$\sum_{S_j} t_i = 0 \qquad j = 1, 2, \ldots, z_1, \text{ and}$$

$$t_i = 0 \qquad \text{for each } i \text{ such that } r_i = 0,$$

that is, if the sum of the treatments associated with each set of connected blocks is zero, and each treatment that is mssing is assumed to be zero, then the solution $\hat{t} = C^{*} Q$ with C and Q as previously defined, satisfies the constraints. The analysis of variance table (3) remains the same except that the degrees of freedom for treatments, Rank $C = v - z_1 - z_2$.

Now, in the factorial case, the problem is to find the relationships resulting from the additional constraints on the t_i's, select a set of $(v - z_1 - z_2)$ independent a_X's, and compute the corresponding sums of squares and degrees of freedom for the analysis of variance table.

To determine the relationships on the a's in addition to those in (6) let

$$\bar{\hat{a}} = \begin{bmatrix} \bar{\hat{a}}_{X_1} \\ \bar{\hat{a}}_{X_2} \\ \vdots \\ \bar{\hat{a}}_{X_{2^n-1}} \end{bmatrix}_{v-1 \times 1} \qquad \bar{M} = \begin{bmatrix} \bar{M}_{X_1} \\ \bar{M}_{X_2} \\ \vdots \\ \bar{M}_{X_{2^n-1}} \end{bmatrix}$$

That is, $\bar{\hat{a}}$ consists of all the elements in the $\hat{a}_y$ vectors, as defined in (10) and $\bar{M}$ consists of the corresponding rows of each $\bar{M}_X$. Then the system of equations to calculate the (v-1) $\bar{\hat{a}}$ elements is:

$$\bar{\hat{a}} = \frac{1}{v}\bar{M}\,\hat{t} = \frac{1}{v}\bar{M}\,C^+\,Q \qquad (11)$$

Linear relationships among the rows of $\bar{M}\,C^+$, and hence among the elements of $\bar{\hat{a}}$, can be determined by numerical techniques. If the rows of $\bar{M}\,C^+$ are arranged so that rows corresponding to elements of main effects are first, then those of second-order interaction terms, then third-order, etc. and a pivotal-method is used in which rows are interchanged only when necessary to remove a zero element from the diagonal, elements of terms of lowest order possible can be selected for the independent elements, and the remainder expressed in terms of those elements.

The linear relationships so determined can be used to categorize terms involving the dependent elements as being aliased with independent terms for which the coefficients are non-zero, or unestimable if all coefficients are zero.

Having so determined a set of linearly independent elements, reduce $\bar{\hat{a}}$ by eliminating the dependent elements, getting

$$\bar{\bar{\hat{a}}} = \begin{bmatrix} \bar{\bar{\hat{a}}}_{X_1} \\ \vdots \\ \bar{\bar{\hat{a}}}_{X_k} \end{bmatrix} = \frac{1}{v}\,\bar{\bar{M}}\,C^+\,Q$$

where $\bar{\bar{M}}$ contains the rows of $\bar{M}$ corresponding to the elements in $\bar{\bar{\hat{a}}}$, and where the X_i, $i = 1, \ldots, k$, represent those interaction terms for which at least one element is among the final set of independent elements. Some terms may not appear at all (if all elements associated with that term have been eliminated), while others may have degrees of freedom less than $\prod_{i=1}^{n}(m_i - 1)^{x_i}$ (if only part of the elements have been eliminated). Then the following relationships hold;

$$\begin{aligned}
\bar{\bar{\hat{a}}}_{X_i} &= \frac{1}{v}\,\bar{\bar{M}}\,\hat{t} = \frac{1}{v}\,\bar{\bar{M}}\,C^+\,Q \qquad i=1, \ldots, k \\
\mathrm{Var}(\bar{\bar{\hat{a}}}_{X_i}) &= \frac{\sigma^2}{v^2}\,\bar{\bar{M}}_{X_i}\,C^+\,\bar{\bar{M}}'_{X_i} = \sigma^2\,\bar{\bar{\Sigma}}_{X_i} \\
\mathrm{Cov}(\bar{\bar{\hat{a}}}_{X_i}, \bar{\bar{\hat{a}}}_{X_j}) &= \frac{\sigma^2}{v^2}\,\bar{\bar{M}}_{X_i}\,C^+\,\bar{\bar{M}}'_{X_j} \qquad i \neq j \qquad (12) \\
SS(a_{X_i}) &= \bar{\bar{\hat{a}}}'_{X_i}\,\bar{\bar{\Sigma}}^{-1}_{X_i}\,\bar{\bar{\hat{a}}} \\
f_{X_i} &= \mathrm{Rank}\,\bar{\bar{\Sigma}}_{X_i}
\end{aligned}$$

If the design is orthogonal, then $\sum_{i=1}^{k} SS(a_{X_i}) = \hat{t}'Q = SS(\text{treatments})$.

The analysis of variance table is given by:

Source of Variation	Sums of Squares	Degrees of Freedom
X_1	$SS(a_{X_1})$	Rank $\bar{\bar{\Sigma}}_{X_1}$
X_2 (adjusted for blocks)	$SS(a_{X_2})$	Rank $\bar{\bar{\Sigma}}_{X_2}$
$\vdots$	$\vdots$	$\vdots$
X_k	$SS(a_{X_k})$	Rank $\bar{\bar{\Sigma}}_{X_k}$
Blocks (unadjusted)	$B'K^{-1}B - G^2/w$	$b-1$
Error	$Y'Y - \hat{t}'Q - B'K^{-1}B$	w - Rank $C - b$
Total	$Y'Y - G^2/w$	$w - 1$

(12)

DESIGN OF FIELD TEST PROGRAMS AND STATISTICAL TECHNIQUES FOR ANALYSIS OF THE PERFORMANCE OF NAVIGATION AND POSITIONING SYSTEMS*

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GLOSSARY AND DEFINITIONS

1. Analysis of Variance - A statistical technique based on a linear model and the application of least squares for subdivision of the total variability in a sample into components specified by the model.

2. ACE(S) - Along course error of the system position.

3. b_{EO} - Slope of the orthogonal regression line describing the system's path based upon external position data.

4. CRD - Completely randomized design, the simplest type of experimental design or pattern of experimentation. The treatment combinations are randomly assigned to the entire set of experimental units.

5. CCE(S) - Cross-course error of the system position.

6. CCE(SP) - Cross-course error based upon the system's estimate of its own position.

7. CCE(E) - Cross-course error (external) = length of a perpendicular from point (X_E, Y_E) to the programmed path as determined by external measurements.

8. Correlation - A measure of linear association between two random variables: $\rho = \sigma_{xy}/\sigma_x \sigma_y$, i.e., a ratio of the covariance to the product of the standard deviations. Sample estimator, $r = s_{xy}/s_x s_y$.

9. Chi Square Distribution - The probability distribution of the square of a standard normal variable. Let z_i be N(0,1), then z_i^2 has Chi Square distribution with one degree of freedom.

*This paper also was presented by the senior author at the "Technical Symposium on Navigation and Positioning," 23-25 September 1969, USAECOM, Fort Monmouth, New Jersey.

10. $\Delta x = X_S - X_E$ = deviation of the system's indicated position (X coordinate from an external measure of system's location.

11. $\Delta y = Y_S - Y_E$ = deviation of the system's indicated position (Y coordinate) from an external measure of system's location.

12. d = radial error = $[(\Delta x)^2 + (\Delta y)^2]^{1/2}$ = straight line distance from system's indicated position to position determined by external measuring equipment. Note that if Δx and Δy are normal random variables, then $d^2\sigma^2$ is distributed as Chi Square with two degrees of freedom.

13. Duplicate = A subsample of an experimental unit; one of two measures of system performance for the same experimental unit.

14. Degrees of freedom = Formally, a parameter of the Chi Square probability distribution. In application, the number of independent deviations available for estimating a variance or mean square.

15. Experiment = Study of system performance over a set of experimental units.

16. Experimental error = A mean square or quadratic measure of system variability about its average performance measured over a set of homogeneous experimental units.

17. Experimental unit = A period or segment of system operation for which an independent measure of system performance can be obtained.

18. F ratio = A ratio of two independent estimates of variance for which under the "null hypothesis" both numerator and denominator are distributed as (Chi Square) (σ^2) with degrees of freedom, say f_1 and f_2.

19. Interaction - A situation in which the observed results for the simultaneous application of two or more factors cannot be explained by addition of the direct effects of each factor separately estimated.

20. Local Control - (Blocking) a subdivision of the total set of available experimental units into relatively homogeneous subsets. Each subset is called a block. A complete block contains one experimental unit for each treatment combination. Two or more such blocks then comprise a RCB.

21. Median - A value which divides a population or a sample into two equal parts.

22. Orthogonal regression line - A least squares fitted line such that the sum of squares of normal distances from points to the line is minimized.

23. Position error (refer text)

24. Quartile - The quartiles are values that divide a population or a sample into four equal parts. The second quartile is the median.

25. Radial error = d (see 12 above).

26. RCB - Randomized complete block design (see 20 above).

27. Response - A measure of system performance. May be univariate but is often multivariate.

28. Replication - In a simple measurement situation a single independent observation of system performance. Otherwise, one replicate comprises one observation of system performance for each treatment combination of the entire set of treatment combinations being investigated.

29. Regression mean square - The mean square of deviations of points from a regression line (based upon division by the degrees of freedom).

30. Sample size - The number of observations on each treatment combination or the number of complete replicates. Note: this is <u>not</u> the total number of experimental units.

31. Standard deviation - Square root of the variance; a measure of variability about the average of observations from a homogeneous set of experimental units.

32. Structure of a Test Program or Experiment - The overall arrangement of a test program which includes the treatment combinations to be investigated, the environments and locations in which the system is to be operated and the experimental design imposed.

33. Treatments (and Treatment Combination) - If a system is to be tested at altitudes, say Low and High, we say that altitude is a factor at two levels. We also refer to altitude as a treatment imposed on the system. Suppose we also wish to test the system over land and over water. Then water and low altitude and water and high altitude are two different treatment combinations. Two factors each at two levels provide a total of four treatment combinations.

Field Test programs are fraught with many difficulties. Developmental equipment just never seems to perform as well as desired by its producers or as hoped for by the Army. Characteristics of the field environment may not have been adequately anticipated by the development engineers. Often an extensive shakedown period is required before a system is really ready to be entered into a field test program.

Even before the shakedown trials are started a complete TEST PLAN must be developed for the field test program. The field test envisaged may comprise several parts with each part designed to exercise the system in a different way. When this is the case, a specific TEST PLAN should be developed for each part.

It has been our experience that field test programs are often inadequate or incomplete in several respects. Therefore, we need to consider the question, "What are the GENERAL FEATURES OF A TEST PROGRAM?" These features are set out as a list of ten items (prepared by the senior author at a time when he first came in contact with the study of navigation and positioning systems) [1].

GENERAL FEATURES OF A TEST PROGRAM

1. Careful delineation of the problem and thorough understanding of the system or systems to be examined.

2. Definition of the phenomena to be studied. (Including "What are the requirements?")

3. Selection of the response (i.e., performance characteristics) and the technique of measurement for each response. Know the standards that should be applied.

4. Determination of a suitable experimental unit.

5. Selection of treatments to be studied (i.e., equipment parameters to be varied.

6. Selection of environmental conditions or parameters to be varied.

7. Choice of a pattern of experimentation (suitable combinations of experimental units, treatments and environment). Result is an experimental plan or design that includes the randomization procedures, adequate local controls and sufficient replication.

8. Complete layout of the plan for analysis of the responses or measurements to be obtained (before the data are taken).

9. Interpretations to be made from all possible experimental results.

10. What is the next experiment that may be relevant after the currently proposed one is completed?

Let us consider these ten items in turn. Item 1, we leave to the engineers although much questioning is often required to obtain a clear statement of the problem. Item 2, we also regard largely as an engineering area. Spelling out what is expected of the system in realistic and useful terms is a major step. Later, the question is to be asked and answered, "Does the system fulfill the requirements?" To a considerable extent the answer will depend on the data acquired and our analysis of these data. Examples of requirements might be, "Take off from Dulles International for Paris; make landfall in France with cross-course deviation less than 5 miles with respect to a designated point." Or, "Take off from Field A; fly over point X, Y with an average radial error not to exceed 20 meters; land at Field B."

Succeeding items on the list lead us more into the statistical and experimental design problems. Determination and definition of the relevant response (Item 3) for judging the performance of the system is basic to all that follows. Yet many "test programs" have been written without having the performance measures for the system quantified and the methods of measurement clearly stated. Related to the performance measure is the selection of the standard for assessment of that performance. With respect to navigation and positioning systems we may ask,

(1) Do we need photo-theodolite data?, or

(2) Is a radar network required?, or

(3) Will the measurements from a single radar such as the FPS-16 be sufficient?, or

(4) Can we rely on a higher resolution non-radar electronic network?, or

(5) Will cruder methods, simple photographs or visual observation, be sufficient?

Depending on the stated requirements, we may select one or more of these alternatives.

After much though about item 4, we have reached the conclusion that for an electronic system mounted in a land vehicle, a ship or an A/C, the <u>entire mission</u> on a given day must be regarded as the <u>experimental unit</u>. In this mission we include -

starting up the system
warm-up
check out
calibrations
departure from base
calibrations enroute
traversing selected courses
return to base
checking calibrations
shutting down the system
complete return to ambient conditions.

This view of the experimental unit means that any repetition maneuvers performed by the system within the same mission must be regarded as duplicates and not as replicates. Of course, we are interested in the variation among duplicates but major interest centers on the replicates, that is, the repeated performance of the system over a set of experimental units that we regard as similar or sufficiently homogeneous for the problem under study. By definition, experimental error is the failure of a system to produce identical responses over a set of independent trials (or experimental units). The key word here is independent; we believe that repeated maneuvers in any one mission are likely to be highly correlated. Therefore, we insist that an independent trial for an electronic system include the complete sequence given above from "starting up the system" through "return to ambient conditions."

A system may have several "modes" of operation, threshold settings may be required and variation of dial settings may affect the performance of the system. All these equipment parameter variations we include under the set of treatments that may be investigated (Item 5). Further, we usually extend our concept of the treatments of interest to include the variations external to the system which may or may not affect (hopefully not) the performance of the system. Under Item 6, we include weather, altitude, day or night operation, electromagnetic disturbances (natural or man-made), terrain, direction over a course, etc.

The result of considering Items 4, 5, and 6 leads us to selection of a pattern or program for the system test. The structure of the test program is determined by the factors (conditions and parameter settings) which we wish to investigate. The simpler this structure can be made, the easier it will be to:

(1) Cope with the inevitable modifications of the test program that arise due to revision of test objectives, unexpected equipment limitations, or failure to obtain adequate data for some courses; and,

(2) Analyze the data.

The simplest form of experimental pattern is called a Completely Randomized Design (CRD). This design is preferred when it is feasible. A simple description is that we write down on slips of paper each combination of conditions and parameter settings that is to be included in the test program. Then we put the slips into a hat, mix thoroughly, draw them out one at a time and write out a complete list of the consecutive drawings. Suppose altitudes of one thousand and 12,000 feet were included in the test program for an airborne system. If any part of the consecutive sequence of drawings came out with altitudes (in thousands of feet) 12, 1, 12, 1, 12 for the sequence of courses to be flown the pilots would object; hence, we regard a CRD as not feasible for such a situation. Therefore, split-plot structures or nested designs must be worked out when some of the treatments cannot be submitted to complete randomizations.

Performance of systems tends to vary with time, or for a development item prototype the performance is even likely to deteriorate with time. Such results are to be expected when the "bugs" are not all ironed out, and the test program covers a 3 to 6 month period. Because of this time variability in performance, it is highly desirable to introduce a "blocking" with respect to time. Such blocking is a form of what is generally known in experimental design as "local control." This local control permits the removal of (or elimination) of time variation so that any two treatment combinations (choice of parameter settings) can be compared without time bias. What this means in practice is that if two particular combinations are run in, say, the second week of the test program, and if one or the other is scheduled again for the 7th and 13th weeks of the test program, then if both are run in the 7th and 13th week, then the time differences (if any) among the 2nd, 7th, and 13th weeks can be removed in making the desired comparison. The balancing of the experimental program against time or some other possible source of undesirable variability is accomplished by setting up a Randomized Complete Block design. We regard the use of local control by blocking as a necessary requirement in the study of complex systems used for navigation and position determination. Here, we have assumed one week as comprising a block.

It is to be noted that each block as just described forms one complete replicate for a set of treatment combinations. The time period included in the block can be any reasonably short period of homogeneous test conditions, say, one day, three days, or one week. Thus, the number of blocks completed determines the total number of replicates for this set of treatment combinations. The number of blocks completed then determines the sample size so the natural question is, "How many blocks do we need?"

Two considerations enter into the determination of the desired sample size. First is the requirement of obtaining a stable estimate of the experimental error. It is our experience that an estimate with 10 to 20 degrees of freedom may often be adequate for development test programs. Such an estimate can be obtained with as few as three blocks when eight or more treatment combinations are to be investigated in each block. Larger blocks, however, may introduce other problems; e.g., lack of homogeneity of experimental units. The second consideration is the magnitude of real differences in system performance that may be associated with environmental and/or parametric changes for the system. Again from experience we have found that system developers and system users have limited information on the magnitudes of these differences. It can be shown from theory that for a specified probability a "large" sample is required to detect "small" differences, but that a "modest" sample may detect easily "large" differences. These vague words (large, small, modest) can be given numerical values only when we are able to insert in the available formulae actual values for (1) the standard deviation of our experiment the (experimental error)$^{1/2}$ previously described); and (2) the magnitude of the difference to be detected.

The discussion of Item 7 of the "General Features" has been rather lengthy, but we have tied together in this discussion the preceding Items 4, 5, and 6 with Item 7. In this discussion we have covered some aspects of the choice of experimental pattern and its associated randomization, local control by blocking on time, and the choice of sample size.

Item 8 follows quite easily if we have done our homework well in covering Items 3 through 7. Perhaps, we should note that it is easy only in principle. We recall a paragraph from our abstract as follows:

> When these 'GENERAL FEATURES' have been closely adhered to, then the work of summarization and analysis of data and the final interpretation of results becomes much simpler. An experimental design for the field test program has associated with it a mathematical model; the two together determine the analytical procedures. One of the most useful and severe disciplines to impose on the military personnel and the development contractor is to require that a set of tables be prepared before the field test is started. This set of tables should include the detailed format of the summary data on which the performance of the system is to be judged. Further, the parties should agree that the performance is to be judged on these criteria.

The last two Items, 9 and 10, are essentially self-explanatory. It is usually salutary to give them some consideration, however, before the first experiment is begun. As the test program proceeds, other considerations will appear or come to bear on the problem. Thoughts about 9 and 10 will then take new directions. Without the pre-first-experiment considerations well though out and written down, the new directions may turn out to be undesirable tangents. The "whole forest needs to be kept in view rather than the interesting trees that appear as we walk in the woods." A remark on the use of the term <u>experiment</u> may be added here. Physical scientists often think of an experiment as a <u>single trial</u> under carefully specified conditions. From the analysis point of view, which must be taken by the statistician, an experiment comprises all replicates of a set of treatment combinations among which comparisons are to be made. The TEST PLAN (or test program) for a given system may consist of only one, or two or more experiments.*

We now turn to the consideration of the second area of our paper as indicated by our title. Analysis of the performance of a navigation and positioning system must describe this performance quantitatively in terms of precision and accuracy [5]. Various statistical techniques may be required to describe this performance. In order to give concreteness to this section of the paper we shall base our discussion upon the analysis of the performance of an airborne navigation system in which we were engaged several years ago [6].

The field test program for this system included a requirement that the system depart from a base, fly over a calibration check point, and then proceed to maneuver the A/C over a series of six parallel flight paths whose end points were defined by specified longitude and latitude coordinates (see Figure 1). In Figure 1, we show two series of six parallel lines, sets 1A and 1B. The set 1B was actually laid over the same ground area as set 1A. Each line of a set of six we refer to as a LEG, so that the total flight course comprised six LEG's. Starting with LEG 1 in series 1A as shown we refer to this pattern as a Zero Degree flight forward over the course (AO,F). Beginning with LEG 6 and reversing direction over each LEG is called (AO, R). Using series 1B in the direction shown starting with LEG 1 is designated as 90 degrees forward (B90, F). Similarly, reversing course beginning with LEG 6 is designated as (B90, R). Other designations are possible such as starting at other end of LEG 1 in each series, which gives (A180, F) and (B270, F).

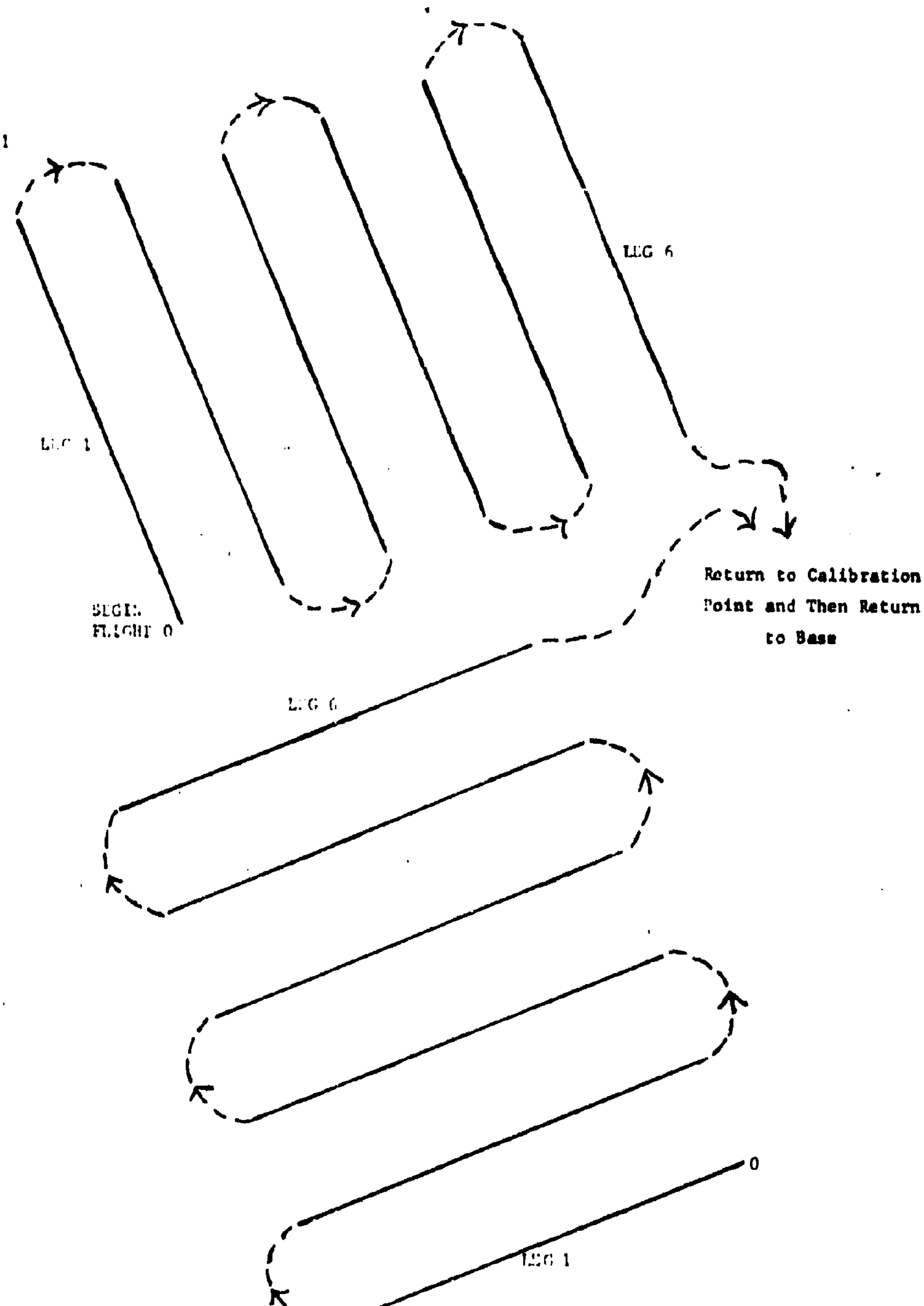
With this view of the flight area pattern we may approach the details of describing the system performance. Assessment of the system performance will be based largely on a <u>position error</u>; i.e., the difference in location of the system as determined by an external measuring system and the system's own indication of its location (at a given time). This position error information is to be analyzed by averaging and/or decomposition to provide descriptions of system performance. Among these descriptions are:

(1) The difference between the average location of the system over a number of repetitions under essentially similar conditions for a programmed flight over a point or a course and the desired point or course is a <u>measure of system accuracy</u> [7]. This accuracy, however, may vary over the flight area (1A & 1B) for a variety of reasons. Thus, it may be useful to speak of the system's <u>predictability</u> or <u>reproducibility</u> for a group of points or LEG's in the assessment of accuracy,

*Appropriate references for this first section of the paper are [2], [3], and [4].

FIGURE 1

1A



1B

(2) For precision or repeatability assessment we may describe the performance along a specific LEG or segment of a course which was programmed for the system, or

(3) We may describe the repeatability of the system in flying over the same programmed course a number of times, each time appearing on a different day.

Thus, it is seen from (2) and (3) that we can describe precision over experimental units (replicates), which is of greatest interest, and also in terms of within replicates over a segment of a LEG, a whole LEG or the set of LEG's. Replications of any LEG or part of a LEG on the same day may be regarded as duplicates from the viewpoint of sampling the system performance. We note that within a single programmed flight on the same day all individual position determinations made by the system must be regarded as inherently correlated to some greater or lesser but unknown degree. This point of view is conceptually correct in regarding the output of a single programmed flight as one "realization" in the sense of the theory of stochastic processes. The degree of correlation, of course, depends on the time and/or distance separation between any two position determinations. The actual magnitude or form and shape of this correlation function may be quite relevant for system design but need not be of major concern for evaluation of system performance. The realization of its presence, however, requires the definition of a single trial or experimental unit in the way already described and then it guides our analysis.

The discussion thus far has been general in the evaluation of system performance. It will be helpful to list some of the actual variables measured in relation to the determination of position error. These random variables were:

(1) $\Delta x = X_S - X_E$

(2) $\Delta y = Y_S - Y_E$

(3) CCE(S) = Cross-course error for the system

(4) ACE(S) = Along course error for the system

(5) $d = [(\Delta x)^2 + (\Delta y)^2]^{1/2}$ = Radial error for the system.

A rectangular grid system was laid out over the area indicated in Figure 1 with the point (0,0) arbitrarily selected. At time t_i, (S_{si}, Y_{Si}) was the system's indicated position while (X_{Ei}, Y_{Ei}) was the actual position of the system as determined by an external means. Thus, d_i was the radial error at time t_i. The time interval from t_i to $t_i + 1$ was five seconds.

The assessment of repeatability is most easily begun by examining the performance within a single LEG. For each LEG a number of summary statistics were computed for the variables just listed. These statistics included:

(1) Average value for the variable.

(2) Mean square deviation of the individual values from the average. Note that this quantity although calculated like a sample variance does not have the usual Chi Square distribution with n - 1 degrees of freedom because of the correlation of data points within a given LEG (as already discussed).

(3) Minimum value.

(4) Maximum value.

(5) Median value.

(6) First and third quartiles.

In this paper we can illustrate only a few analyses of these many statistics. A mere tabular summary, of course, gives some description of repeatability. A further analysis considers the behavior of these LEG statistics from LEG to LEG, from (programmed) flight to flight, at different altitudes, orientations (or direction of flight), and even over different areas. The statistical technique used for this further analysis is known as the analysis of variance. This technique has been well described by Kempthorne and Scheffe in its application to the analysis of experimental data [8, 9]. Briefly, the technique may be described as a procedure for evaluating the variation of averages and the variation of individual observations. These evaluations, called mean squares, may be compared by forming Snedecor's F ratio in order to make inferences about the magnitude of the variations of the averages. Specific assumptions, of course, are made in the application of the technique. Currently, most attention is given to these assumptions: (1) the specified linear model adequately represents the experimental structure; and, (2) independence, i.e., the data comprise a random sample from the universe of interest.

A simplified example will illustrate the application of the analysis of variance to a possible set of data from the flight program described above. Let us suppose a series of flights made over the area of Figure 1 with variations in altitude and heading. The series of flights is carried out in a completely randomized design with the results obtained as in Table 1. There are two replicates of each combination of conditions. Note that only average results for each entire flight are presented.

The analysis of variance appears in Table 2.

TABLE 1

ILLUSTRATIVE EXAMPLE OF POSSIBLE RESULTS
FOR EIGHT FLIGHTS FOR COMBINATIONS OF
TWO ALTITUDES AND TWO HEADINGS

Flight No.	Altitude	Heading (Degrees)	Average Radial Error (Meters)
1	7,000	0	80
2	7,000	90	90
3	15,000	0	50
4	7,000	0	90
5	15,000	90	40
6	7,000	90	100
7	15,000	0	50
8	15,000	90	60
			560

TABLE 2

Analysis of Variance of Average Radial Error

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square
Total	8	42800	----
Average	1	39200	----
Altitude	1	3200	3200
Heading	1	50	50
Altitude by Heading Interaction	1	50	50
Remainder	4	300	75

The model upon which this analysis is based is written as -

$$Y_{ijk} = \mu + A_i + H_j + (AH)_{ij} + \epsilon_{ijk}$$

where Y_{ijk} is an average radial error as shown in Table 1 and the terms on the right in order are -

a general mean,
an altitude effect,
a heading effect,
an altitude-heading interaction, and
a random component associated with the ijk th experimental unit (flight).

It is the variation among these eight averages given in Table 1 which is to be subdivided into parts associated with the sources of variation present. We note that the altitude means are: 1/4 (80+90+90+100) = 90 at 7,000', and 1/4 (50+40+50+60) = 50 at 15,000'. Similarly, the Heading means are: 67.5 at 0 degrees and 72.5 at 90 degrees.

Thus, the 2 x 2 table of means for average radial error is

	Altitudes		
Headings	7	15	Averages
0	85	50	67.5
90	95	50	72.5
Averages	90	50	70

Details of the calculations, the assumptions underlying the analysis and interpretation of the results are given in most modern texts on statistical theory or techniques [10, 11]. We cannot consider these matters further here, but we point out two aspects of this hypothetical example: (1) The "Remainder" with 4 degrees of freedom is an appropriate estimate of experimental error, so that $(75)^{1/2} = 8.66$, is a standard deviation that estimates the repeatability of the System over repeated flights; and, (2) that the Mean Square for Altitude, 3200, when compared with the Remainder Mean Square provides a basis for assessing the effect of Altitude. If Altitude variation did not affect the System, we would expect these Mean Squares to be about equal. From Table 2 we would conclude by looking at the interaction component (F ratio = 50/75<1) that the Altitude effect does not vary with Heading. The Heading effect appears negligible (F ratio $\approx$ 50/75 1). Finally, we would conclude that performance differs with altitude (F ratio = 3200/75 $\approx$ 43) ($P < <0.01$). From the averages, we see that the radial error is much smaller at the higher altitude.

In reference to the description given above for the analysis of variance as a technique for studying the variation of averages in contrast to the variation of individual observations, there is a point to be noted in relation to the hypothetical example just given. In the example, the individual values analyzed are themselves averages. Thus, there is a further component of variation associated with individual observations or points within LEGS, that has been suppressed in the example. Generally, in analyzing data for studying the system we followed this same procedure of studying averages. Thus, a simple LEG average provided a single datum and we analyzed the variation of these averages in relation to other factors.

There are several reasons for following this procedure. First, this approach, of course, has simplified some problems in analysis due to unequal numbers of observations within LEGS. Second, even though numbers of observations on a given LEG varied from as low as 80 to around 200, there was no reason for giving more weight to one flight over a given LEG than another if a reasonable set of data were obtained to represent that flight over that LEG. Thus, using averages and giving each average equal weight seemed a proper procedure for assessing the overall performance. Third, the use of averages, even though each average is computed from data with considerable correlation, will provide values of a random variable which more closely approach the assumptions of the analysis of variance technique. In analyzing the repeatability within LEGS as measured by the variances of designated random variables (Mean Square Deviations from Average or from an Orthogonal Regression Line), these variances may also be considered as "averages." Because of the greater apparent dispersion of these variances, it seemed desirable to analyze the natural logarithms of these quantities to obtain a transformed variable more suitable for the analysis of variance technique. Fourth, and last, this approach in terms of further analysis of original statistics (averages, variances, slopes of regression lines and deviations from such lines) is in keeping with the spirit of Professor John W. Tukey's suggestions [12].

The preceding example was made small in order to be easy to follow. The conclusions stated relate only to the hypothetical data of Table 1 as if they were real data. We now present in Table 3 some real data for six flights over the area represented by Figure 1. These flights were flown at three altitudes with zero degree heading (i.e., AO, F as noted above). Table 3 gives averages of Δx, Δy and d = radial error for each LEG of each flight. Hence, 36 averages are shown for each variable. The

Table 3

Tabulation of Leg Means for Selected Variables
from Six Flights Over the Figure 1 Area
(Units are Meters)

Flight Number	(Altitude/ 1000)	Leg Number 1	2	3	4	5	6	
		Variable: $\Delta x = X_S - X_H$						
5	7.5	143	32	54	109	120	116	
6	7.5	103	32	51	91	108	115	
1	11	30	-105	-9	51	71	86	
7	11	95	70	74	101	124	139	
3	15	121	81	107	88	125	116	
4	15	91	89	59	119	97	140	
		Variable: $\Delta y = Y_S - Y_H$						
5	7.5	-70	-135	-335	-341	-144	+11	
6	7.5	-75	-101	-233	-257	-80	+45	
1	11	+35	-339	-77	-130	+18	+185	
7	11	-92	-230	-341	-373	-172	+3	
3	15	-109	-139	-345	-324	-95	+3	
4	15	-103	-273	-331	-391	-122	-3	
		Variable: Radial Error = $[(\Delta x)^2 + (\Delta y)^2]^{1/2}$						
5	7.5	319	187	349	370	219	159	267*
6	7.5	277	139	233	288	195	142	216
1	11	278	378	176	169	168	212	230
7	11	262	253	356	402	231	175	280
3	15	264	186	370	344	208	138	252
4	15	260	298	357	412	209	196	287
		277**	240	310	331	204	170	
								255***

*Flight Averages;
**Leg Averages:
***Overall Average.

Table 4

Analysis of Variance of the Leg Averages
for the Variable: Radial Error

Source of Varation	Degrees of Freedom	Sum of Squares	Mean Square
Total	36	2582140	
Average	1	2346156	
Altitudes	2	4778	2389
Flights at Same Altitude	[illegible]	19092	6374
(Pooled Variation for Flights)	(5)	(23870)	(4774)
Legs Over All Flights	5	115968	23194
Legs x Altutudes Interactions	10	40300	4030
Legs x Flights at Same Altitudes	15	55847	3723
(Grouped Legs x Flights)	(25)	(16147)	(3846)

analysis of variance for one of the variables, radial error, is given in Table 4. Therefore, Table 3 also shows the marginal averages for this variable, that is, over all LEG's of the same flight, and over all flights for the same LEG.

The model for the analysis of Table 4 is written as -

$$Y_{ijk} = \mu + A_i + \epsilon_{ij} + L_k + (AL)_{ik} + \delta_{ijk}$$

where Y_{ijk} is average radial error as given in Table 3 and the terms on the right are -

a general mean,
an altitude effect,
an error component for flights at same altitude,
a LEG effect,
an ALTITUDE x LEG interaction, and
a residual which measures failure to obtain same results for
  a LEG when a repeated flight is made at the same Altutude.

Major interest in Table 4 first centers on the Altitude Comparison. The mean square for Altitudes is 2389 while the mean square for repeated flights made at the same altitude is 6364. The latter is our measure of experimental error for Altitudes; hence, the F ratio is 2389/6364< 1. We conclude that altitude variation did not affect the performance of the system over the range of altitudes selected (our choice of altitudes was limited by the performance capability of the A/C carrying the navigation system).

Next, we examine the variation within flights or between LEGs. The "LEGs over all flights" mean square is 23194, a large value relative to all other mean squares in Table 4. Thus, we are inclined to conclude that there are large differences among the six LEGs of the programmed flight pattern. The remaining two mean squares, LEGs x Altitudes = 4030 and LEGs x Flights at same Altitude = 3723, indicate the consistency of these large LEG differences. Pooling of the last two sources of variation yields a mean square of 3846 with 25 degrees of freedom. An approximate F ratio for comparing LEGs could be formed by F = 23194/3846 ≈ 6. We regard this ratio as an approximate F in distribution because of the correlation of LEGs within the same flight although this may be small because of the apparently large LEG differences. Perhaps, a multivariate test could be devised for comparing LEGs; we have not considered this approach. In view of the consistency of these LEG differences over different days throughout the test program it seemed reasonable to us to conclude that natural electromagnetic field variations over the six LEGs affected the system performance.

We shall not endeavor in this paper to summarize our conclusions about the repeatability or precision of the system which produced the data given in Table 3. It is our purpose here merely to present statistical methods and techniques for securing such information about any navigation system. It will be useful to give one more table, however, to show another aspect of the repeatability. In Table 5, we give the mean square deviation of the radial errors from the average radial error (given in Table 3) arranged by Flights and LEGs as in Table 3. We shall not give the analysis of variance for the data in Table 5 but we note that natural logarithms of these mean square deviations were taken before computing the analysis of variance. This log transformation is usually applied before analyzing variances of observations.

Although we have given only a small sample of the large amount of repeatability information obtained for the system we have been using for our discussion, we turn now to the system accuracy. If the system exhibits accurate performance we may say that it has predictability or reproducibility. In addition to the variables listed above, which, were used for examining the system repeatability, we also obtained the cross course error of the system location from the external measurements, CCE(E), which was the distance of the point (X_E, Y_E) from the programmed path (lines shown in Figure 1). An average of these values would show the bias or systematic error of the system in flying the programmed course. If this bias were negligible over all flights we would regard the system as accurate or that its performance is reproducible.

Further, from the (X_{Ei}, Y_{Ei}) data we obtained a derived quantity, the slope of the orthogonal regression line, b_{EO}, through the points traversed by the system. The slope of this line for each LEG of the programmed path then could be compared with the actual slope, β, of the programmed path in terms of the arbitrary X, Y coordinate system imposed on the area of Figure 1. Departures of the observed slopes, b_{EO} from the desired slope, β, then give further information on the system predictability.

Before presenting actual results it will be helpful to discuss briefly the use of the orthogonal regression line. From the above, it is clear that both X_E and Y_E are random variables. The usual regression models consider X_E to be an independent variable observed or measured with no error or negligible error. Natrella in "Experimental Statistics" gives a good discussion of regression analysis for functional and statistical relations [13]. It is clear that none of the standard models apply to this navigation system analysis. After considerable study, we concluded that fitting the orthogonal regression line would best describe the system performance. Derivation of the normal equations for fitting this line was given by Coleman in 1932 [14]. To our

TABLE 5

Mean Square Deviations by LEGs for Six Flights
Variable: Radial Error

Flight Number	Altitude	LEG 1	2	3	4	5	6
5	7.5	53240	15630	12740	5215	12720	6883
6	7.5	41050	7854	26780	5716	7835	1017
1	11	22790	155100	16280	4622	3332	6394
7	11	39490	27780	33510	6991	12890	4270
3	15	36400	23990	26200	9361	5372	1227
4	15	34170	28880	25420	3196	7804	8589

TABLE 6

Tabulation of Altitude Averages and General Average for Variables Which Describe System Predictability (based on LEG AVERAGES weighted equally)

(units are meters)

Variable	General Average*	7.5	11	15
		Altitudes/(1000)(feet)** AVERAGES		
CCE(E)	-20.2	-7.4	-22.5	-30.8
CCE(SP)	6.4	10.9	7.0	2.2
Slope, b_{FO}***	-1.3001	-1.2982	-1.3030	-1.2990
Reg. Mean Square	5800	3500	7600	6200

* Based on average of 36 values; six flights of six LEGs.

** Based on average of 12 values; two flights of six LEGs.

*** Note: [illegible] = -1.3032

TABLE 7

Tabulation of Flight Averages for Variables Which Describe System Predictability

(based on Leg Averages weighted equally)

(units are meters)

	Altitudes 7500		11,000		15,000		
Flights	5	6	1	2	3	4	Standard Deviation**
Variables							
CCE-(E)	-13.9	-0.9	-9.4	-35.7	-19.3	-30.8	19.9
CCE(SP)	16.9	4.1	7.1	6.0	9.4	-5.0	6.9
Slope, b_{FO}	-1.2990	-1.2973	-1.3075	-1.2985	-1.3000	-1.2980	0.0038
Reg. M. Sq.	3900	3150	11500	3700	5500	6900	3100

* Based on average over six Legs within each flight.

** This standard deviation is derived from the flight mean square in the analysis of variance, i.e., $\sqrt{(\text{Flight Mean Square}/6)}$. The divisor six arises from the averaging over six legs within each flight.

knowledge, however, the sampling theory for the slope of this orthogonal regression line has never been presented (approximations could be obtained, perhaps). In a replicated experiment, this lack of adequate sampling theory does not create an impasse. Independent estimates obtained from repeated flights will permit direct estimation of the variability of the orthogonal regression slopes.

Along with computation of the orthogonal regression line we also present the regression mean square for deviations from the regression line. The magnitude of this mean square indicates the scatter of the X_E, Y_E points about the fitted line. The system whose data we have been presenting also provided an estimate of its own position which we designate as X_S, Y_S. From this data series we calculated a CCE(SP) = Cross Course Error of the system's indicated position. Table 6 presents average values of these four statistics for the six flights over the area of Figure 1.

As averages, these numbers in Table 6 speak for themselves. With respect to cross course error, if the system actually was on the left side of the programmed path, the deviation was designated as negative. Thus, we see that the system generally directed the flight slightly to the left of the programmed path. On the other hand, the system's indication of its position on the average was an even smaller deviation but to the right of the programmed path. For reference, the slope of the parallel lines comprising the programmed path was -1.3032. Thus, the average slopes shown in Table 6 agree quite well with the desired direction.

Overall averages, however, do not tell the whole story. Hence, we present average values for the six individual flights for these same variables in Table 7. The right hand column in Table 7 shows the standard deviations of these averages as obtained from the analyses of variance for these four variables. Again, these data need little explanation. We note that for Flight No. 4 the average of the systems' indicated position also was to the left of the programmed path. Furthermore, the largest CCE(E) was observed for this flight. We have discussed the estimation of sampling error for the regression orthogonal slope. Here we see the application of this estimation even though we have no direct sampling theory for b_{EO}. The values shown for each flight are based on average slopes for all six LEGs. The estimated pooled standard deviation for these slopes is only 0.0038.

SUMMARY. In this paper we have considered the assessment of the performance of navigation and positioning systems. Such assessment comprises two parts: (1) the development of a comprehensive TEST PLAN; and, (2) adequate statistical analyses of the data collected. We emphasize that no more information can be extracted from data than has been built into the structure of a test program [15]. This structure is created by the TEST PLAN.

Our discussion of the TEST PLAN has been developed from an Outline of the "General Features of a Test Program." Among the ten items included in this outline we have directed particular attention to the following:

(1) Selection of the system response or performance measures.

(2) Definition of an experimental unit.

(3) Selection of the treatment combinations.

(4) Determination of the pattern of experimentation or choice of experimental design.

(5) Blocking of the test program against time or other sources of variation in the test program, and,

(6) The sample size or how many experimental units should be completed.

For analysis of the test data we have considered the assessment of both precision and accuracy. There are many ways of presenting data summaries to provide information on both of these characteristics of system performance. We have illustrated the application of the analysis of variance in different ways. Generally, we prefer this approach because of the ability to subdivide the total experimental variation into sources associated with the structure of the TEST PLAN. In using some results obtained from the flight test program for a navigation system we have been able to give only a small sample of the many analyses performed for measuring both precision and accuracy. The latter we note also has been referred to as: (1) predictability; and, (2) reproducibility. For one measure of predictability, the slope or direction of a flight path, we showed how to measure directly the variability of the slope estimates.

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A UNIFIED PROCEDURE FOR SELECTING ALTERNATE EXPERIMENTAL DESIGNS

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Considerable knowledge has been developed in the literature that provides for the more effective design of experiments using, primarily, certain statistical techniques for analysis purposes. Such methods are concerned, for the most part, with analyzing the degree of dependence between the variables. These techniques have exerted a significant influence upon the amount of precision and accuracy that is realized in many experiments.

Additional impact on the optimization of experiments is potentially possible through the application of modeling techniques in the synthesis of experiments. Such techniques are concerned with the design of the experimental model, providing a basis for systematic optimization of the design criteria.

Design Criteria

As in any engineering design problem, the ultimate character of the final design is dictated by certain design criteria. Some typical criteria for an experimental design are as follows:

1. The number of factors to be varied
2. The number of levels to be measured for each factor
   (a) Are levels qualitative or quantitative?
   (b) Are nonlinear effects to be measured?
   (c) Are deviations to be measured from a nominal?
   (d) Are all factors to be set at an equal number of levels?
3. The number of measurements of the response variable to be taken
   (a) Are interactions to be measured?
   (b) Are there any physical limitations on the number of measurements in the experiment?
   (c) What precision is required for measuring experimental error?

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The synthesis of an experimental model will be discussed in three steps:

1. Design of the <u>structural</u> model
2. Design of the <u>functional</u> model
3. Design of the <u>experimental</u> model

The first two criteria are important in the determination of the structural model. Criteria 3 (a) and 3 (b) are important in the design of the functional model. Criterion 3 (c) is the major consideration in the design of the experimental model. The ultimate experimental model is the objective of the design process discussed in this paper. Alternate standard experimental designs are compared to the developed experimental model so that a design choice can be made that will optimize compliance with design criteria.

Such an optimization effort differs with the traditional type in statistical design of experiments. This traditional optimization process is typically concerned with a trade-off between the cost of experimentation and the statistical decision. Such optimization would provide the design criterion in 3 (c); i.e., a determination of the number of measurements required to provide a certain precision in estimating experimental error so that certain risk and/or cost requirements can be met. Optimization of the experimental design in this paper is concerned with the selection of the design that will best meet the design criteria established for the experiment. One of these design criteria usually consists of the number of measurements to be made in order to optimize certain statistical and cost requirements.

<u>The Structural Model</u>

The structural model of an experiment is described by

$$N_s = k_1 \cdot k_2 \cdot k_3 \cdot \cdot \cdot k_m \cdot \cdot \cdot k_p \qquad (1)$$

N_s = number of cells (defined below) in the experiment

k = number of levels for a factor or independent variable

$$j = 1, 2, 3, \ldots, k$$

m = the m^{th} factor or independent variable; 1, 2, ..., p

p = the total number of factors in an experiment

The simplest form of an experiment is the case of one factor, for example x_1, at one level, so that

$$p = 1, \qquad k_1 = 1$$

and thus, from Eq. (1), the structural model becomes

$$N_s = k_1 = 1$$

This model is called a <u>cell</u>, the basic structural unit of an experiment. The next form of an experiment is the case of one factor at two or more levels, so that

$$p = 1, \qquad j_1 = 2, 3, \ldots, k_1$$

and thus, the structural model becomes

$$N_s = 2, 3, \ldots, k_1 \text{ cells}$$

The next form or level of an experiment is illustrated by a case in which there are two factors at two or more levels. Thus

$$p = 2, \qquad j_1 = 2, 3, \ldots, k_1, \qquad j_2 = 2, 3, \ldots k_2$$

and the structural model becomes

$$N_s = k_1 \cdot k_2 \text{ cells}$$

Consider another example. A three-factor experiment is described by

$$p = 3, \quad k_1 = 2, \quad k_2 = 3, \quad k_3 = 3$$

and

$$N_s = k_1 \cdot k_2 \cdot k_3 = 2 \cdot 3 \cdot 3 = 18 \text{ cells}$$

A special case of the structural model occurs when the experiment is symmetrical, meaning that all factors have an equal number of levels. Therefore, when

$$k_1 = k_2 = k_3 = \cdots = k_m = \cdots = k_p$$

Eq. (1) becomes

$$N_s = k_1 \cdot k_2 \cdots k_m \cdots k_p = k^p \qquad (2)$$

To illustrate, let us consider an experiment with two factors, each at two levels, described by

$$p = 2, \qquad k_1 = k_2 = k = 2$$

Thus,

$$N_s = k^p = 2^2 = 4 \text{ cells}$$

For another example consider a case of the symmetrical model with three factors, each at two levels.

$$p = 3, \qquad k = 2$$

$$N_s = N_f = 2^3 = 8$$

Thus, Eq. (2) determines the number of cells for any symmetrical model with p factors each, at an equal number of k levels.

The design criteria that are described by the structural model are:

1. The number of factors
2. The number of levels per factor

These criteria are determined by the objectives of the experiment, the measurability of the factors, the interest in nonlinear effects, etc. They should not be dictated by any limitations upon the total number of measurements that can be made of the response variable. Such limitations, or lack of them, is the concern of the functional model.

The Functional Model

The functional model determines how many cells in the structural model will contain a response measurement. Such functional models are either complete or incomplete. A functional model is considered to be complete when all cells contain a response. A functional model is incomplete when the number of responses are systematically limited, so that the number of responses is less than the number of cells. Each of these basic types of functional models will now be discussed.

The necessary and sufficient conditions for a complete functional model are:

$$N_f = N_s = k_1 \cdot k_2 \cdots k_m \cdots k_p \qquad (3)$$

where:

N_s = the number of cells in the experiment.

N_f = the number of responses in the experiment.

k = the number of factor levels ≥ 2.

p = the total number of factors ≥ 1.

m = 1, 2, 3, ..., p.

For the special case of symmetry where

$$k_1 = k_2 = k_3 = \cdots = k_p$$

Equation (3) can be written as

$$N_f = N_s = k^p \qquad (4)$$

In both Eq. (3) and (4), it can be observed that the number of cells in the structural model (N_s) and the number of responses in the functional model (N_f) are equal. This equality is the basic characteristic of a complete model. In other words, for every cell there is a response, or

$$N_s = N_f$$

For example, given the experiment with two factors, x_1 and x_2, one at two levels and the other at three levels, we have

$$p = 2, \qquad k_1 = 2, \qquad k_2 = 3$$

$$N_s = N_f = k_1 \cdot k_2 = 2 \cdot 3 = 6$$

A functional model is incomplete when

$$N_f < N_s$$

or when the number of responses in the experimental model are determined, in some systematic manner, to be less than the number of cells. Our concern at this point is to consider the fundamental methods that are involved in designing such an incomplete model.

Functional models can be made incomplete in three fundamental ways. The first of these is the restriction of responses exponentially, so that the number of excluded responses are determined by restriction with the factors in the model. The second method for designing incomplete models is to restrict the responses linearly, so that the number of excluded responses in a model are determined by restriction with a certain number of levels of a single factor in the model. The third method consists of a combination of the first two, in which case the restriction of responses is accomplished by both exponential and linear methods. Each of these methods will now be discussed.

From Eq. (3), the necessary and sufficient conditions for an incomplete functional model whose responses are restricted with factors are

$$N_f = \frac{k_1 \cdot k_2 \cdot \cdot \cdot k_m \cdot \cdot \cdot k_p}{k_1 \cdot k_2 \cdot \cdot \cdot k_l \cdot \cdot \cdot k_q} \qquad (5)$$

where: q = the number of factors restricting the number of responses in the model. $l = 0, 1, 2, \ldots q$ (Non-negative integers.)

$m = 1, 2, \ldots, p$

When q is equal to zero, no restriction on responses exists. Consider the case of a structural model with three factors, x_1, x_2, and x_3, with

$$N_s = k_1 \cdot k_2 \cdot k_3 = 2 \cdot 4 \cdot 2 = 16$$

in which the number of responses is to be restricted by one factor, for example x_2. Therefore, we have one restricting factor, making

$$q = 1$$

and, thus from Eq. (5)

$$N_f = \frac{k_1 \cdot k_2 \cdot k_3}{k_2} = \frac{2 \cdot 4 \cdot 2}{4} = 4$$

giving four responses that are contained in the sixteen cells of the structural model.

Consider another case. Suppose that a structural model contained four factors x_1, x_2, x_3, x_4, with

$$N_s = k_1 \cdot k_2 \cdot k_3 \cdot k_4 = 4 \cdot 5 \cdot 4 \cdot 4 = 320$$

Suppose that the number of responses in the functional model is to be restricted by the two factors, x_2 and x_3. Thus, we have the l^{th} factors ($l = 2, 3$) restricting, so that

$$p = 4, \quad q = 2; \qquad k_2 = 5, \quad k_3 = 4$$

and from Eq. (5)

$$N_f = \frac{k_1 \cdot k_2 \cdot k_3 \cdot k_4}{k_2 \cdot k_3} = \frac{4 \cdot 5 \cdot 4 \cdot 4}{5 \cdot 4} = 16$$

giving that 16 responses will be contained in the 320 cells.

For the symmetrical functional model, the exponential characteristic of this restriction method becomes more apparent. From Eq. (5), when

$$k_1 = k_2 = k_m = k_p$$

and

$$k_1 = k_2 = k_l = k_q$$

then:

$$N_f = \frac{k_1 \cdot k_2 \cdots k_m \cdots k_p}{k_1 \cdot k_2 \cdots k_l \cdots k_q} = k^{p-q} \qquad (6)$$

where: $q < p$ and is a non-negative integer.

The q restriction becomes a negative exponent of the number of equal factor levels. An example is a case in which a symmetrical model contains three factors, $p = 3$, each at two levels, $k = 2$. The structural model is

$$N_s = k^p = 2^3 = 8$$

Suppose that the functional model is to be incomplete by restricting the number of responses with one factor, so that, from Eq. (6)

$$q = 1$$

$$N_f = k^{p-q} = 2^{3-1} = 4$$

giving the functional model containing a total of four responses in eight cells.

Consider another example. Suppose that for the structural model

$$N_s = k^p = 3^4 = 81$$

it is desirable to limit the number of responses in the functional model to nine. The value for q to accomplish this is determined as follows:

$$N_f = k^{p-q} = 3^{4-q} = 9 = \frac{3^4}{3^q}$$

$$3^q = \frac{81}{9} = 9$$

$$q = 2$$

The second method for restricting the responses in an incomplete functional model limits the responses within the levels of a particular factor rather than with q number of factors. This is done by subtracting the total number of blank cells for a particular factor from the total number of cells in the structural model. Thus, from Eq. (3), we have

$$N_f = k_1 \cdot k_2 \cdot \cdot \cdot k_m \cdot \cdot \cdot k_p - c_m k_m \tag{7}$$

where: c_m = the number of blank cells in each level of the m^{th} factor. (a non-negative integer)

k_m = the number of levels of the m^{th} factor.

$c_m < k_m$

The m^{th} factor can be any one of the p factors in the model. For example, a $p = 2$ model can be systematically limited by arbitrarily determining the number of blank cells to exist in each level of one of the two factors, x_1 and x_2. This is c. The number of responses, N_f, is then calculated from Eq. (7). Consider an example in which the levels for the first factor are six, $k_1 = 6$, and the levels for the second factor are three, $k_2 = 3$. If we choose to restrict the first factor, x_1, so that each factor level has one blank cell, then,

$$k_1 = 6, \quad k_2 = 3, \quad m = 1, \quad c_1 = 1$$

The number of cells are

$$N_s = k_1 \cdot k_2 = 6 \cdot 3 = 18$$

and the number of responses are, from Eq. (7)

$$N_f = k_1 \cdot k_2 - c_1 k_1 = k_1 (k_2 - c_1) = 6 (3 - 1) = 12$$

In the case of a symmetrical model, we determine the incomplete functional model from Eq. (7) to be

$$N_f = k_1 \cdot k_2 \cdots k_m \cdots k_p - c_m k_m$$

and since:

$$k_1 = k_2 = k_m = k_p$$

$$N_f = k^p - ck^{p-1} \tag{8}$$

where: c = the number of blank cells in the level of any p factor.

Eq. (8) gives emphasis to the <u>linear</u> feature of this method. In the case of the model

$$N_s = k^p = 3^4 = 81$$

we could limit the number of responses by creating blank cells in the factors. For example, with $c = 1$, we can calculate from Eq. (8)

$$N_f = k^p - ck^{p-1}$$
$$= 3^4 - (1)\ 3^{4-1} = 81 - 27 = 54$$

The model can be used in a different, and more useful, way from a design standpoint. As an example, what value of c is required to reduce the model

$$N_s = k^p = 5^5 = 3125$$

to the functional model of

$$N_f = 625$$

This is determined from Eq. (8) thus:

$$N_f = k^p - ck^{p-1} = 625$$

$$5^5 - c5^4 = 625$$

$$c = \frac{3125 - 625}{625} = 4$$

Therefore, the functional model can be restricted to 625 responses by providing for four blank cells in each factor level.

In order to further increase the possible combinations of N_f values, the third method utilizes both the q and c criteria. This can be accomplished by Eq. (6) and Eq. (8) to become

$$N_f = k^{p-q} - c\,k^{p-q-1} \qquad (9)$$

so that the number of restricting factors, q, and the number of blank cells per factor level, c, can be used to determine a particular number of responses for a given model. The application of Eq. (9) will be illustrated by an example. Suppose that it is desirable to restrict the number of responses for the model

$$N_s = k^p = 4^3 = 64$$

to eight responses. This can be done by using Eq. (9), and following a systematic procedure. First, assume c = 0, and q = 1

$$N_f = k^{p-q} - ck^{p-q-1}$$
$$= 4^{3-1} - 0 = 16$$

which is greater than the desired number. Next, keep c = 0 and assume q = 2

$$N_f = 4^{3-2} - 0 = 4$$

which is less than the desired number. Therefore, hold q = 1, and assume c = 1

$$N_f = 4^{3-1} - 4^{3-1-1} = 16 - 4 = 12$$

which is more than desired. Next, hold q = 1 and set c = 2

$$N_f = 4^{3-1} - (2)\,4^{3-1-1} = 16 - 8 = 8$$

which is the desired number of responses.

Suppose that we wanted to determine how to design a functional model with nine responses for the model

$$N_s = k^p = 3^6 = 729$$

with $k = 3$, the desired number of responses is $k^2 = 9$. It can be seen that such a value for N_f is possible in two ways. First, $N_f = k^2$ for the case when

$$q = p - 3 = 6 - 3 = 3$$

$$c = k - 1 = 3 - 1 = 2$$

Therefore, in this problem

$$q = 3, \qquad c = 2$$

$$\begin{aligned} N_f &= k^{p-q} - c\,k^{p-q-1} \\ &= 3^{6-3} - (2)\,3^{6-3-1} \\ &= 27 - 18 = 9 \end{aligned}$$

The same number of responses can be obtained with a different combination of q and c. $N_f = k^2$ is possible with

$$q = p - 2 = 6 - 2 = 4$$

$$c = 0$$

Therefore, the model becomes

$$N_f = k^{p-q} = 3^{6-4} = 3^2 = 9$$

A complete functional model is the same as a factorial experiment, with a single response in each cell. An incomplete functional model is desirable when there is no interest in interaction effects and the total number of measurements required is less than N_s. An incomplete model is necessary when the total possible number of measurements is less than N_s. More specifically, the design of the functional model is made to meet the following design criteria:

1. The total number of possible measurements is equal to or less than N_s.

2. The total number of measurements is limited to some number less than N_s because of some physical limitation of the experimental situation or equipment.

The Experimental Model

The final step in the synthesis of an experiment is to design the experimental model. The experimental model is described by

$$N = n\, N_f \qquad (10)$$

where:

n = the number of replications of the experiment

N_f = the number of responses in the functional model

N = the total number of responses in the experiment

From Eq. (9) and Eq.(10) we get

$$N = n\, N_f = n\,(k^{p-q} - c\, k^{p-q-1}) \qquad (11)$$

which provides a general expression for a symmetrical experimental model.

Eq. (10) thus defines the experimental model as follows:

> The total number of responses in an experiment is a function of the number of factors (p), the number of factor levels (k), the number of factor restrictions (q), the number of cell restrictions (c), and the number of replications (n).

Given the number of factors and factor levels, the number of possible values for N can be determined by certain combinations of values for n, q, and c. For example, if it is desirable to design an experimental model with 54 responses of the type

$$N_s = k^p = 3^4 = 81$$

we can set q = 0, c = 1, and n = 1 and get

$$\begin{aligned} N &= n\,(k^{p-q} - c\,k^{p-q-1}) \\ &= 1\;3^{4-0} - (1)\;3^{4-0-1} \\ &= 81 - 27 = 54 \end{aligned}$$

Table 1 provides a general tabulation of the experimental model in Eq. (11)

TABLE 1. Values of N for All Values of q, c, and n in a Symmetrical Experimental Model

q	c					
	k	$k - 1$	$k - 2$	$k - 3$	. . .	0
p	0	n/k	$2\,n/k$	$3\,n/k$	. . .	n
$p - 1$	0	n	$2\,n$	$3\,n$	. . .	nk
$p - 2$	0	nk	$2\,nk$	$3\,nk$	. . .	nk^2
$p - 3$	0	nk^2	$2\,nk^2$	$3\,nk^2$	. . .	nk^3
⋮	⋮	⋮	⋮	⋮		⋮
0	0	nk^{p-1}	$2\,nk^{p-1}$	$3\,nk^{p-1}$	. . .	nk^p

The number of responses, N, for a symmetrical experimental model can be determined if given the values for p, k, q, c, and n. As an example of its use, suppose that we have a model with $p = 5$ factors and each factor has $k = 5$ levels so that

$$N_s = 5^5 = 3,125$$

Assume that the experimental model is to contain forty-five responses. The responses are first limited by

$$q = p - 2 = 5 - 2 = 3 \text{ factors}$$

Also, the responses are further restricted by

$$c = k - 3 = 5 - 3 = 2 \text{ blank cells per factor}$$

When $q = p - 2$ and $c = k - 3$

$$N = 3\,nk$$

and with $k = 5$

$$N = 3\,n(5) = 15\,n = 45$$

$$n = 3$$

<u>Selection of Optimal Alternate Designs</u>

The experimental model provides the specifications necessary for the final experimental design to meet the established design criteria, as to total number of responses. Such a selection is not concerned with the problems of balancing the responses in the cells or randomizing the arrangement of the responses. These are considerations made in certain standard designs with which the subject design procedure is not concerned.

The synthesis of any experiment can be described by its experimental model. For example, a <u>complete factorial</u> experiment is described by the following necessary and sufficient conditions:

$$n = 1, 2, 3, \ldots, \quad k = 2, 3, 4, \ldots, \quad p = 2, 3, 4, \ldots, \quad q = 0, \quad c = 0$$

A specific case of such a factorial experiment is

$$n = 1, \quad p = 3, \quad k = 4, \quad q = 0, \quad c = 0$$

and from Eq. (11)

$$N = n\,(k^{p-q} - c\,k^{p-q-1}) = (1)\,(4^3) = 64$$

Examples of other models for certain traditional experiments are listed as follows:

1. A <u>one-way classification experiment</u> with five responses in each of four columns of a single factor is described by

$$n = 5, \quad p = 1, \quad k = 4, \quad q = 0, \quad c = 0$$

$$N = n\,(k^{p-q} - c\,k^{p-q-1})$$

$$= (5)\,(4^{1-0}) = 20$$

Such a model thus explains the one-way classification experiment as a single factor experiment that is replicated.

2. Consider a <u>nested</u> experiment with three factors: x_1, with two levels, x_2, with four levels, and x_3 with two levels. Factor x_2 is such that only half of its levels are crossed with each of the two levels of x_1. Thus:

$$p = 3, \quad k_1 = 2, \quad k_2 = 4, \quad k_3 = 2$$

From Eq. (1)

$$N_s = k_1 \cdot k_2 \cdot k_3 = 2 \times 4 \times 2 = 16$$

Since factor x_1 restricts the number of responses in the experiment

$$q = 1, \quad k_1 = k_1 = 2$$

and, thus, from Eq. (5)

$$N_f = \frac{k_1 \cdot k_2 \cdot k_3}{k_1} = \frac{2 \cdot 4 \cdot 2}{2} = 8$$

with only one replicate

$$N = n\,N_f = (1)(8) = 8$$

A hierarchical layout becomes

Factor X_1															
1								2							
Factor X_2								Factor X_2							
1		2		3		4		1		2		3		4	
X_3		X_3		X_3		X_3		X_3		X_3		X_3		X_3	
1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
X	X	X	X									X	X	X	X

A matrix layout becomes

		X_1			
		1		2	
		X_2			
		1	2	3	4
X_3	1	X	X	X	X
	2	X	X	X	X

3. Consider a <u>Latin Square</u> experiment. The necessary and sufficient conditions for this symmetrical restricted model are

$$p = 3, \quad q = p - 2, \quad k \geq p - 1, \quad c = 0$$

The minimum case occurs when there are three factors, each at two levels, with

$$p = 3, \quad k = 2, \quad q = p - 2 = 1$$

and the number of cells and responses are

$$N_s = k^p = 2^3 = 8$$

$$N_r = k^{p-q} = 2^2 = 4$$

so the experiment contains a total of four responses in eight cells.

A hierarchical layout becomes

X_1							
1				2			
X_2				X_2			
1		2		1		2	
X_3		X_3		X_3		X_3	
a	b	a	b	a	b	a	b
X			X		X	X	

A matrix layout becomes

		X_1	
		1	2
X_2	1	a	b
	2	b	a

4. A <u>Graeco-Latin Square</u> experiment is described by the following necessary and sufficient conditions

$$c = 0, \quad k \geq 3, \quad p = 4, \quad q = p - 2 = 2$$

For the case in which $k = 3$, the number of cells in the structural model would be

$$N_s = k^p = 3^4 = 81$$

and the number of responses would be

$$N_f = k^{p-q} = 3^{4-2} = 9$$

Only one replicate is taken. Thus,

$$N = n\,N_f = (1)\,(9) = 9$$

5. An <u>incomplete block</u> experiment is represented by the incomplete functional model whose necessary and sufficient conditions are

$$N = n\,(k_1 \cdot k_2 \cdots k_m \cdots k_p - c_m k_m)$$

where:

$$p = 2, \quad k_1 \geq 3, \quad k_2 \geq 3, \quad n = 1$$

Thus:

$$N = k_1 \cdot k_2 - c_m k_m$$

where: $m = 1$ or 2

c_m = the number of black cells in the j_mth level.

One of the two factors is a block.

An example would be a model as follows:

$$k_1 \text{ (blocks)} = 6 \qquad k_2 \text{ (treatments)} = 3$$

$$c_1 = 1, \quad c_2 = \frac{k_1}{k_2}\, c_1 = \frac{6}{3}(1) = 2$$

Therefore:

$$N_s = k_1 \cdot k_2 = 6 \cdot 3 = 18 \text{ cells}$$

$$n = 1$$

$$N = k_1 \cdot k_2 - c_2 k_2 = 6 \cdot 3 - 2(3) = 12$$

so the twelve responses are to be balanced in the eighteen cells of the incomplete block design.

6. A symmetrical incomplete block experiment is described by the necessary and sufficient conditions from Eq. (8)

$$k \geq 3, \quad p = 2, \quad c = 1, 2, \cdots, k-1$$

$$n = 1$$

$$N = k^2 - ck$$

where: One of the two factors is a block.

7. A Youden Square experiment is described by the following necessary and sufficient conditions from Eq. (9)

$$N = n\,(k^{p-q} - c\,k^{p-q-1})$$

$$k \geq 2, \quad p = 3, \quad q = 1, \quad c = 1, 2, \cdots, k-1$$

$$n = 1$$

A specific example is a case in which

$$p = 3, \quad k = 4, \quad q = 1, \quad c = 1, \quad n = 1$$

$$N_s = k^p = 4^3 = 64$$

$$N = k^{p-q} - c\,k^{p-q-1}$$

$$= k^{p-q}\,(1 - \tfrac{c}{k}) = 4^{3-1}\,(1 - \tfrac{1}{4}) = 12$$

8. A <u>lattice square experiment</u> is described as a type of incomplete block (See 6 above) that is replicated. Thus

$$N = n\,(k_1 \cdot k_2 - c_2 k_2)$$

where: $2 \leq n \leq t + 1; \quad t \geq 3$

$k_1 \geq m\,t$ (blocks)

$k_2 = t^2$ (treatments)

$c_2 = k_1 - 1$

Example:

$$t = 3, \quad n = 4, \quad k_1 = 12, \quad k_2 = 9, \quad c_2 = 11$$

$$N_s = 12 \cdot 9 = 108$$

$$N = 4\,(12 \cdot 9 - 11 \cdot 9) = 36$$

Another example of a lattice square will demonstrate the relationship between the structural, functional, and experimental model more clearly. A 13 x 13 balanced lattice square will be used to illustrate. ($t = 13$, $n = 7$.)

Structurally speaking, the experiment consists of two factors: blocks at $k_1 = 21$ levels and treatments at $k_2 = 169$ levels. Thus

$$N_s = k_1 \cdot k_2 = 21 \cdot 169 = 3{,}549 \text{ cells}$$

The functional model is incomplete and its characteristic feature is that only one treatment can occur in each replicate. Thus, the functional model is restricted so that for each treatment all cells are empty except one. Thus $c_2 = 20$, giving

$$N_f = k_1 \cdot k_2 - c_2 k_2 = 3,549 - 20(169) = 169$$

The experiment is replicated 7 times, thus the experimental model becomes

$$N = n N_f = 7(169) = 1{,}183 \text{ responses}$$

which is the total number of responses to be balanced.

9. Following are a number of miscellaneous <u>incomplete</u> <u>block</u> <u>designs</u> with their corresponding structural, functional, and experimental models.

BALANCED DESIGN FOR 9 TREATMENTS IN BLOCKS OF 3 UNITS

Block	Rep. I		Rep. II		Rep. III		Rep. IV
(1)	1 2 3	(4)	1 4 7	(7)	1 5 9	(10)	1 8 6
(2)	4 5 6	(5)	2 5 8	(8)	7 2 6	(11)	4 2 9
(3)	7 8 9	(6)	3 6 9	(9)	4 8 3	(12)	7 5 3

$$p = 2, \quad k_1 = 9, \quad k_2 = 12$$

$$N_s = k_1 \cdot k_2 = 9 \cdot 12 = 108$$

$$c_1 = 11$$

$$N_f = N_s - c_1 k_1 = 108 - (11)\,9 = 9$$

$$n = 4$$

$$N = n N_f = 4(9) = 36$$

BALANCED DESIGN FOR 7 TREATMENTS IN BLOCKS OF 3 UNITS

Block							
(1)	1 2 4	(3)	3 4 6	(5)	1 5 6	(7)	1 3 7
(2)	2 3 5	(4)	4 5 7	(6)	2 6 7		

$$p = 2, \qquad k = 7$$

$$N_s = k^p = 7^2 = 49$$

$$c = 4$$

$$N_f = N_s - c\,k^{p-1} = 49 - (4)\,7 = 21$$

$$n = 1$$

$$N = n\,N_f = (1)\,21 = 21$$

BALANCED DESIGN FOR 9 TREATMENTS IN 4 LATTICE SQUARES

	Rep. I		Rep. II		Rep. III		Rep. IV
				Columns			
Rows	(1)(2)(3)		(4)(5)(6)		(7) (8) (9)		(10)(11)(12)
(1)	1 2 3	(4)	1 4 7	(7)	1 6 8	(10)	1 9 5
(2)	4 5 6	(5)	2 5 8	(8)	9 2 4	(11)	6 2 7
(3)	7 8 9	(6)	3 6 9	(9)	5 7 3	(12)	8 4 3

$$p = 3, \quad k_1 = 9, \quad k_2 = 12, \quad k_3 = 12$$

$$N_s = k_1 \cdot k_2 \cdot k_3 = 9 \cdot 12 \cdot 12 = 1{,}296$$

$$c_1 = 143$$

$$N_f = N_s - c_1 k_1 = 1{,}296 - 143\,(9) = 9$$

$$n = 4$$

$$N = n\,N_f = 4\,(9) = 36$$

BALANCED DESIGN FOR 7 TREATMENTS IN AN INCOMPLETE LATIN SQUARE

	Columns (Blocks)						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Rows							
(1)	1	2	3	4	5	6	7
(2)	2	3	4	5	6	7	1
(3)	4	5	6	7	1	2	3

$$p = 3, \quad k = 7$$

$$N_s = k^p = 7^3 = 343$$

$$q = 1, \quad c = 4$$

$$N_f = k^{p-q} - c\, k^{p-q-1} = 7^{3-1} - (4)\, 7^{3-1-1} = 21$$

$$n = 1$$

$$N = n\, N_f = 21$$

10. A <u>composite design</u> is used to estimate the regression coefficients for a second degree ploynomial. These designs are traditionally constructed by adding further treatment combinations to those obtained from a 2^p factorial. Such designs are described here as replicated incomplete models with $N_s = 3^p$. Such an approach recognizes the necessity for three factor levels to measure second degree effects. Thus

$$q = p - 1, \quad c = 2$$

$$N_f = 3^{p-p+1} - (2)\,(3)^{p-p} = 3 - 2 = 1$$

$$n = 2^p + 2\,p + 1 \quad \text{(central design)}$$

$$n = 2^p + 2\,p + p \quad \text{(noncentral design)}$$

$$N = n\, N_f = 2^p + 2p + 1 \quad \text{(central)}$$

$$N = n\, N_f = 2^p + 2p + p \quad \text{(noncentral)}$$

<u>Examples</u>

1. Design Criteria. A flight vehicle trajectory is to be designed so that a multistage rocket may place a payload into a circular orbit about the earth. An experiment is to be designed to determine how the first stage booster thrust program affects the amount of mass which is injected into a circular orbit with an altitude of 100 miles. The thrust program of the booster consists of programmed adjustments in the angle of attack for a given time period. The length of the first stage burn, t_b, is determined by the propellant loading of the first stage. The parameters that control the thrust program are (1) the initial rate of increase of the angle of attack, R, and (2) the length of time this rate is flown, t_1.

The factor, R, is to be set at six different rates of increase and t_1 is set at four time levels. The flight is to be simulated on a digital computer. Previous experience in similar studies indicates that each run (response) requires about 3/4 of a minute of computer time. The computer "turn around" time is very slow. It is necessary to obtain the maximum priority time required on the computer, which is 10 minutes. Therefore, a maximum of 13 runs can be made. There is no interest in interaction effects.

Model Synthesis. The structural model, according to the above design criteria, is determined to be

$$p = 2 \text{ factors}, \quad k_1 = 6 \text{ levels}, \quad k_2 = 4 \text{ levels}$$

$$N_s = k_1 \cdot k_2 = 6 \cdot 4 = 24 \text{ cells}$$

The maximum number of responses requires an incomplete functional model restricted by the c criteria. Thus

$$N_f = k_1 \cdot k_2 - c_1 k_1$$

Setting $N_f = 13$ (maximum) we calculate

$$c_1 = \frac{6 \cdot 4 - 13}{6} = 11/6$$

which is not an integer. Using $N_f = 12$ we get

$$c_1 = 2$$

Therefore, the number of blank cells per level of factor R is 2, providing a total of 11 degrees of freedom. With the two main effects requiring 5 and 3 degrees of freedom, respectively, the experimental error is estimated with three degrees of freedom since the experiment is not replicated. Thus

$$n = 1$$

$$N = n N_f = (1)(12) = 12$$

2. Design Criteria. Two different analogue-to-digital converters are contained in test stations used in checking out a particular instrument unit. An experiment is designed to determine the causes of variation in the digital output of these converters. The response variable is the difference between input voltage and output voltage. The variables to be measured are (1) input voltage, (2) converter units, and (3) adjustments. The input voltage is to be set at two levels, -10 volts and +10 volts. The number of converters are limited to two. The adjustments consist of gain and balance settings as specified by the manufacturer. Four different adjustments will be made. The adjustments are unique with each unit and, therefore, they cannot be duplicated between the two converters. Thus, the first two adjustments will be unique with the first unit and the second two adjustments will be unique with the second unit. All possible interactions are to be measured. The optimal degrees of freedom for the error estimate, considering cost of experimentation and desired decision confidence levels, has been determined to be 16 in a previous study.

Model Synthesis. The structural model is

$$p = 3, \qquad k_1 = 2, \qquad k_2 = 2, \qquad k_3 = 4$$

$$N_s = k_1 \cdot k_2 \cdot k_3 = 2 \cdot 2 \cdot 4 = 16$$

The converter factor restricts the adjustment factor thus providing conditions for an incomplete functional model, restricted by the q criteria. Thus

$$c = 0, \qquad q = 1 \text{ (converters, } k_2 = 2)$$

$$N_f = \frac{k_1 \cdot k_2 \cdot k_3}{k_2} = \frac{2 \cdot 2 \cdot 4}{2} = 8$$

The optimal degrees of freedom for the error estimate, 16, is provided by replicating the functional model. Thus

$$n = 3$$

$$N = nN_f = 3(8) = 24$$

The degrees of freedom are partitioned as follows:

Source of Variation	Degrees of Freedom
Converters (C)	1
Adjustments (A)	2
Voltage (V)	1
C V	1
A V	2
Error	16
Total	23

A layout of the selected experiment, which is a nested factorial, is shown in Table 2.

	Converters			
	1		2	
	Adjustments		Adjustments	
Voltage	1	2	3	4
-10	X	X	X	X
	X	X	X	X
	X	X	X	X
+10	X	X	X	X
	X	X	X	X
	X	X	X	X

Table 2. Layout of Analogue-to-Digital Converter Experiment

3. Design Criteria. An electronic manufacturer has designed a component board using four capacitors to establish a time base. He wishes to test five different brands of the capacitors in the component boards. Four capacitors are placed in parallel and then connected through a resistor to an input plug where a fixed voltage may be applied. The voltage across the capacitors is connected to an output jack. The test is made by applying a fixed voltage to the plug at the input of the component board. The output jack is monitored with an oscilloscope to measure the time required for the output voltage to rise to a specified amplitude.

The response variable (T) is the time required for the output of the component board to rise to a specified amplitude upon application of a fixed input voltage. The factors of interest are capacitor brands (C) and component boards (B). Since both factors are qualitative, nonlinear effects are not applicable. Also, past experience in tests of this type has shown negligible interaction between the capacitors and component boards. Since information is desired on the capacitors only, the same resistor will be used for each test. A set of terminals allows the resistor to be plugged in or removed from the component board. Five different capacitor brands are, therefore, to be tested in a circuit that is limited to four capacitors. A minimum of 10 degrees of freedom is required to make an error estimate.

Model Synthesis. Since five brands are being tested, it would seem reasonable to test these brands in five different component boards. We, therefore, have a symmetrical model. The structural model is

$$p = 2, \qquad k = 5$$

$$N_s = k^p = 5^2 = 25$$

Since there are four capacitors in the circuit but five different brands we will have one missing value in each level of capacitor brand. Thus, the functional model is incomplete with

$$q = 0, \quad k = 5, \quad p = 2, \quad c = 1$$

$$N_f = k^p - c\, k^{p-1} = 5^2 - (1)\, 5^{2-1} = 20$$

The degrees of freedom are

Source of Variation	Degrees of Freedom
Capacitors (C)	4
Boards (B)	4
Error	12
Total	20

Only one replicate is required since the minimum of 10 degrees of freedom is met. The experimental model is

$$n = 1$$

$$N = n N_f = (1)(20) = 20$$

A balanced layout of the experiment is shown in Table 3, as an incomplete block design.

Component Boards	Capacitor Brands				
	1	2	3	4	5
I	X		X	X	X
II	X	X	X	X	
III	X	X		X	X
IV		X	X	X	X
V	X	X	X		X

TABLE 3. Incomplete Block Design for Capacitor Experiment

<u>Summary</u>

The modeling of experiments has been described as a three-phase process, namely

1. Designing the structural model
2. Designing the functional model
3. Designing the experimental model

The structural model determines the number of cells in the experiment as a function of the number of factors and the levels for each factor. For the symmetrical case the structural model is

$$N_s = k^p$$

The functional model determines the number of responses to be taken in the structural model. A complete symmetrical functional model is expressed as

$$N_f = N_s = k^p$$

A functional model can be incomplete in three ways. First, if the responses are restricted by q number of factors, the symmetrical functional model becomes

$$N_f = k^{p-q}$$

Second, if the responses are restricted by c cells within a factor, the symmetrical functional model becomes

$$N_f = k^p - c\,k^{p-1}$$
$$= k^p\,(1 - \frac{c}{k})$$

Third, if the responses are restricted by both q and c the symmetrical functional model becomes

$$N_f = k^{p-q} - c\,k^{p-q-1}$$
$$= k^{p-q}\,(1 - \frac{c}{k})$$

The final experimental model is defined as

$$N = n\,N_f$$

for the symmetrical case, where n is the number of replications. All types of matrix experiments can be described by such models.

The unified procedure for selecting alternate experimental designs can be summarized as

1. Determine experimental design criteria
2. Synthesize the experimental model
3. Compare model to standard experimental designs and choose the optimal design.

A PROBLEM IN CONTINUOUS SAMPLING VERIFICATION

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There are basically two types of sampling inspection procedures in use today. These are lot-by-lot and continuous sampling procedures. In addition to the two types of sampling inspection, there are also two different methods of inspection, namely, by attributes and by variables. Inspection by attributes is on a go-no-go basis. That is, a unit of product is inspected and determined to be either satisfactory or unsatisfactory with respect to the characteristic under consideration. Under inspection by variables, the actual value of the measurement of a measurable characteristic is recorded. Several of these measurements might then be used together to estimate some parameter upon which a lot of product may be judged relative to its conformity to specification requirements. Our discussion shall be limited to inspection by attributes.

Under inspection by attributes, the inspection can be performed on a lot-by-lot basis or continuously. Let us first consider the lot-by-lot case. The units of product are divided into identifiable lots, and a lot is judged either conforming or nonconforming on the basis of the number of defective units found in a sample from the lot.

One of the most widely used Military Standards listing sampling plans for this type of inspection is MIL-STD-105D, "Sampling Procedures and Tables for Inspection by Attributes," 29 April 1963. When using this Standard, a sampling plan is determined by the following:

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a. the size of the lot,

b. the specified acceptable quality level (AQL),

c. the specified Inspection Level (when none is specified, Inspection Level II is used), and

d. the type of plan specified or approved for use (single, double, or multiple).

The size of the lot may be stated in the specifications, or it may be up to the supplier, subject to approval by the consumer, to determine a suitable lot size. The AQL is the maximum percent defective of product which can be considered satisfactory for the process. For example, in this Standard possible AQL values are .010%, 1.0% and 10%. Once a plan has been determined, the plan parameters (sample sizes and acceptance and rejection numbers) can be found.

As an example of a lot-by-lot plan, consider a single sampling plan where the lot size is 1000, the sample size is 100, the acceptance number is 3 and the rejection number is 4. Then, under this plan a random sample of 100 units would be selected from the lot. The number of defective units would be counted, and if the number were 3 or less the lot could be submitted to the consumer for acceptance. If, however, the number of defective units in this sample were 4 or more, then the lot could not be submitted to the consumer for acceptance, and it must be rejected.

Let us now turn our attention to continuous sampling inspection. A limited Standard which defines various types of these sampling plans is MIL-STD-1235(ORD), "Single and Multilevel Continuous Sampling Procedures and Tables for Inspection by Attributes," 17 July 1962. It is a limited Standard in that it is applicable only to the Army. This Standard is a composite of Inspection and Quality Control Handbooks (Interim) H106, "Multi-level Continuous Sampling Procedures and Tables for Inspection by Attributes," 31 October 1958 and H107, "Single-level Continuous Sampling Procedures and Tables for Inspection by Attributes," 30 April 1959.

In order to use these plans the following criteria must be met:

a. the units of product must be moving, which means that they must pass by the inspection station by means of a conveyor belt or some other conveyance, such as a tote box or skid,

b. the process must produce homogeneous material or be capable of producing homogeneous material,

c. there must be relative ease of inspection, and

d. there must be ample physical facilities for rapid 100% inspection.

All continuous sampling plans are characterized by periods of screening and sampling. The simplest CSP plan is designated CSP-1 and was developed by Dodge (See Annals of Mathematical Statistics, Sept., 1943). Under this plan, 100% inspection (screening) is performed until i consecutive good units have passed inspection. The prescribed value "i" may be some value between 4 and 2000, depending upon the specific plan being used. After i consecutive good units have passed inspection, sampling is begun at a certain prescribed frequency, f. The value of f may be some value between 1/2 and 1/200, again depending upon the specific plan being used. Since each unit of product should have an equal chance of being selected, the interval between the sampled units should vary somewhat. Sampling is continued until a defective unit is found. When this occurs, screening (100% inspection) begins and continues until i consecutive good units have passed inspection, at which time sampling will again be introduced.

A sampling plan under MIL-STD-1235(ORD) is also determined by the following factors:

a. the number of units in a production interval,

b. the specified AQL, and

c. the specified Inspection Level (when none is specified, Inspection Level II is used) and

d. the type of continuous sampling plan specified or approved for use (CSP-1 or one of the other types of plans provided in the Standard).

The production interval is that period of time, usually a day or shift, during which conditions of manufacture can reasonably be expected to remain stable. Of the four continuous sampling procedures provided in MIL-STD-1235(ORD) CSP-1 is the simplest. It will be the only one considered here.

As an example of a CSP-1 plan, consider one in which $i=20$ and $f=1/10$. Screening would be performed until 20 consecutive good units had passed inspection. When this had been accomplished, sampling could begin at the rate 1 in 10. This means that the sampling inspector would select 1 out of 10 units but would vary the interval between these selected units to give each unit of product an equal chance of being included in the sample. Sampling would continue until a defective unit is found. At that time screening would again be instituted, and it would be necessary to screen 20 consecutive good units before sampling could be resumed again.

Verification of the supplier's inspection records is advantageous to the consumer because he would like to ascertain that the supplier is following the inspection plan and classifying inspected units properly. That is, inspected units which are defective should be classified defective and inspected units which are non-defective should be classified non-defective. In order to achieve this aim, AMSMU-P-715-503, "Army Ammunition Plant Quality Assurance Procedures," December, 1966, describes the appropriate procedures to be used by Army Ammunition Plants for verification purposes. This document is designed to be used in conjunction with either lot-by-lot or continuous sampling inspection, and can therefore be used with MIL-STD-105D or MIL-STD-1235(ORD). In the lot-by-lot case, it is a relatively easy matter to perform verification. First, the supplier selects a random sample from the lot in question and counts the number of defective units in this sample. He then compares the number of defective units to the acceptance number for his specified sampling plan from MIL-STD-105D. If the number of defectives is equal to or less than the acceptance number, the lot may be submitted to the consumer for acceptance. The consumer takes a sample from the lot, and counts the number of defective units. The consumer is then ready to compare his results with those of the contractor using Table I of Quality Control and Reliability Handbook (Interim) H109, "Statistical Procedures for Determining Validity of Suppliers' Attributes Inspection," 6 May 1960. For purposes of this comparison, it is assumed that the consumer has classified all of his sample units properly. The H-109 comparison is in effect a test of significance between the number of defectives found by the supplier and the number of defectives found by the consumer, given a certain value r, which is the ratio of the supplier's sample size to the consumer's sample size. Rejection under this test will cause the supplier's data to be considered invalid.

Verification of inspection results when the sampling inspection is done by continuous sampling procedures is more complicated. Under the provisions of MIL-STD-1235(ORD) and AMSMU-P-715-503, the supplier performs checking inspection at rate f during all periods of screening, in order to ascertain that the screening crew is doing an efficient job. The units inspected during this checking inspection plus the units inspected by the supplier's sampling inspector form the supplier's sample for comparison purposes, where the period under consideration is a production interval.

Concurrently with the inspection by the supplier described above, the consumer is performing verification inspection at rate (1/r)f, where r is the ratio of comparison sample sizes described previously and f is the prescribed sampling frequency. The method of determining the particular value of r (1, 2, 3, 5 or 8) to be used is outlined in AMSMU-P-715-503 and is not important to our discussion here, since we will only concern ourselves with the case r=8.

The various types of inspection described above are summarized in Table I. Reviewing the Table, and from the preceding discussion, it can be noted that only one type of inspection is performed by the consumer, namely, verification inspection, and this is done at a definite sampling frequency which is proportional to that used by the supplier (in the case to be considered here, the proportion is one-eighth). The units inspected in this manner constitute the consumer's verification sample which is used for comparison purposes with that compiled by the supplier. Again, the sample, which for convenience shall be called the comparison sample hereafter, is composed of units which may have come from the screening or sampling phase with the proportion of units from any phase for a production interval dependent upon the amount of time spent on this phase by the supplier. The consumer usually has no knowledge as to which units came from which phase since verification inspection might be performed at a place far removed from the inspection conducted by the supplier.

Let us consider how these inspections function. Since we are considering only continuous operations under CSP-1 procedures, the units of product will be moving past the various inspection stations via conveyor belts, tote boxes or some other conveyance. Let us first consider the supplier's function. As the operation begins, the product is inspected 100% to remove any defective units and to see if i consecutive good units can be found. Concurrently with this initial product inspection is checking inspection which is performed at a rate f (the specified sampling frequency) and is a means of checking the effectiveness of the screening operation. The units sampled during this checking inspection will form part of the supplier's comparison sample. Once i consecutive good units have been found, sampling inspection of the product is initiated. This sampling of the units of product is done in a random manner at some specified sampling frequency, f. The units sampled form the remainder of the supplier's comparison sample.

Let us now review the consumer's inspection function. As can be seen from Table I, there is only one type of inspection which the consumer performs, namely, verification inspection. This inspection is done concurrently with the supplier's inspection. The point at which the consumer conducts this inspection may be far removed from the site of the supplier's inspection operations. Since the units of product are not marked or designated as to which units came from which phase, the consumer generally is ignorant of this information. The consumer samples the units in a random manner at a sampling frequency which is proportional to the sampling frequency used by the supplier. This value of the sampling frequency is $[(1/r)(f)]$, where $1/r$ is the proportional factor (one-eighth for purposes of discussion here) and f is the prescribed sampling frequency. Because the sampling is done in a random manner without requiring a certain number or percentage of the inspected units to be from any one phase, there might be a considerable difference in the proportion of units from one of the phases for the consumer and supplier during the production interval.

To use Table I of H-109 to compare $d_s (= d_{s,100} + d_{s,f})$ with d_c $(= d_{c,100} + d_{c,f/8})$, the probability of accepting the hypothesis of validity should remain the same as reflected on the O.C. curves (See Figure I, extracted from H-109) for the test to be of the level α and probability of acceptance over the parameter space as shown on the O.C. curves. By way of explanation the parameter under consideration is the ratio of fractions defective, p_c/p_s, which can be thought of as

$$\frac{\text{Prob (defective inspected unit will be classified defective by consumer)}}{\text{Prob (defective inspected unit will be classified defective by supplier)}}$$

This, then, is our problem: To show that the probability of accepting the hypothesis of validity over the parameter space is approximately the same as that shown on the O.C. curves.

To simplify the remainder of the discussion and the problem definition, the notation below shall be used.

Let

$n_{s,100}$ = number of units in supplier's comparison sample coming from the screening phase,

$n_{s,f}$ = number of units in supplier's comparison sample coming from the sampling phase,

$n_{c,100}$ = number of units in consumer's comparison sample coming from the screening phase,

$n_{c,f/8}$ = number of units in consumer's comparison sample coming from the sampling phase.

Let d, subscripted as above, refer to the number of defective units found in the portion of units identified by the subscripts.

Let us now reflect on some aspects of the problem.

Since there are two phases, namely, the screening phase and the sampling phase, from which the verification sample as well as the supplier's comparison sample can come, there is a possibility of considerable variation between the two in the proportion of units from any one phase. That is, for example,

$$\frac{n_{s,100}}{n_{s,100} + n_{s,f}}$$ might be considerably

different from $$\frac{n_{c,100}}{n_{c,100} + n_{c,f/8}} .$$

Let us now consider only one value of the parameter space, $p_c/p_s = 1$, which is equivalent to saying that the supplier has perfect inspection efficiency. Then no defectives should be found in the samples from the screening phase since these should have been removed during the screening phase of product inspection. Hence, any defective which would be found in either of these samples would come from the sampling phase.

As reflection will show, if the units comprising the samples were selected completely independently of order or position in the production interval, we would have a situation equivalent to a lot-comparison situation, and the O.C. curves would be exactly as defined for H-109. Further, if the proportions described previously were exactly the same, that is, the fraction of the supplier's comparison sample coming from the screening phase were exactly the same as the fraction of the consumer's phase, we would have essentially a stratified sampling problem, and again the O.C. curves would be exactly as defined in H-109.

Since the prescribed method of sampling, however, is to take about one out of every 1/f units, allowing the interval between inspected units to vary somewhat, we have neither of the situations described above. This brings us to the reason why we are only considering the case r=8. It is reasonable to assume that the greatest variation from the O.C. curves of H-109 is possible for the largest value of r. Therefore, if this variation is insignificant for r=8 it should be insignificant for the lower value of r. Let us now consider a specific example.

Since screening need only be done at the initiation of production, and thereafter only when a defect is found during a period of sampling inspection, it is not necessary in our example to assume that screening is initiated at the start of the production interval, but for sake of discussion let us assume that it does. Suppose the supplier is sampling at a frequency of 1/10, and the consumer is using a ratio of r=8. Therefore, the consumer would be sampling at a frequency of 1/80. First, the supplier's screening crew inspects all units of product until the appropriate number of consecutive good units has been cleared. At the same time, the checking inspector is selecting one unit out of ten in a random manner to see if the screening crew is doing its job properly. After the necessary number of consecutive good units has been cleared, sampling inspection is begun whereby one out of ten units is selected for inspection. There is no checking inspection during this phase.

During the entire production interval, the consumer's verification inspector selects one out of eighty units in a random manner. At the completion of the production interval, the supplier's and consumer's comparison sample inspection results can be compared. The supplier's sample consists of those units inspected by the checking inspector during the screening phase plus the units inspected by the supplier during the sampling phase. The consumer's sample consists of all units inspected by the verification inspector, whether these units came from the screening or sampling phase.

Let us assume that the production interval encompasses 80 units and 76 of these units were subjected to screening while the remaining 4 units were part of the sampling phase. Let us suppose the sampling frequencies are as above, namely, f=1/10 for the supplier and f=1/80 for the consumer.

Reflection will show that there are many possible variations in the values of $n_{s,100}$, $n_{s,f}$, $n_{c,100}$, and $n_{c,f/8}$. It is possible, for example, that all of the units for the supplier's sample came from the screening phase while the single unit composing the consumer's sample came from the sampling phase. In this case, the proportion of units in the supplier's sample from the screening phase is 1.0 whereas the corresponding proportion of units in the consumer's sample from the same phase is 0.

Since the probability of each possible variation is not known, since strict probabilistic sampling is not performed, the effective O.C. curve cannot be determined simply.

Ideally then, a mathematical model describing the O.C. curves would be desirable.

In lieu of such a mathematical model, we conducted a Monte Carlo simulation of the process. Twenty different simulations of various CSP-1 and CSP-2 plans were considered. A few selected AQL's ranging from 0.01% to 4.0% were used, with production intervals ranging from 70 units to 1000 units. The value of p was set equal to the AQL in each case on these first attempts. Ten production intervals were considered for each simulation. Finally, it was assumed that the screening crew was 100% efficient, i.e., all defective units were removed during the screening phase.

Random numbers were used to designate the defective units. Once this had been determined, the inspection processes could be simulated. First, the units from the initial screening phase were identified, and then random numbers were used to select the first unit to be sampled by the supplier.

For convenience on these first attempts, a systematic sample followed the random selection of the first unit. When all ten production intervals had been completed in this manner, the units inspected by the checking inspector during the screening phase needed to be specified. Random numbers were again employed to designate the initial units sampled during these phases and systematic sampling ensued. When this had been completed, the proportion of units from the screening phase for each production interval and for the ten production intervals as a whole could be calculated. Then, the consumer's inspection had to be simulated. Since the sampling was done at a specified sampling frequency without regard as to which phase the supplier was on, a random number was used to indicate the first unit of the sample, and a systematic sampling followed for the duration of the ten production intervals. Upon the completion of the ten production intervals, the proportion of units from the screening phase for each production interval and for the ten production intervals as a whole could be tallied. These proportions could then be compared to the corresponding one for the supplier. Table 2 shows the results of one of these simulations. For this simulation, the production size was 70; the AQL was 2.5%; the i value was 25; the supplier's sampling frequency was 1/5, and the consumer's sampling frequency was 1/40. There does not appear to be too much difference between the proportions except for the seventh production interval where the supplier's proportion was .357, and the consumer's proportion was 0.

In order to use the O.C. curves from H-109, some calculations needed to be performed. The fractions defective for the supplier and consumer as well as the expected number of defective units in the supplier's sample needed to be specified. Since it was assumed that the screening crew was 100% efficient, theoretically no defective units should have appeared in either the supplier's or the consumer's sample from the screening phase. Therefore, the fraction defective for either the supplier or consumer is the proportion of units from the screening phase times the appropriate AQL (since p was set equal to the AQL, as mentioned previously). Then, the ratio of the consumer's fraction defective to the supplier's fraction defective was calculated. Finally, the expected number of defective units in the supplier's sample was estimated by the number of units in the production interval times the fraction defective described above. The results of these computations for each of the ten production intervals and for the ten production intervals as a whole are summarized in Table 3. The last two columns are of more interest. It will be noted that most of the ratios are around 1.0 except for production interval #7 where the ratio is 2.6040.

Note that all of the expected number of defective units in the supplier's sample for our example are considerably less than the smallest value, indexing the H-109 curves (see figure at end of paper), namely, 0.75. Hence, the O.C. curves for these figures would be above that for 0.75. Also, some of our ratios are less than 1.0 which is the smallest ratio given on the chart. This means that the probability of acceptance for these ratios would be even greater than 0.95 which is the corresponding value when the ratio is 1.0.

While we were unable to develop a suitable model to determine whether the probability of acceptance over the long run would be of any important difference from that yielded by the H-109 O.C. curve formula described by Ellner (see Technometrics, February 1963, pp. 23–46) it seemed reasonable to assume that if the variation of individual simulation results from the H-109 value were small, the probability of acceptance under the continuous sampling verification method could be adequately described by the Ellner formula.

To simplify our work, we arbitrarily decided to concern ourselves only with the frequency of simulation for which the probability of acceptance was less than .90. This would allow us to get a quick picture of the results without having to compute an O.C. curve point for each simulation.

If we consider all of the production intervals, it can easily be seen that they meet the criterion of having a probability of acceptance of greater than .90. Therefore, in this example, it seems reasonable to assume that the O.C. curve under the continuous sampling assumption is probably close to the range of values (94%–96%) provided by the Ellner formula.

Thus, it is possible to study this problem using simulation methods. However, it obviously would be preferable to have a mathematical model. Therefore, to reiterate the problem: a mathematical model describing the operating characteristic of the procedure described is desired.

TABLE I

INSPECTION REQUIRED UNDER CSP-1 AND ASSOCIATED CHECKING AND VERIFICATION INSPECTION

TYPE OF INSPECTION	PERFORMED BY	PHASES WHEN PERFORMED	FREQUENCY	COMPARISON SAMPLE OF	
				SUPPLIER	CONSUMER
1. Product screening	Supplier	Screening	100%		
2. Product sampling	Supplier	Sampling	f	X	
3. Checking	Supplier	Screening	f	X	
4. Verification	Consumer	Screening and Sampling	(1/r)f		X

TABLE II

PROPORTION OF UNITS SUBJECTED TO 100% INSPECTION

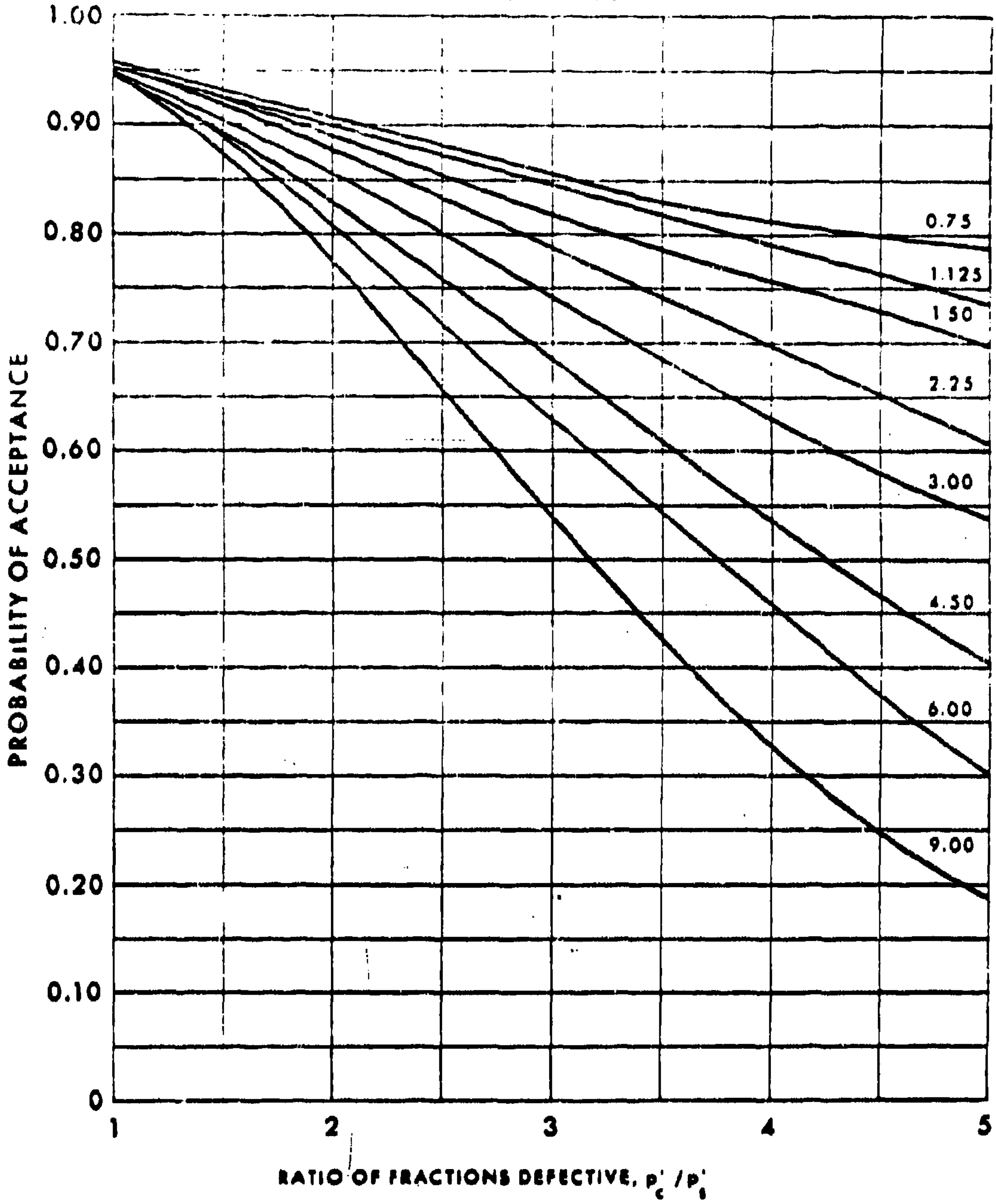
Production Interval	Screening Phase Supplier	Consumer
1	0.357	0.500
2	0.000	0.000
3	0.000	0.000
4	0.000	0.000
5	0.000	0.000
6	0.000	0.000
7	0.357	0.000
8	0.357	0.500
9	0.067	0.000
10	0.308	0.500
Cumulative	0.143	0.150

TABLE III

Production Interval	P_{eff_c}	P_{eff_S}	$\frac{P_c}{P_s}$	$n_s P_s$ (expected number of defectives in supplier's sample)
1	.0125	.0161	.7760	.2250
2	.0250	.0250	1.0000	.3500
3	.0250	.0250	1.0000	.3500
4	.0250	.0250	1.0000	.3500
5	.0250	.0250	1.0000	.3500
6	.0250	.0250	1.0000	.3500
7	.0250	.0096	2.6040	.1340
8	.0125	.0096	1.3020	.1340
9	.0250	.0230	1.0900	.3450
10	.0125	.0173	.7225	.2250
Cumulative	.021	.021	1.00	.294

OPERATING CHARACTERISTIC CURVES OF TWO-SAMPLE TEST FOR HOMOGENEITY

SAMPLE SIZE RATIO: r=8



NOTE:
Figures on curves are the expected numbers of defectives (defects) in the supplier's sample

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TOWARD A STOCHASTIC MODEL OF TERRAIN

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ABSTRACT

We present an account of an attempt to find useful random models of terrain. Measurements have shown that the distribution of slopes is what has been called the bilateral exponential distribution, definitely not normal. The problem is to find a convenient random function of geographical positions of two real variables which has this distribution for slopes and fits, in some approximations, the dependence of slopes in various directions at neighboring points. A family of random functions, the probability distributions in function space which are spherically symmetric in a Hilbert norm suitable to the purposes of the study, was introduced with an enormous latitude in the choice of parametric functionals. We felt sure that random functions with the required properties must be included. Sad to relate further mathematical developments which we deem intrinsically interesting have shown it not to be so. We know not how to proceed. Help!

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We have found it easier to separate our contributions to this exposition although they are interdependent. Peterson has written the first paragraphs and Taylor the later ones as is indicated in the text.

Terrain, being the medium of ground combat, has been the subject of many investigations by analysts in the field of weapon systems analysis. Most of these studies have been focused on the particular role played by terrain in the particular problem at hand. Others have been more general in nature with a goal of giving more insight into the quantitative aspects of describing terrain.

I would like to indicate a sample of the type of problems that arise involving terrain and its influence on the outcome of combat that have received attention. In order to lend some semblance of order to such a listing I have attempted this simple two way classification of these roles. (Figure 1) For lack of better terms I have labeled them scales and mechanisms. For scales I have fallen back on the vernacular of micro and macro, micro generally referring to distances of up to a few meters and macro from there on out to perhaps several kilometers. Mechanisms I have broken into a clear dichotomy of contact and non-contact. By contact I mean that the terrain is actually supporting the objects whether they be vehicles or other pieces of equipment being considered. By non-contact I mean we are concerned with the existence of a line-of-sight. I have listed those roles of terrain which can be fairly well categorized but I will also try to indicate problem areas where there is not a clear distinction or there are strong interactions.

Under contact with the micro structure of terrain I have listed V_{RIDE} and posture. V_{RIDE} has come to be used as an indication of the speed of a vehicle that is tolerable to both the occupants and to the vehicle itself due to roughness of terrain. It is concerned with dynamics of the vehicle over the terrain.

FIGURE 1

ROLES OF TERRAIN IN COMBAT

Scale / Mech.	Micro	Macro
Contact	V_{RIDE} Posture	Routes
Non-Contact	Fragment Shielding	Ranges of Engagement

Posture refers more to the static role of the vehicle and is concerned with the capability of a weapon system. As an example, if a tank is canted and the gun is elevated, this elevation will introduce a horizontal component of error into the aim of the gun.

Under contact with the macro characteristics of terrain I have listed simply routes. Factors other than slopes influence the routes taken by a vehicle, of course.

Under non-contact in the micro regime I have indicated the fragment shielding which has been quantified in terms of "cover functions". The non-contact aspects of the macro relief are closely tied up with the ranges of engagement. A defender may choose fields of fire to get his opponent out into the open and yet there may be draws and gullies which can allow the attacker to approach under cover.

As an example of the multiple interaction of all of these roles, we might consider the case of a tank hastily taking up a firing position. The tank is advancing along some preplanned axis -- his route has been established. The enemy is encountered -- the approximate range of engagement has been established. The tank may stop or head for a nearby rise in the terrain to get into hull defilade -- V_{RIDE} and shielding come into play. How the tank is canted in position may influence his accuracy -- hence the role of posture.

We see then that there are a number of properties of terrain that are of concern to the military OR analyst and, as I have mentioned at the outset, there are a number of ways that terrain has been categorized, measured, stored in machine memories for retrieval, generated by Monte Carlo means, etc. In order to state the problem which we bring to this clinical session I'd like to discuss two observations concerning the nature of terrain which we feel have not been exploited to their fullest in dealing with this problem area. One concerns the results of a statistical study of terrain slope. The other concerns the underlying geometry of the nature of terrain.

The statistical study to which I refer was conducted to determine the posture of tanks as measured by the pitch and cant of the trunnion after taking up simulated firing positions. A sampling of widely varying terrain types was obtained in that the test was run at Fort Knox, Fort Bragg, Fort Hood, Camp Pickett and Camp Erwin. The pattern that emerged indicated that the distributions of slopes in these firing positions were not normal but seemed to be much more like the bilateral exponential distribution. (Figure 2) Moreover the mean absolute slope varied greatly from one test site to another. In order to check out the possibility that this non-normal characteristic of these distributions was due to the selection of the firing positions sample profiles of each of the test sites were constructed from maps of each of the installations and the distributions of slopes measured over 200 yard intervals were obtained. Here again, the bilateral exponential distribution seemed to be the natural means of describing these slopes.

A detailed map study of the type mentioned above was made of the region around Houffalize, Belgium (based on a map we happened to have available). It showed that the distributions of north-south slopes and of east-west slopes both seemed to fit the bilateral exponential. The inadequacy of the normal distribution for generating profiles from which lines-of-sights can be determined was demonstrated some 20 years ago by people in the U. K. (personal communication from Mr. Eddie Benn then at the Armament Research and Development Establishment). This finding has seemed to influence their subsequent investigations along this line. (See Forbes, "The Generation of Terrain on an Electronic Computer," A.R.D.E. Memorandum (B) 75/60).

In several of the studies mentioned above, attempts were made at establishing distributions of the height or elevation of terrain itself. The results were erratic and no pattern was observed. Such behavior is probably due to general trends which can be attributed to near-zero frequency components in the spectrum.

FIGURE 2

THE FREQUENCY FUNCTION OF THE BILATERAL EXPONENTIAL DISTRIBUTION

$$f(s) = \frac{1}{2a} e^{-|s|/a}$$

where

a = mean absolute deviation

$$a^2 = \sigma^2/2$$

characteristic function

$$\phi(\alpha) = [1 + \sigma^2\alpha^2/2]^{-1}$$

In addition to the non-normal nature of terrain there is the problem of dimensionality. Many of the existing schemes for generating random terrain profiles proceed in the same manner that one would treat a time series. This approach cannot be used to generate a surface, as two neighboring rays say emanating from a point, will be completely independent. Put in terms of statistically describing terrain rather than generating it we must think in terms of the gradient of a surface rather than the slope of a curve. We know from vector analysis that the curl of a gradient is zero. In other words there are constraints between the two perpendicular components of the gradient at a point. In the one dimensional case, as typified by a time series, the random function or stochastic process is readily expressed in terms of Fourier series, i.e., sines and cosines. In the two dimensional case the functions which replace the trigonometric functions in a natural way are the Bessel functions. Other areas of endeavor on which reference to two dimensional random functions have been found include windblown waves, agricultural productivity and images both photographic and video. The household term of snow as applied to a television picture is just an adoption of the television engineer's term "white snow" which is his extension to two dimensions of the concept of "white noise" in the one dimensional process. (We might add in passing that the most well known application of three dimensional random functions is in the field of turbulence.)

We have briefly stated two characteristics of terrain which we believe to be pertinent to the statistical description of terrain. One based on data analysis that, whereas terrain height itself does not seem to have any pattern to its distribution, its difference field as measured over a few meters or a few hundred meters has a common non-normal distribution which can be expressed in terms of a single parameter. The other based on geometrical reasoning indicates that the tools developed for one dimensional processes are not adequate for describing a two dimensional random surface.

We are now at a point of being able to state the problem which has plagued us for a number of years. Is it possible to construct a meaningful stochastic model of terrain which embodies these two considerations?

Questions for which we would like to get more insight, include the following: 1) are what we categorize as rough, rolling or flat in fact simply manifestations of the same basic model with different scale factors in the horizontal and vertical directions, 2) to what extent can we use easily obtained information for a region and infer the details from the model and/or 3) can we build a composite model from which we can infer both the micro and macro characteristics of a given terrain type?

In closing my part of this presentation I want to stress that we are not posing the general question as to how to statistically categorize terrain but as to what extent the theory of two dimensional random functions can contribute to our basic understanding of the statistical properties of terrain.

Dr. Taylor will now describe one approach we have taken to this problem along with its triumphs and pitfalls.

A Class of Random Functions

After careful consideration of some requirements on a random function that it be eligible for consideration as a random terrain, Peterson was led to propose the following wide class of random functions as candidates for investigation. Let

$$P:\ x = (x_1, x_2) \tag{1.1}$$

be rectangular coordinates of a point P in a horizontal datum plane. Let u(x) be the height of a terrain above the datum plane at the geographical point P. For our purpose u(x) is a complete description of the terrain. We are concerned with a random function U(x), a probability distribution on certain subsets of a set, say B, of functions u(x). We consider a linear set B′ of linear functionals $\ell(u(\cdot))$ and suppose that the expection

$$E(\ell(U)) = 0 \tag{1.2}$$

for all ℓ of the set. By the variance of ℓ we mean

$$E(\ell(U)^2) = \text{Var}\,\ell \tag{1.3}$$

and by the characteristic functional of ℓ we mean the expectation of the exponential

$$E(\exp i\ell(U)) = \text{Ch } \ell \tag{1.4}$$

The proposal is to limit our discussion to those random functions for which there exists a complex valued function of a real positive argument g(z) such that, for all ℓ,

$$\text{Ch}\ell = g(\text{Var } \ell) \tag{1.5}$$

Example: For a gaussian random function U,

$$E(\exp i\ \ell(U)) = \exp(-\tfrac{1}{2} E(|\ell(U)|^2)),$$

since $E(\ell(U)) = 0$.

<u>Spherical Symmetry</u>

We may introduce also the inner product

$$<\ell_1,\ell_2> = E(\ell_1(U)\ell_2(U)) \tag{2.1}$$

and

$$\begin{aligned} \ell^2 &= <\ell,\ell> \\ &= E(\ell(U)^2) \\ &= \text{Var } \ell. \end{aligned} \tag{2.2}$$

It is but a small step to extend our discussion to the Hilbert space, H, of linear functionals and to suppose further that this space is sufficient in the following sense: For any u(x) under discussion

$$\ell(u(x)) = 0 \text{ for all } \ell \in H \tag{2.3}$$

implies $u(x) \equiv 0$. This is not necessary for the rather loose discussion we are presenting but it may ease the reader's way. Now a function u(x) defines on H a linear functional whose value at the element ℓ is $\ell(u)$. Whether every linear functional in B′ is thus represented by some function u(x) is of no importance to our discussion. What is very important is to realize that the linear functional defined by u(x) need not be in any fixed sense a bounded linear functional and indeed, for a given $\ell \in H$, $\ell(U)$ need be defined only with probability one.

The preceding discussion of Hilbert spaces has been principally only for orientation. We need at first be concerned only with finite dimensional subspaces defined as follows: disregard all but a finite set of the linear functionals, along with their linear combinations. We define the projection of the measure space, and the measure, into this finite dimensional space by identifying all functions u(x) which agree in the values taken for them by each of this finite set of linear functionals. These finite dimensional spaces are euclidean with the inner product we have introduced. The characteristic functional and the variance of each of these finite dimensional projections of the probability measure will have the same values as when they were considered to be defined on the infinite dimensional space and the characteristic functional defined on the conjugate space will thus be a function only of the distance from the origin. That is to say that it will be spherically symmetric. It follows immediately that the n dimensional measure is spherically symmetric and must be described by a spherically symmetric density -- at least if we assume it to be described by a density at all, and we do. Even though no spheres nor radii are defined on our infinite dimensional space (at least not with positive probability) we may nonetheless define spherical symmetry of the measure: A measure is spherically symmetric if all its projections into finite dimensional subspaces are spherically symmetric.

Characterization of Spherically Symmetric Measures on Infinite Dimensional Spaces*

In each finite dimensional projection of a spherically symmetric measure the density, if supposed to exist, must be the same function of the distance from the center as in any other projection of the same dimension. In n dimensions, let the density at distance r from the center

*We are indebted to J. Feldman and R. M. Dudley for the information that this result concerning spherically symmetric measures in infinite dimensional spaces is not new. It was published in 1962 by Umemura, who obtained it in a more recondite context.

be $p_n(r^2)$. Then, considering the projection of the measure from n + 2 dimensions to an n dimensional subspace, an easy argument shows that

$$p_n'(r^2) = -\pi\, p_{n+2}(r^2) \tag{3.1}$$

From this it follows that the derivatives of each of the p's alternate in sign. Such functions are called completely monotone. There is a theorem of S. Bernstein [see e.g., Feller, Th. of Probability, Vol II, p 415] which states that a completely monotone function $p(z)$, $0 \leq z \leq \infty$, with $p(\infty) = 0$ can be expressed as a linear aggregate of decreasing exponentials with positive coefficients:

$$p(z) = \int_0^\infty e^{-\lambda z}\, d\phi(\lambda), \quad 0 < z < \infty, \tag{3.2}$$

with $d\phi(\lambda) \leq 0$.

Setting $z = r^2$, $\lambda = 1/2\sigma^2$ and re-defining the measure $d\phi(\lambda)$, we may then write

$$p_n(r^2) = \int_0^\infty p_{n\sigma}(r^2)\, d\phi(\sigma) \tag{3.3}$$

where

$$p_{n,\sigma}(r^2) = (\sigma\sqrt{(2\pi)})^{-n}\, e^{-\, r^2/2\sigma^2} \tag{3.4}$$

is the n dimensional gaussian density. This formula, once obtained for any value of n, implies the same formula for all lower dimensional densities, as is seen by successive integration with respect to each of an orthogonal set of coordinates. The integrals are all absolutely convergent and may be integrated freely in any order. The same statement is then true for all n. Further the corresponding statement may be asserted expressing the given measure m similarly in terms of the gaussian measures m_σ:

$$m = \int_0^\infty m_\sigma\, d\phi(\sigma) \tag{3.5}$$

Adjusting the Parameters of the Model

The procedure we are to follow is now quite clear. Whatever may be the distribution of the individual linear functionals, we shall adjust the density $p_1(r^2)$ to it by choosing the weights $d\phi(\sigma)$ in (3.3). A necessary condition is of course that the density be a completely monotone function of r^2. But, as Peterson has pointed out above, it lies very near an exponential function e^{-ar}, which, fortunately, satisfies this condition. We shall need only to be firm with the small residue, if there be any, and its derivatives, and insist that it conform. There will then be the task of fitting the remaining free element, the variance of a linear functional. Here there is a great deal more freedom. There is a functional to be adjusted to approximate as best we can the statistical interdependence of the values of $U(x)$ at neighboring values x. (We want them to become independent at distant points.) But this is just the same problem to be faced in fitting a gaussian random function. For any $\ell(u)$ we need only go to the samples we wish to fit and estimate $E(\ell(U)^2)$, or what is simpler to tabulate, for some linear basis $\ell_1, \ell_2, \ldots$ of the linear functionals we estimate $E(\ell_i(U)\ell_j(U))$ from the samples for all pairs i,j. There is no arbitrary decision left to be made. It's just a question of whether it works or not! or how well it works!

Sad to say, it doesn't work at all. We shall see this without any further examination of samples. The reason lies in an additional significant difference between the finite and infinite dimensional cases.

Lack of Ergodicity

We shall see that (3.5) is, in a reasonable sense, an orthogonal representation of the measure.* For this purpose it is convenient (and

*For the source of the train of thought which led to this analysis, we are indebted to Jacob Feldman for a lucid and provocative briefing on relatively singular measures, a briefing which grew out of a discussion some years ago of the application of information theory to empirical functions. But the simple case with which alone we need be concerned here was known to us as well as many other people long ago. It appears, for example, in a paper of W. T. Martin and R. H. Cameron in the 1940's.

perhaps something equivalent is also necessary) to introduce a sequence of linearly independent bounded linear functionals $\ell_i(u)$, $i=1,2,\ldots$, which we can then as well suppose to have been replaced by an orthonormal sequence, so that

$$E(\ell_i(u)\ell_j(u)) = \delta_{ij}$$

$$= \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \qquad (5.1)$$

The question of when and in what sense does a sequence of numbers w_i, $i=1,2,\ldots$, represent a function u such that

$$w_i = \ell_i(u) \qquad (5.2)$$

will not be discussed.

The random variables

$$W_i = \ell_i(U), \quad i=1,2,\ldots, \qquad (5.3)$$

are uncorrelated but not necessarily independent. However, for any one of the gaussian measures, m_σ, calling the random function U_σ, the random variables

$$W_{\sigma i} = \ell_i(U_\sigma) \qquad (5.4)$$

are uncorrelated gaussian variables and hence independent. Since

$$E_{m_\sigma}(W_{\sigma i}^2) = \sigma^2, \qquad (5.6)$$

we have, with probability 1,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} W_{\sigma i}{}^2 = \sigma^2. \qquad (5.7)$$

The measure m_σ is not essentially altered if we trim its space to the set A_σ of sequences $w_1, w_2, \ldots$ for which (5.7) is true and to those functions u(x) which give rise to such sequences. We restrict our measure, supposed to exist and to be given, to the set

$$A = \sum A_\sigma \qquad (5.8)$$

of functions u(x). We do not discuss which subsets of A have probability and how it is defined since this subject seems either too trivial or too difficult.

Suppose now that we test the distribution of slopes in A in the same fashion that was described above. That is, we draw a single sample function and measure slopes at many points on it. Further, for simplicity, suppose these points are far enough apart that we may ignore statistical dependence of the slopes. Each sample function from A is, for some σ, taken from A_σ. Slopes at distant points on it are then independent, identical, gaussian variables and the sample values of a large number of them will characterize their common distribution as gaussian with whatever assurance their number permits. But haven't we brought this about by artificial tampering with the ensemble? No. We have only turned a statement true with probability one into a true statement. Devise a statistical test for the normality of the distribution from which a sample is taken, using statistics whose distribution is independent on the variance of the ensemble. The result of the test will (at least at any specified stage) depend on only a finite sample. A finite set of linear functionals has the same distribution in A as in the original probability measure, on the space we have called B, and the distribution of the statistics of the test will thus have the same distribution in B as in A and as in a gaussian ensemble.

In short then, these random functions fail to represent a random terrain since an orthonormal sequence of linear functionals read off any one sample function have values distributed like independent samplings from a univariate gaussian ensemble. We demand of our model of terrain on the contrary that slopes read at widely separated points have a different distribution, approximately the one described by a bilateral exponential density. More generally, in order to make sense our random function model must have the ergodic property: independent identical functionals (such as slopes at widely separated points) must show the same distribution whether read from a single sample or each from a different randomly chosen one.

A SUGGESTED PROCEDURE FOR ANALYZING MISSILE PERFORMANCE BY A LEAST SQUARES FIT TO A GENERALIZED LINEAR STATISTICAL MODEL AND A QUICK CHECK FOR NORMALITY OF THE DATA

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ABSTRACT

The data taken in a series of missile tests are often in the form of a variable of interest (such as radial miss distance from a given target) and several dependent variables (e.g., range, temperature, type of missile modification) for each test made. In such cases, it may be possible to construct a linear statistical model relating the main variable, y, to the others, x_1 through x_k. The coefficients of this model can be estimated by a least squares procedure.

The difference between each measured y and the y predicted by the linear model is called a residual. If the set of residuals is normally distributed, several well-known tests of statistical hypotheses and methods of setting confidence intervals are applicable. A procedure for graphically validating the normality of the residuals has also been developed.

1. INTRODUCTION

In the summer of 1969 the Systems Evaluation Branch* had the task of determining which, if any, of three modifications of a certain missile was "best." A modification was considered "best" if the average radial miss distance measured from the center of a target of fixed size was significantly less for the modification than for the other two modifications.

There was no lack of data for this project; in fact, data had been recorded for over 1000 firings of the missile. For each firing, the following had been recorded: radial miss distance (y), target altitude at intercept (v_1), range of the target at launch (v_2), range of the target at intercept (v_3), target closing velocity at intercept (v_4), missile modification (v_5), target type (v_6), and radar power (v_7).

The data were sorted for duplications and missing values. There remained data on over 900 firings. Of these, approximately 6 percent were Mod 1 firings, 15 percent were Mod 2 and 79 percent were Mod 3. For this paper, 100 firings were chosen from the total; 6 of Mod 1, 15 of Mod 2 and 79 of Mod 3. Since the original data were classified, the values were coded or transformed to nonstandard, undefined "units." The coded data are shown in Table I.

The following simple procedure was considered: divide the data into three groups according to modification. Calculate the sample average and sample variance of the radial miss distances for each group. Test these values for equality using the F and t statistical tests. This procedure was rejected for the following reason: the testing procedure was not planned in advance to insure sets of comparable conditions for each modification. For example, most of the firings for Mod 1 were with the second target type $(v_6 = 2)$. Thus, if the above test procedure had been used, the effect on the radial miss distance of the modification would have been confounded with the effect of the target type. The conclusions would then be questionable at best.

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TABLE I. CODED MISSILE DATA, EXAMPLE I

y	v_1	v_2	v_3	v_4	v_5	v_6	v_7
3.8	1.11	27.1	24.3	110.0	3.0	1.0	1.0
4.7	1.88	27.1	20.1	108.2	3.0	1.0	1.0
5.0	2.99	31.3	24.3	187.4	2.0	2.0	2.0
5.0	1.24	24.3	20.1	103.7	2.0	1.0	2.0
5.0	1.67	15.9	13.1	127.1	3.0	1.0	1.0
5.0	1.55	29.9	24.3	101.0	3.0	1.0	2.0
5.6	2.32	45.3	35.5	160.4	3.0	2.0	1.0
6.2	6.06	42.5	27.1	191.0	3.0	2.0	2.0
6.8	6.06	36.9	27.1	204.5	3.0	2.0	2.0
7.1	7.05	46.7	32.7	174.8	2.0	2.0	2.0
7.4	1.55	32.7	27.1	108.2	3.0	1.0	1.0
7.4	1.33	29.9	25.7	110.0	3.0	1.0	2.0
7.7	1.73	34.1	25.7	128.0	1.0	2.0	2.0
7.7	3.64	31.3	25.7	123.5	3.0	1.0	1.0
8.0	2.34	45.3	35.5	112.7	2.0	1.0	2.0
8.0	2.32	29.9	21.5	108.2	3.0	1.0	2.0
8.3	1.11	31.3	25.7	107.3	3.0	1.0	1.0
8.3	10.02	43.9	31.3	191.9	3.0	2.0	1.0
8.9	2.21	24.3	20.1	123.5	3.0	1.0	1.0
9.2	1.66	38.3	29.9	108.2	2.0	1.0	2.0
9.2	1.88	28.5	20.1	110.0	3.0	1.0	1.0
9.2	1.22	28.5	22.9	108.2	3.1	1.0	2.0
9.5	2.87	31.3	25.7	108.2	3.0	1.0	1.0
10.1	1.34	28.5	21.5	182.0	3.0	2.0	1.0
10.1	1.11	29.9	20.1	114.5	3.0	1.0	1.0
10.1	1.67	35.5	25.7	188.3	3.0	2.0	2.0
10.4	6.02	29.9	24.3	123.5	2.0	1.0	2.0
10.7	1.88	22.9	17.3	174.8	1.0	2.0	2.0
10.7	1.66	28.5	24.3	110.0	3.0	1.0	2.0
11.0	1.55	24.3	20.1	108.2	3.0	1.0	2.0
11.0	1.11	38.3	32.7	114.5	3.0	1.0	1.0
11.3	1.55	32.7	27.1	108.2	3.0	1.0	1.0
11.6	2.32	25.7	21.5	110.0	3.0	1.0	2.0
11.9	2.98	24.3	21.5	107.3	3.0	1.0	1.0
11.9	1.33	29.9	22.9	108.2	3.0	1.0	2.0
12.2	1.24	35.5	28.5	108.2	2.0	1.0	2.0
12.2	2.21	27.1	20.1	98.3	3.0	1.0	[illegible]
12.5	1.66	28.5	24.3	110.9	3.0	1.0	1.0
12.8	1.88	32.7	25.7	114.6	3.0	1.0	1.0
13.1	1.50	31.3	25.7	107.3	2.0	1.0	2.0

TABLE I. CODED MISSILE DATA, EXAMPLE I (Continued)

y	v_1	v_2	v_3	v_4	v_5	v_6	v_7
13.1	2.98	27.1	21.5	95.6	3.0	1.0	1.0
13.4	2.10	45.3	32.7	164.0	3.0	2.0	2.0
13.7	1.67	50.9	36.9	161.3	1.0	2.0	2.0
13.7	2.65	25.7	22.9	121.7	3.0	1.0	1.0
14.0	2.98	34.1	29.9	105.5	2.0	1.0	2.0
14.0	1.11	28.5	21.5	123.5	3.0	1.0	1.0
14.3	1.33	25.7	21.5	117.2	3.0	1.0	2.0
14.9	6.06	49.5	29.9	107.3	3.0	2.0	2.0
14.9	1.88	17.3	14.5	122.6	3.0	1.0	1.0
15.5	1.67	41.1	28.5	209.0	2.0	2.0	2.0
15.5	1.77	31.3	27.1	108.2	3.0	1.0	1.0
16.1	3.42	28.5	18.7	144.2	3.0	1.0	1.0
16.1	4.30	49.5	35.5	211.7	3.0	2.0	1.0
16.7	1.88	28.5	20.1	117.2	3.0	1.0	1.0
17.0	1.55	27.1	22.9	101.0	3.0	1.0	2.0
17.6	3.64	49.5	32.7	200.9	2.0	2.0	2.0
17.6	2.10	25.7	22.9	108.2	3.0	1.0	1.0
17.9	2.00	25.7	21.5	81.2	3.0	1.0	2.0
18.2	1.67	28.5	24.3	85.7	3.0	1.0	2.0
18.5	1.22	28.5	24.3	108.2	3.0	1.0	1.0
18.5	1.55	29.9	24.3	108.2	3.0	1.0	1.0
18.8	1.33	35.5	25.7	198.2	3.0	2.0	2.0
19.4	2.65	28.5	18.7	121.7	3.0	1.0	1.0
19.4	1.55	28.5	24.3	110.0	3.0	1.0	2.0
20.0	1.99	24.3	14.5	316.1	1.0	2.0	2.0
20.3	1.77	21.5	0.5	108.2	3.0	1.0	1.0
20.3	5.95	31.3	25.7	114.5	3.0	1.0	2.0
20.6	1.25	22.9	18.7	112.7	2.0	1.0	2.0
21.2	2.65	21.5	17.3	101.0	3.0	1.0	1.0
21.2	1.33	28.5	25.7	108.2	3.0	1.0	2.0

TABLE I. CODED MISSILE DATA, EXAMPLE I (Concluded)

y	v_1	v_2	v_3	v_4	v_5	v_6	v_7
21.8	2.32	31.3	25.7	96.5	3.0	1.0	1.0
22.1	5.40	34.1	27.1	114.5	3.0	2.0	1.0
22.4	1.77	31.3	25.7	108.2	3.0	1.0	1.0
23.0	1.11	28.5	21.5	114.5	3.0	1.0	1.0
23.3	2.32	35.5	28.5	115.4	3.0	1.0	2.0
23.9	1.11	27.1	18.7	110.9	3.0	1.0	1.0
23.9	1.88	20.1	17.3	103.7	3.0	1.0	1.0
24.2	1.34	22.9	18.7	120.8	2.0	1.0	2.0
24.2	1.99	25.7	18.7	114.5	3.0	1.0	1.0
24.8	2.98	24.3	18.7	107.3	3.0	1.0	1.0
24.8	1.22	29.9	25.7	101.0	3.0	1.0	2.0
26.9	1.99	34.1	25.7	183.8	3.0	2.0	1.0
28.4	4.08	22.9	17.3	137.0	3.0	1.0	1.0
29.3	1.11	28.5	24.3	107.3	3.0	1.0	1.0
30.2	2.00	45.3	31.3	181.1	1.0	2.0	2.0
30.2	5.07	42.5	28.5	225.2	3.0	2.0	1.0
31.1	1.55	27.1	21.5	108.2	3.0	1.0	2.0
31.4	1.29	34.1	28.5	101.9	2.0	1.0	2.0
32.3	1.77	32.7	27.1	108.2	3.0	1.0	2.0
34.1	1.33	43.9	29.9	210.8	3.0	2.0	2.0
35.0	3.20	43.9	31.3	184.7	3.0	2.0	1.0
37.1	1.55	21.5	18.7	110.0	3.0	1.0	2.0
38.6	1.88	20.1	17.3	101.0	3.0	1.0	2.0
41.3	1.88	28.5	24.3	119.0	3.0	1.0	1.0
41.6	1.11	46.7	27.1	374.6	1.0	2.0	2.0
45.8	5.73	32.7	24.3	141.5	3.0	2.0	1.0
48.5	1.99	32.7	28.5	110.0	3.0	1.0	2.0
57.5	1.13	56.5	31.3	386.3	3.0	2.0	2.0
66.5	1.22	24.3	18.7	174.8	3.0	2.0	2.0
69.5	6.02	27.1	22.9	108.2	2.0	1.0	2.0

2. THE LINEAR STATISTICAL MODEL FOR THIS MISSILE

It was decided to set up a linear statistical model [1] relating the radial miss distance, y, to functions of the 7 other variables, v_1 through v_7. Through engineering considerations, the model chosen was:

$$y = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5 + b_6x_6 + b_7x_7 + b_8x_8 + b_9x_9 + b_{10}x_{10} + e$$

$x_1 = v_1 =$ target altitude at intercept

$x_2 = v_2 =$ range of target at launch

$x_3 = x_2^2 = v_2^2$

$x_4 = v_3 =$ range of target at intercept

$x_5 = x_4^2 = v_3^2$

$x_6 = v_4 =$ target closing velocity at intercept

$$x_7 = \begin{cases} -0.5 \text{ if Mod 1} \\ 0.5 \text{ if Mod 2} \\ 0.0 \text{ if Mod 3} \end{cases}$$

$$x_8 = \begin{cases} 0.0 \text{ if Mod 1} \\ -0.5 \text{ if Mod 2} \\ 0.5 \text{ if Mod 3} \end{cases}$$

$$x_9 = \begin{cases} -0.5 \text{ if target type 1} \\ 0.5 \text{ if target type 2} \end{cases}$$

$$x_{10} = \begin{cases} -0.5 \text{ if low intensity radar} \\ 0.5 \text{ if high intensity radar} \end{cases}$$

$e =$ random error

3. GENERAL LINEAR STATISTICAL MODELS

Frequently the results of experiments or measurements are given as a set of independent variables and as associated result or dependent variable. The data discussed above provides one example. As another example, the velocity of the vehicle could be measured at various time points.

The result or observation, y, is considered as a function of the independent variables, v_1, v_2, . . . , v_m, and random noise e and written:

$$y = y\ (v_1, v_2, \ldots v_m, e)\ .$$

The observation noise or measurement noise e is a result of the inaccuracy of the measuring devices and of variables which are not included in the model but which do affect the observation. If the model is correct, e is the random fluctuation of y for the fixed values of v_1 through v_m.

The most convenient and frequently used model is the linear statistical model:

$$y = b_0 + b_1 x_1 + b_2 x_2 + \ldots + b_k x_k + e\ .$$

Here the x_i's are functions of the basic variables v_i, e.g., $x_1 = v_1$, $x_2 = v_1^2$, $x_3 = v_2$, $x_4 = v_1 v_2$. One restriction on the x_i's is that they be linearly independent; e.g., if $x_1 = v_1$ and $x_2 = v_2$, then x_3 cannot be set to $(v_1 + v_2)$. The other restriction is that the x_i's be known or measured without error. (Both restrictions can be relaxed in more advanced work.) The model is termed "linear" because it is linear in the coefficients b_i. The b_i's are considered to be fixed but unknown and must be estimated from the data.

It should be noted that this is not the only statistical model possible and may not apply in some cases. However, it can be used successfully in a large number of situations and it does possess manipulative ease. The model should be constructed from physical and engineering considerations. As will be seen later, statistical tests can be used to determine which terms can be dropped from the model without seriously affecting the accuracy; however, they give no indication of which new terms should be added to the model.

The results on the missile discussed above, hereafter known as Example I, are of concern here. However, in order to illustrate the method with a small, uncomplicated case, a simple example (Example II) was concocted. The calculations of Example II can be done by hand in a "reasonable" (compared to Example I) length of time.

In Example II, the amount of catalyst added to each of two vats in a chemical plant was varied from 0 to 5 units. The resulting yields are listed in Table II.

TABLE II. CHEMICAL YIELD, EXAMPLE II

v_1 Amount of Catalyst	y	
	Yield for Vat 1	Yield for Vat 2
0	8.81	7.02
1	10.00	10.02
2	13.25	10.15
3	14.51	13.43
4	11.36	10.40
5	8.58	4.33

The yields are plotted in Figure 1 as functions of the amount of catalyst.

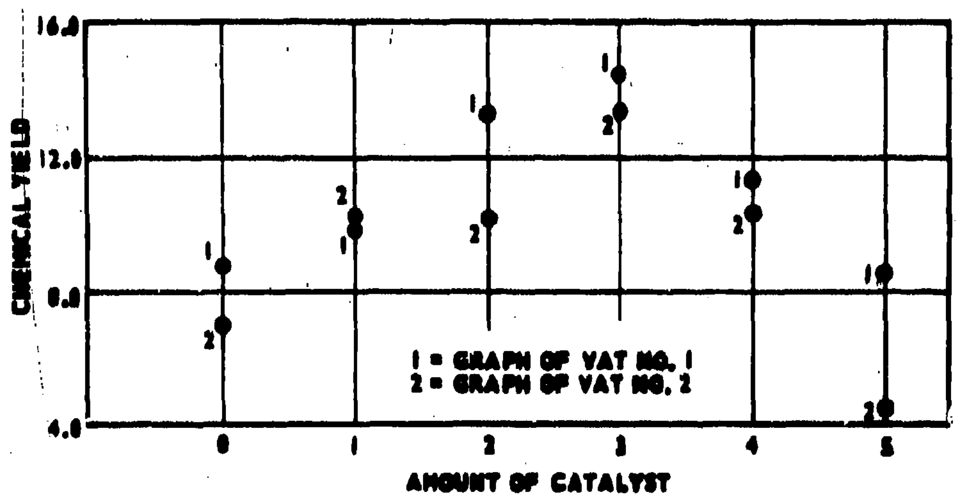

FIGURE 1. CHEMICAL YIELD VERSUS AMOUNT OF CATALYST, EXAMPLE II

The plot suggests that an appropriate model would be a second degree polynomial:

$$y = b_0 + b_1 v_1 + b_2 v_1^2 + e$$
$$= b_0 + b_1 x_1 + b_2 x_2 + e \, ,$$

where

$$x_1 = v_1, \; x_2 = v_1^2 \, .$$

Without loss of generality, it can be assumed that the noise e has zero mean [if not, the mean, E (e), could be included in the term b_0 so that the redefined noise $e^* = e - E(e)$ has zero mean].

If a total of n observations are taken, then the model can be written as:

$$y_j = b_0 x_0 + b_1 x_{1,j} + b_2 x_{2,j} + \ldots + b_k x_{k,j} + e_j, \; j = 1, \ldots, n$$

where $x_0 = 1$ and $x_{i,j}$ is the value of x_i for the j^{th} data point. To shorten the above equations, the following vectors and matrix are defined:

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}, \underline{b} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \end{pmatrix}, \underline{e} = \begin{pmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{pmatrix}, X = \begin{pmatrix} 1 & x_{1,1} & x_{2,1} & \ldots & x_{k,1} \\ 1 & x_{1,2} & x_{2,2} & \ldots & x_{k,2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & x_{1,n} & x_{2,n} & \ldots & x_{k,n} \end{pmatrix}.$$

The above equation becomes

$$\underline{y} = X\,\underline{b} + \underline{e} \; .$$

4. <u>ESTIMATION OF THE COEFFICIENTS</u>

A linear statistical model has been postulated in Section 2. In addition, the noise $\underline{e}$ is assumed to have zero mean and covariance matrix $= \sigma^2 I$, where I is the identity matrix and σ^2 is a constant that may be unknown. That is, for i, j = 1, ..., n, $E(e_j) = 0$, var $(e_j) = \sigma^2$ and cov $(e_i, e_j) = 0$ if $i \neq j$. If this assumption is not met, the proper transformation of variables, in most cases, will reduce the model to one in which the assumption does hold.

A method must be found for determining $\hat{\underline{b}}$, the estimate of the coefficients $\underline{b}$. There is usually a loss incurred when the estimate $\hat{\underline{b}}$ is not the true value $\underline{b}$. Usually, the further $\hat{\underline{b}}$ lies from the true value $\underline{b}$, the greater the loss becomes. Since the values of X and $\underline{y}$ are given, it is desirable to choose $\hat{\underline{b}}$ so that the predicted value of $\underline{y}$, $\hat{\underline{y}} = X\hat{\underline{b}}$ will be close, in some sense, to the actual observation vector $\underline{y}$. A convenient way of doing this is to choose $\hat{\underline{b}}$ so that the quadratic loss

$$\text{Loss} = (\underline{y} - X\,\hat{\underline{b}})^T\;(\underline{y} - X\,\hat{\underline{b}})\;,$$

where $(\underline{y} - X\,\hat{\underline{b}})^T$ is the transpose of $(\underline{y} - X\,\hat{\underline{b}})$, is minimized. This is equivalent to

$$\text{Loss} = \sum_{j=1}^{n} \left(y_j - \overline{x}_j\,\hat{\underline{b}}\right)^2\;,$$

where $\overline{x}_j$ is the row vector

$$\overline{x}_j = \left(1,\; x_{1,j},\; x_{2,j},\; \ldots,\; x_{k,j}\right)\,.$$

Because of the above form, the estimate $\hat{\underline{b}}$ which minimizes the quadratic loss is called the <u>least squares estimator</u>.

The quadratic loss can be expanded

$$(\underline{y} - X\,\hat{\underline{b}})^T\;(\underline{y} - X\,\hat{\underline{b}}) = \underline{y}^T\,\underline{y} - 2\,\hat{\underline{b}}^T\,X^T\,\underline{y} + \hat{\underline{b}}^T\,X^T\,X\,\hat{\underline{b}}\;.$$

If this quantity is differentiated by $\hat{\underline{b}}$ and set equal to the zero vector, the result is

$$-2\,X^T\,\underline{y} + 2\,X^T\,X\,\hat{\underline{b}} = \underline{0}$$

$$X^T\,X\,\hat{\underline{b}} = X^T\,\underline{y}$$

$$\hat{\underline{b}} = (X^T\,X)^{-1}X^T\,\underline{y}\;.$$

In Example II, the quantities of interest are

$$\underline{y} = \begin{bmatrix} 8.81 \\ 10.00 \\ 13.25 \\ 14.51 \\ 11.36 \\ 8.58 \\ 7.02 \\ 10.02 \\ 10.15 \\ 13.43 \\ 10.40 \\ 4.33 \end{bmatrix},\; X = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 1.0 & 1.0 & 1.0 \\ 1.0 & 2.0 & 4.0 \\ 1.0 & 3.0 & 9.0 \\ 1.0 & 4.0 & 16.0 \\ 1.0 & 5.0 & 25.0 \\ 1.0 & 0.0 & 0.0 \\ 1.0 & 1.0 & 1.0 \\ 1.0 & 2.0 & 4.0 \\ 1.0 & 3.0 & 9.0 \\ 1.0 & 4.0 & 16.0 \\ 1.0 & 5.0 & 25.0 \end{bmatrix},\; \underline{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}\;.$$

The quantities $X^T\underline{y}$ and X^TX are calculated to be

$$X^T\underline{y} = \begin{pmatrix} 121.86 \\ 302.23 \\ 1035.99 \end{pmatrix}, \quad X^TX = \begin{pmatrix} 12.0 & 30.0 & 110.0 \\ 30.0 & 110.0 & 450.0 \\ 110.0 & 450.0 & 1958.0 \end{pmatrix} .$$

The inverse of X^TX is

$$(X^TX)^{-1} = \begin{pmatrix} 0.4107 & -0.2946 & 0.0446 \\ -0.2946 & 0.3634 & -0.0670 \\ 0.0446 & -0.0670 & 0.0134 \end{pmatrix} .$$

The estimate of $\underline{b}$ is

$$\hat{\underline{b}} = \left(X^TX\right)^{-1} X^T\underline{y} = \begin{pmatrix} 7.249 \\ 4.549 \\ -0.924 \end{pmatrix} .$$

The prediction equation for y is thus

$$\hat{y} = 7.249 + 4.549\, x_1 - 0.924\, x_2$$

$$\hat{y} = 7.249 + 4.549\, v_1 - 0.924\, v_1^2 \quad .$$

Listed below are $\underline{y}$, $\hat{\underline{y}}$, and the error in the prediction of $\underline{y}$, $\underline{y}-\hat{\underline{y}}$.

$\underline{y}$	$\hat{\underline{y}} = X\hat{\underline{b}}$	$\underline{y}-\hat{\underline{y}}$
8.81	7.249	1.561
10.00	10.874	-0.874
13.25	12.652	0.598
14.51	12.582	1.927
11.36	10.667	0.693
8.58	6.904	1.676
7.02	7.249	-0.229
10.02	10.874	-0.854
10.15	12.652	-2.502
13.43	12.582	0.848
10.40	10.667	-0.267
4.33	6.904	-2.574

Plotted in Figure 2 are the data points and the prediction equation.

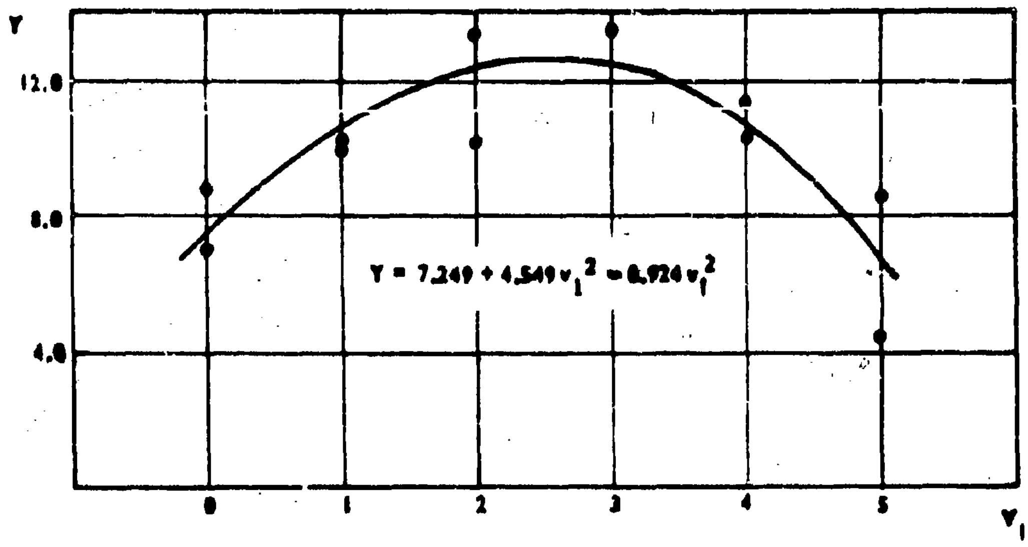

FIGURE 2. PREDICTED YIELD, EXAMPLE II

Notice that no mention has been made of the probability distribution of the measurement noise $\underline{e}$ except that the covariance matrix is $\sigma^2 I$ and the mean is the zero vector. Thus, the formula for the least squares estimator is free of the distribution of $\underline{e}$. Also, no matter what the distribution of $\underline{e}$, if $E(\underline{e}) = \underline{0}$ and $\text{cov}(\underline{e}) = \sigma^2 I$, then

$$E(\underline{y}) = E(X\underline{b} + \underline{e}) = X\underline{b}$$

$$\text{cov}(\underline{y}) = \text{cov}(X\underline{b} + \underline{e}) = \sigma^2 I$$

$$E(\hat{\underline{b}}) = E\left([X^T X]^{-1} X^T \underline{y}\right) = [X^T X]^{-1} X^T E(\underline{y}) = \underline{b}$$

$$\text{cov}(\hat{\underline{b}}) = [X^T X]^{-1} X^T \text{cov}\,\underline{y}\, X [X^T X] = \sigma^2 [X^T X]^{-1} .$$

Thus, no matter what the distribution, the least squares estimator is unbiased $[E(\hat{\underline{b}}) = \underline{b}]$ and has covariance matrix $\text{cov}(\hat{\underline{b}}) = \sigma^2 (X^T X)^{-1}$.

An appealing estimate of the variance σ^2 is the "average" loss. After the coefficient vector $\underline{b}$ has been estimated by $\hat{\underline{b}}$, the predicted value of the dependent variable at the j^{th} point is $\hat{y}_j = \underline{x}_j \hat{\underline{b}}$. The difference between the actual or measured value of y_j and the predicted $\hat{y}_j$ is called the j^{th} residual, $r_j = \left(y_j - \underline{x}_j \hat{\underline{b}}\right)$. The sum of the squares of the residuals is called the sum of squares for error (SSE) and can be shown to equal:

$$SSE = \sum_{j=1}^{n} r_j^2 = \sum_{j=1}^{n} \left(y_j - \tilde{x}_j \hat{b}\right)^2 = (\underline{y} - X \underline{\hat{b}})^T (\underline{y} - X \underline{\hat{b}}) .$$

One estimate of the variance σ^2 is then

$$s^2 = \frac{SSE}{n-k-1} .$$

How well s^2 estimates σ^2 depends upon the forms of the probability distribution of the noise e.

In Example II,

$$SSE = 25.02$$

$$s^2 = 2.78 .$$

The covariance matrix of $\underline{\hat{b}}$ is $(X^T X)^{-1} \sigma^2$ and is estimated by

$$(X^T X)^{-1} s^2 = \begin{pmatrix} 1.14 & -0.82 & 0.12 \\ -0.82 & 1.01 & -0.19 \\ 0.12 & -0.19 & 0.037 \end{pmatrix} .$$

5. TWO TYPES OF INDEPENDENT VARIABLES — QUANTITATIVE AND QUALITATIVE

For the missile model (Example I), y is the dependent variable, v_1 through v_7 are the basic independent variables; x_1 through x_{10} are the expanded variables. The expanded variables x_1 through x_6 are quantitative variables and x_7 through x_{10} are qualitative variables.

A quantitative variable is one to which such units as meters, degrees, and pounds can be attached. The quantitative variables include velocity, time, angle measurement, distance and amount.

The other kind of variable is the assigned or qualitative variable which represents such things as missile modification, type of stimuli, which of several measuring devices were used to obtain the data, etc. These variables must be assigned values and cannot logically be given units. Certain conventions for the assigning of values have been set up for this paper.

In Example II, the model could be expanded to include a term for the vat used. The expanded model is

$$y = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + e \ ,$$

where

$$x_1 = v_1$$

$$x_2 = v_2$$

$$x_3 = \begin{cases} -0.5 \text{ if the first vat is used} \\ 0.5 \text{ if the second vat is used.} \end{cases}$$

The matrix X becomes

$$X = \begin{bmatrix} 1.0 & 0.0 & 0.0 & -0.5 \\ 1.0 & 1.0 & 1.0 & -0.5 \\ 1.0 & 2.0 & 4.0 & -0.5 \\ 1.0 & 3.0 & 9.0 & -0.5 \\ 1.0 & 4.0 & 16.0 & -0.5 \\ 1.0 & 5.0 & 25.0 & -0.5 \\ 1.0 & 0.0 & 0.0 & 0.5 \\ 1.0 & 1.0 & 1.0 & 0.5 \\ 1.0 & 2.0 & 4.0 & 0.5 \\ 1.0 & 3.0 & 9.0 & 0.5 \\ 1.0 & 4.0 & 16.0 & 0.5 \\ 1.0 & 5.0 & 25.0 & 0.5 \end{bmatrix} .$$

The vector $X^T \underline{y}$ and the matrices $X^T X$ and $(X^T X)^{-1}$ are

$$X^T \underline{y} = \begin{pmatrix} 121.86 \\ 302.23 \\ 1035.99 \\ -5.585 \end{pmatrix}, \quad X^T X = \begin{pmatrix} 12.0 & 30.0 & 110.0 & 0.0 \\ 30.0 & 110.0 & 450.0 & 0.0 \\ 110.0 & 450.0 & 1958.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 3.0 \end{pmatrix},$$

$$(X^T X)^{-1} = \begin{pmatrix} 0.4107 & -0.2946 & 0.0446 & 0.0 \\ -0.2946 & 0.3634 & -0.0670 & 0.0 \\ 0.0446 & -0.0670 & 0.0134 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.3333 \end{pmatrix} .$$

The estimate $\hat{\underline{b}}$ is thus

$$\hat{\underline{b}} = \begin{pmatrix} 7.249 \\ 4.549 \\ -0.924 \\ -1.860 \end{pmatrix} .$$

The prediction equation is

$$\hat{y} = 7.249 + 4.549\ x_1 - 0.924\ x_2 - 1.860\ x_3 .$$

The estimate of the variance is

$$s^2 = 1.83 .$$

The covariance matrix and the correlation matrix of $\hat{\underline{b}}$ are given below:

$$\text{cov}\ (\hat{\underline{b}}) = \begin{pmatrix} 0.75 & -0.54 & 0.082 & 0.0 \\ -0.54 & 0.66 & -0.12 & 0.0 \\ 0.082 & -0.12 & 0.24 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.61 \end{pmatrix}$$

$$\text{cor}\ (\hat{\underline{b}}) = \begin{pmatrix} 1.00 & -0.77 & 0.65 & 0.00 \\ -0.77 & 1.00 & -0.96 & 0.00 \\ 0.65 & -0.96 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{pmatrix} .$$

In this case, two curves are predicted

$$\hat{y} = 8.179 + 4.549\ v_1 - 0.924\ v_1^2 \text{ if vat 1 is used}$$

$$\hat{y} = 6.319 + 4.549\ v_1 - 0.924\ v_1^2 \text{ if vat 2 is used.}$$

Thus, the difference between the predicted yields from vat 1 and vat 2 with the same amount of catalyst is estimated as $\hat{y}_{\text{vat 2}} - \hat{y}_{\text{vat 1}} = \hat{b}_3 = -1.860$. The two curves are plotted in Figure 3.

This fit may be compared with the preceding fit without the term for vat differences.

In the previous example, there were two vats used and the values of -0.5 and 0.5 were rather arbitrarily assigned to represent the vat used. It is noticeable that the qualitative variable occupies one place in the model and one column in the X matrix; this corresponds to the one difference between two factors.

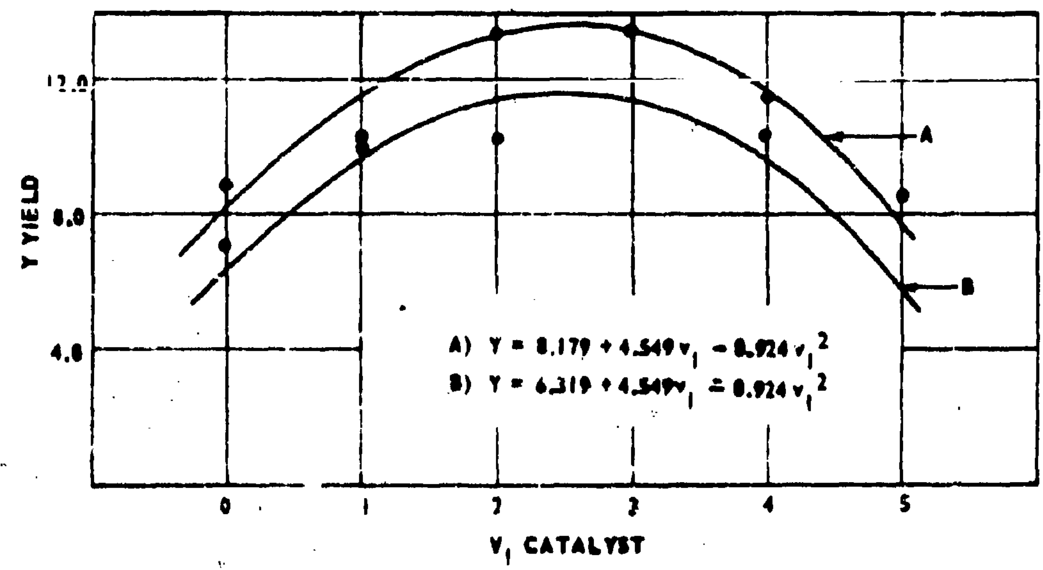

FIGURE 3. PREDICTED YIELD FOR THE EXPANDED MODEL, EXAMPLE II

In Example I, there were three modifications of the missile. There are two linearly independent differences among the three effects, A, B, and C of the modifications. Thus, one could choose B-A and C-B; in this case, C-A is a linear combination of the others, C-A = C-B + B-A. Another choice of linearly independent differences is B-A and 2C-B-A. In this case, two terms are added to the model and two vectors are added to the matrix.

In this work the following values were assigned to the expanded variables for the modifications:

Modification	x_7	x_8
1	-0.5	0.0
2	0.5	-0.5
3	0.0	0.5

If there are four types for a qualitative variable, then there are three independent vectors. They could be assigned the following values:

Type	x_1	x_2	x_3
1	-0.5	0.0	0.0
2	0.5	-0.5	0.0
3	0.0	0.5	-0.5
4	0.0	0.0	0.5

The same pattern is followed for other numbers of types.

6. THE ESTIMATED COEFFICIENTS FOR THE MISSILE MODEL

The data for the missile (Example I) were input into the Generalized Least Squares Fit (GELSF) digital computer program [2], which was written to do the above calculations. The results are shown in Tables III through VI. The predicted model is

$$\hat{y} = 24.103 + 0.086x_1 - 1.496x_2 + 0.020x_3 + 0.666x_4 - 0.016x_5 + 0.087x_6$$
$$+ 6.719x_7 + 6.413x_8 - 2.061x_9 + 3.753x_{10}.$$

a. The Advantages of Normal Noise

In the special case of Gaussian or normal noise, the least squares estimator is also the maximum likelihood estimator. The likelihood function is the joint probability density of the observations. Assuming X is known perfectly, $\underline{b}$ is fixed but unknown, the noise $\underline{e}$ Gaussian with mean $\underline{0}$ and covariance matrix $\sigma^2 I$, the observations will be Gaussian with mean $X\underline{b}$ and covariance matrix $\sigma^2 I$. Thus, the likelihood function is given by

$$\text{Lh.} = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}(\underline{y} - X\underline{b})^T(\underline{y} - X\underline{b})\right].$$

If the derivative of the likelihood function with respect to $\underline{b}$ is set to zero, the value $\hat{\underline{b}}$ which maximizes the likelihood is

$$\hat{\underline{b}} = (X^T X)^{-1} X^T \underline{y}.$$

This is identical to the least squares estimator. If, however, the distribution of the noise is other than Gaussian, the likelihood function and, thus, the maximum likelihood estimator, may be different from the least squares estimator.

Furthermore, if the covariance matrix is of the form $Q = \sigma^2 I$ and the noise is Gaussian, then it can be shown that the ratio $\dfrac{(\underline{y} - X\hat{\underline{b}})^T(\underline{y} - X\hat{\underline{b}})}{\sigma^2}$ has a Chi-square distribution with (n-k-1) degrees of freedom where n is the number of data points or observations and k is the number of x_i's in the model, and thus, $\dfrac{(\underline{y} - X\hat{\underline{b}})^T(\underline{y} - X\hat{\underline{b}})}{n-k-1}$ is an unbiased estimator of σ^2. This is the s^2 discussed in Section 4.

TABLE III. THE ESTIMATED COEFFICIENTS, EXAMPLE I

I	BETA(I)	BASIC TYPE	IDENTIFICATION	NUMBER OF VECTORS
0	0.24103171E 02		CONSTANT	
1	0.66253552E-01	1	TARGET ALT.	1
2	0.14957100E 01	1	RANGE,LAUNCH	2
3	0.19629180E-01			
4	0.66613513E 00	1	RANGE,INTER.	2
5	0.15850848E-01			
6	0.86580925E-01	1	TAR. CL.VEL.	1
7	0.67192430E 01	2	MISSILE MOD.	2
8	0.64127998E 01			
9	0.20606425E 01	2	TARGET TYPE	1
10	0.37527300E 01	2	RADAR POWER	1

THE SUM OF SQUARES FOR ERROR IS 0.13922442E 05

THE ESTIMATE OF THE VARIANCE IS 0.15643194E 03

THE ESTIMATE OF THE STANDARD DEVIATION IS 0.12507275E 02

THE NUMBER OF DEGREES OF FREEDOM IS 89.

TABLE IV. THE RESIDUALS, EXAMPLE I

THE RESIDUALS (ACTUAL OBSERVATIONS - PREDICTED)

1	0.12992613E 02	2	-0.12161280E 02	3	-0.15827706E 02
4	0.13478972E 02	5	-0.15798877E 02	6	-0.13748894E 02
7	0.90875208E 01	8	-0.17314040E 02	9	-0.17530940E 02
10	0.11537761E 02	11	-0.70560862E 01	12	-0.11932177E 02
13	0.35928126E 01	14	-0.88353692E 01	15	-0.53195298E 01
16	0.11606226E 02	17	-0.661453[illegible]E 01	18	-0.11071021E 02
19	0.10677389E 02	20	-0.4729[illegible]E 01	21	-0.72510670E 01
22	-0.10747760E 02	23	-0.56442658E 01	24	-0.10487110E 02
25	0.61851535E 01	26	-0.12793162E 02	27	-0.76293404E 01
28	0.90082127E 01	29	-0.93267230E 01	30	-0.10948503E 02
31	0.13144054E 01	32	-0.31560862E 01	33	-0.98602482E 01
34	0.63506298E 01	35	-0.75681361E 01	36	-0.21880586E 01
37	0.38325926E 01	38	-0.38519159E 01	39	-0.24691151E 01
40	0.25477273E 01	41	-0.27746752E 01	42	-0.64946657E 01
43	0.27930587E 00	44	-0.49562793E 01	45	-0.12232212E 00
46	0.35629303E 01	47	-0.76982490E 01	48	-0.28738209E 01
49	0.86658587E 01	50	-0.56360195E 01	51	0.68971837E 00
52	0.33733943E 01	53	-0.47334766E 01	54	-0.37444960E 00
55	0.29189003E 01	56	-0.41027260E 01	57	0.12090253E 00
58	0.10391162E 01	59	0.27633111E 00	60	0.24198041E 01
61	0.28834528E 01	62	-0.49205871E 01	63	0.19410922E 01
64	0.61723503E 00	65	-0.10700869E 02	66	0.70708198E 01
67	0.53188329E 00	68	0.61561046E 00	69	0.20672664E 01
70	0.15345565E 01	71	0.77161704E 01	72	0.90852933E 01
73	0.73506132E 01	74	0.62162980E 01	75	0.51422264E 01
76	0.69429379E 01	77	0.36491225E 01	78	0.35065422E 01
79	0.62123384E 01	80	0.66303782E 01	81	0.62565392E 01
82	0.73402544E 01	83	0.69008548E 01	84	0.13307215E 02
85	0.14511830E 02	86	0.79642540E 01	87	0.10505017E 02
88	0.17371762E 02	89	0.14072209E 02	90	0.96639019E 01
91	0.16840611E 02	92	0.13396480E 02	93	0.14830160E 02
94	0.24227803E 02	95	0.81107840E 01	96	0.29744796E 02
97	0.30398627E 02	98	0.12325960E 02	99	0.40945884E 02
100	0.51625341E 02				

TABLE V. THE ESTIMATED COVARIANCE MATRIX, EXAMPLE I

COLUMNS 1 THROUGH 4 OF THE COVARIANCE MATRIX

0.93542284E 00	-0.36356692E-01	-0.56652870E-03	-0.56552546E-01
0.36356692E-01	0.19349247E 01	-0.24099465E-01	-0.56836866E 00
0.56652870E-03	-0.24099465E-01	0.35275723E-03	0.95503378E-02
0.56552546E-01	-0.56836866E 00	0.95503378E-02	0.12606248E 01
0.27765373E 02	0.45275652E 02	-0.19510789E 03	-0.28402689E-01
0.86776596E-02	0.67653409E-02	-0.27832166E-03	-0.10729626E-01
0.17877101E 01	-0.14537644E 00	-0.43267103E-02	-0.85554210E 00
0.31085822E 00	-0.12754077E-01	-0.61321814E-02	-0.10186067E 01
0.16707530E 01	-0.99729203E 00	0.93364145E-02	0.52244540E 00
0.40464304E 00	0.21969026E 00	-0.48231689E-02	-0.62695605E 00

COLUMNS 5 THROUGH 8 OF THE COVARIANCE MATRIX

0.27765373E-02	0.86776596E-02	-0.17877101E 01	-0.31085822E 00
0.45275652E-02	0.67653409E-02	-0.14537644E 00	-0.12754077E-01
0.19510789E-03	-0.27832166E-03	-0.43267103E-02	-0.61321814E-02
0.28402689E-01	-0.10729626E-01	-0.85554210E 00	-0.10186067E 01
0.84151427E-03	0.53369615E-03	0.22198053E-01	0.32538614E-01
0.53369615E-03	0.21573451E-02	0.64849563E-01	0.49461851E-01
0.22198053E-01	0.64849563E-01	0.71528088E 02	0.30512465E 02
0.32538614E-01	0.49461851E-01	0.30512465E 02	0.29322505E 02
0.12007927E-01	-0.11804427E 00	0.10935714E 02	0.28468976E 01
0.14573285E-01	0.85635212E-02	0.22797647E 01	0.66327820E 01

TABLE V. THE ESTIMATED COVARIANCE MATRIX, EXAMPLE I (Concluded)

COLUMNS 9 THROUGH 10 OF THE COVARIANCE MATRIX

0.16707530E 01	0.40464304E 00
0.99729209E 00	0.21969026E 00
0.93364145E-02	-0.48231689E-02
0.52244540E 00	-0.62695605E 00
0.12007027E-01	0.14573285E-01
0.11304427E 00	0.85635212E-02
0.10935714E 02	0.22797647E 01
0.28468976E 01	0.66327820E 01
0.26121725E 02	-0.14371421E 01
0.14371421E 01	0.89300860E 01

TABLE VI. THE ESTIMATED CORRELATION MATRIX, EXAMPLE I

COLUMNS 1 THROUGH 4 OF THE CORRELATION MATRIX

0.10000000E 01	-0.28595596E-01	-0.33001279E-01	-0.55106910E-01
0.28595596E-01	0.10000000E 01	-0.92243847E 00	-0.36391936E 00
0.33001279E-01	-0.92243847E 00	0.10000000E 01	0.45288490E 00
0.55106910E-01	-0.36391936E 00	0.45288490E 00	0.10000000E 01
0.10472135E 00	0.11220264E 00	-0.35810149E 00	-0.87203876E 00
0.20440505E 00	0.10471233E 00	-0.31904300E 00	-0.20574622E 00
0.23126270E 00	-0.12357303E-01	-0.27238397E-01	-0.90096964E-01
0.62807087E-01	-0.16932320E-02	-0.60294333E-01	-0.16753790E 00
0.35764954E 00	-0.14027794E 00	0.97261560E-01	0.91043105E-01
0.14814639E 00	0.52850727E-01	-0.85934328E-01	-0.18686000E 00

COLUMNS 5 THROUGH 8 OF THE CORRELATION MATRIX

0.10472135E 00	0.20440505E 00	-0.23126270E 00	-0.62807087E-01
0.11220264E 00	0.10471233E 00	-0.12357303E-01	-0.16932320E-02
0.35810149E 00	-0.31904300E 00	-0.27238397E-01	-0.60294333E-01
0.87203876E 00	-0.20574622E 00	-0.90096964E-01	-0.16753790E 00
0.10000000E 01	0.35605879E 00	0.90478593E-01	0.20714181E 00
0.39609879E 00	0.10000000E 01	0.16598550E 00	0.19665727E 00
0.90478593E-01	0.16598550E 00	0.10000000E 01	0.66625189E 00
0.20714181E 00	0.19665727E 00	0.66625189E 00	0.10000000E 01
0.80384859E-01	-0.49726076E 00	0.25299265E 00	0.10286565E 00
0.16811207E 00	0.61697099E-01	0.90203603E-01	0.40989004E 00

TABLE VI. THE ESTIMATED CORRELATION MATRIX, EXAMPLE I (Concluded)

COLUMNS 9 THROUGH 10 OF THE CORRELATION MATRIX

0.35764994E 00	0.14814639E 00
0.14027794E 00	0.52850727E-01
0.97261560E-01	-0.65934228E-01
0.81343105E-01	-0.18686000E 00
0.90984859E 01	0.16811207E 00
0.49726076E 00	0.61697099E-01
0.25299265E 00	0.90203603E-01
0.10286565E 00	0.40989004E 00
0.10000000E 01	-0.94099556E-01
0.94099556E-01	0.10000000E 01

Further, the assumption of normally distributed noise is used in the statistical tests and confidence intervals to be discussed in Section 10.

The data for Example II was generated under the assumption of normal noise. However, there is a question to be asked about the missile data of Example I.

7 TESTING THE MISSILE DATA FOR NORMAL NOISE

Before conclusions based upon the assumption of normally distributed noise can be drawn for Example I, a test for normality must be made. The residuals (actual y - predicted y) estimate the error e and, thus, should be tested for normality.

It was decided to use the method of normal probability paper and control bands to test the residuals for normality. The Testing for Normality by Control Bands (TEN COB) digital computer program was used.

8. NORMAL PROBABILITY PAPER

The construction of normal probability paper is similar to that of logarithmic paper. Assume that the random variable r has a normal (Gaussian) distribution with mean μ and variance σ^2. Then the reduced variate $v = (r - \mu)/\sigma$ has a standard normal distribution, i.e., v has mean 0 and variance 1. If r is plotted on a horizontal linear scale and v is plotted on a vertical linear scale, the straight line $r = \sigma v + \mu$ will result (Figure 4).

Since v is a standard normal random variable, the cumulative distribution function of v is given by

$$F(v) = \int_{-\infty}^{v} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt .$$

On a second vertical scale the distribution function F (v) is plotted (Figure 5). The values of F (v) = 0 and F (v) = 1 never appear on the scale, since these correspond to values of $v = -\infty$ and $v = \infty$, respectively. If the lines are drawn for the function F (v) instead of v, a nonlinear vertical scale is shown (Figure 6).

a. Plotting of Points on Probability Paper

It is desired to test whether the underlying probability distribution of the n residuals is normal or can be approximated by a normal distribution.

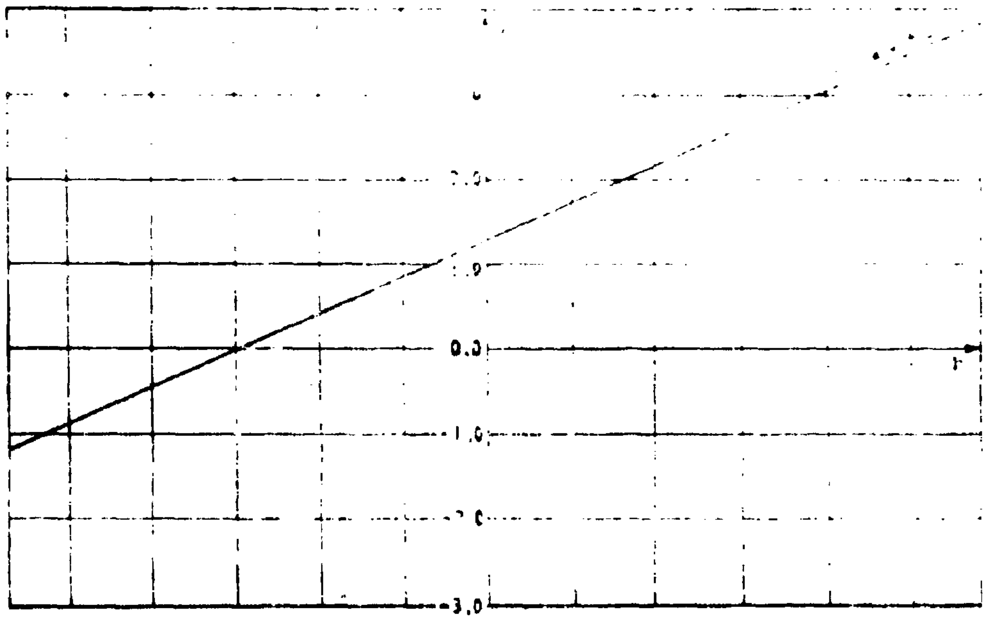

FIGURE 4. THE REDUCED VARIATE

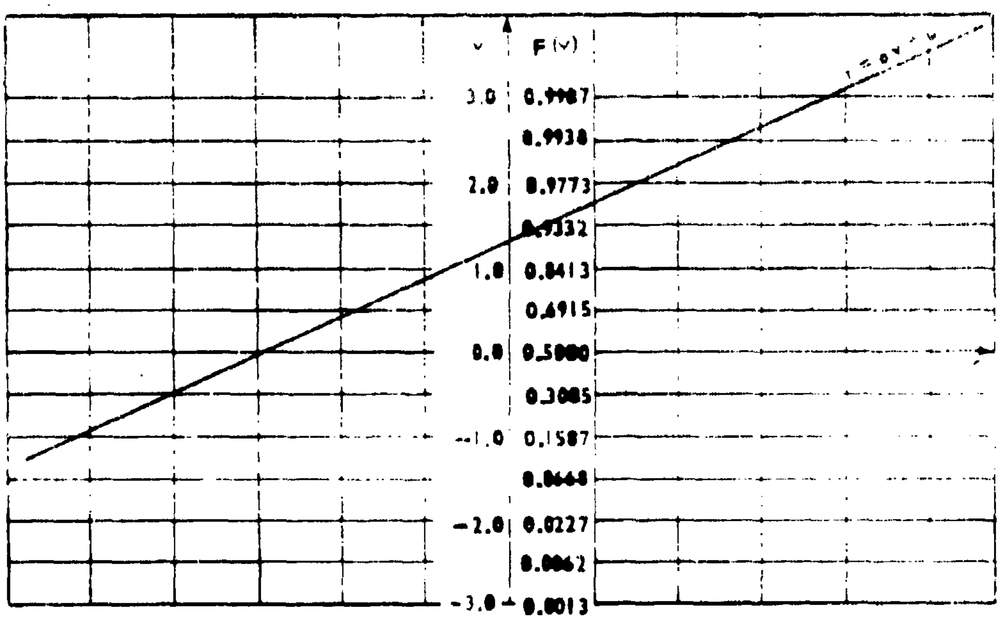

FIGURE 5. REDUCED VARIATE AND DISTRIBUTION FUNCTION

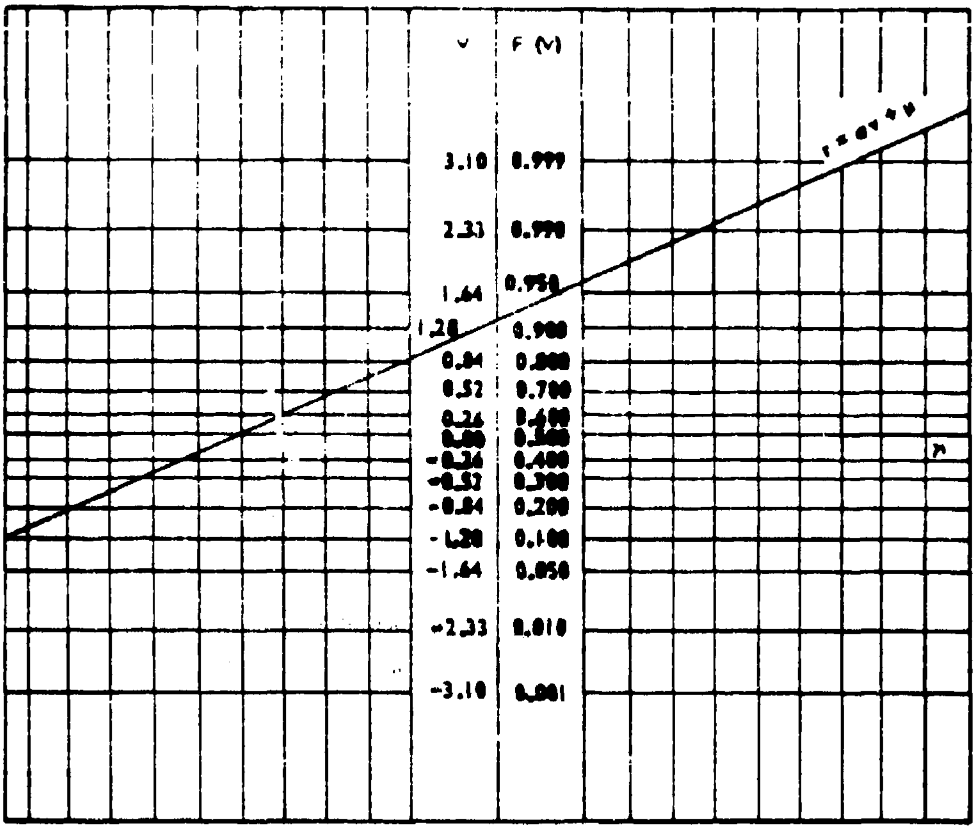

FIGURE 6. EXAMPLE OF NORMAL PROBABILITY PAPER

If the distribution is indeed normal, the residuals, when plotted against F (v) in the manner discussed below, should approximate a straight line, $r = \gamma + v/\delta$. The deviations from the line are caused by the finite random character of the sample of size n and such errors as round-off.

Let $r_1, r_2, \ldots, r_n$ be the n residuals arranged in ascending order. Several methods of plotting this sequence of numbers against F (v) are considered by Gumbel [3]. The best method is that of plotting r_j against $j/(n + 1)$. This method is distribution free and all observations can be plotted. Further, the plotting positions are simple to calculate.

Listed in Table VII are the 100 residuals arranged in ascending order. The corresponding values of $F(v) = j/(n + 1)$ and v are also listed.

[illegible]

[illegible] The Residuals in Ascending Order	J The Rank	[illegible] Distribution Function Plotting Positions	[illegible] Variate Plotting Position
-19.799	1	0.0099	-2.329
-17.531	2	0.0198	-2.05[illegible]
-17.314	3	0.0297	-1.885
-15.828	4	0.0396	-1.755
-13.749	5	0.0495	-1.649
-13.479	6	0.0594	-1.559
-12.993	7	0.0693	-1.4[illegible]
-12.793	8	0.0792	-1.41[illegible]
-12.161	9	0.0891	-1.347
-11.932	10	0.0990	-1.285
-11.606	11	0.1089	-1.232
-11.546	12	0.1188	-1.180
-11.071	13	0.1287	-1.132
-10.949	14	0.1386	-1.086
-10.745	15	0.1485	-1.042
-10.701	16	0.1584	-1.001
-10.677	17	0.1683	-0.960
-10.487	18	0.1782	-0.922
-9.860	19	0.1881	-0.884
-9.328	20	0.1980	-0.848
-9.088	21	0.2079	-0.813
-9.008	22	0.2178	-0.779
-8.835	23	0.2277	-0.746
-8.666	24	0.2376	-0.714
-7.698	25	0.2475	-0.682
-7.629	26	0.2574	-0.651
-7.568	27	0.2673	-0.621
-7.251	28	0.2772	-0.591
-7.065	29	0.2871	-0.561
-6.615	30	0.2970	-0.532

TABLE VII. RESIDUALS OF EXAMPLE I WITH CORRESPONDING REDUCED VARIATE (Continued)

r_j The Residuals in Ascending Order	j The Rank	F (v) = j/101 Distribution Function Plotting Positions	v Reduced Variate Plotting Positions
-6.495	31	0.3069	-0.504
-6.351	32	0.3168	-0.476
-6.185	33	0.3267	-0.448
-5.644	34	0.3366	-0.421
-5.636	35	0.3465	-0.394
-5.319	36	0.3564	-0.367
-4.996	37	0.3663	-0.341
-4.921	38	0.3762	-0.315
-4.733	39	0.3861	-0.289
-4.729	40	0.3960	-0.263
-4.103	41	0.4059	-0.238
-3.852	42	0.4158	-0.212
-3.833	43	0.4257	-0.182
-3.593	44	0.4356	-0.162
-3.563	45	0.4455	-0.136
-3.373	46	0.4554	-0.112
-3.156	47	0.4653	-0.086
-2.919	48	0.4752	-0.062
-2.874	49	0.4851	-0.037
-2.775	50	0.4950	-0.012
-2.548	51	0.5050	0.012
-2.464	52	0.5149	0.036
-2.188	53	0.5248	0.062
-1.814	54	0.5347	0.086
-1.039	55	0.5446	0.112
-0.617	56	0.5545	0.136
-0.374	57	0.5644	0.162
-0.122	58	0.5743	0.187
0.120	59	0.5842	0.212
0.276	60	0.5941	0.238

TABLE VII. RESIDUALS OF EXAMPLE I WITH CORRESPONDING REDUCED VARIATE (Continued)

r_j The Residuals in Ascending Order	j The Rank	F (v) = j/101 Distribution Function Plotting Positions	y Reduced Variate Plotting Positions
0.279	61	0.6040	0.263
0.592	62	0.6139	0.289
0.616	63	0.6238	0.315
0.690	64	0.6337	0.341
1.535	65	0.6436	0.367
1.941	66	0.6535	0.394
2.067	67	0.6634	0.421
2.420	68	0.6733	0.448
2.880	69	0.6832	0.476
3.507	70	0.6931	0.504
3.649	71	0.7030	0.532
5.142	72	0.7129	0.561
6.212	73	0.7228	0.591
6.216	74	0.7327	0.621
6.254	75	0.7426	0.651
6.630	76	0.7525	0.682
6.901	77	0.7624	0.714
6.943	78	0.7723	0.746
7.071	79	0.7822	0.779
7.351	80	0.7921	0.813
7.716	81	0.8020	0.848
7.864	82	0.8119	0.884
7.940	83	0.8218	0.922
8.111	84	0.8317	0.960
9.085	85	0.8416	1.001
9.664	86	0.8515	1.042
10.505	87	0.8614	1.086
12.326	88	0.8713	1.132
13.307	89	0.8812	1.180
13.396	90	0.8911	1.232

TABLE VII. RESIDUALS OF EXAMPLE I WITH CORRESPONDING REDUCED VARIATE (Concluded)

r_j The Residuals in Ascending Order	j The Rank	F (v) = j/101 Distribution Function Plotting Positions	v Reduced Variate Plotting Positions
14.072	91	0.9010	1.285
14.830	92	0.9109	1.347
14.912	93	0.9208	1.410
16.841	94	0.9307	1.480
17.372	95	0.9406	1.559
24.228	96	0.9505	1.649
29.745	97	0.9604	1.755
30.698	98	0.9703	1.885
40.946	99	0.9802	2.058
51.625	100	0.9901	2.329

Figure 7 shows the points plotted on normal probability paper by a modified version of the TEN COB program. The horizontal scale is the r-scale; the horizontal line defining the grid extends from -37.655 to 69.981 for the case. The vertical scale is the reduced variate or v-scale; the vertical line defining the grid extends from v = -4.0 to v = 4.0.

b. Fitting the Straight Line

If the scatter of the plotted points is very small, the best fitting straight line can be found by lining up a ruler through the points. However, in many cases this is not satisfactory. It is then necessary to estimate δ and γ, the two parameters of the straight line;

$$r = \gamma + \frac{v}{\delta}; \quad v = \delta\ (r - \gamma)\ .$$

For the estimation of these parameters, the classical method of least squares will be employed.

In the method of least squares, either the sum of squares of the horizontal deviations of the points from the estimated straight line,

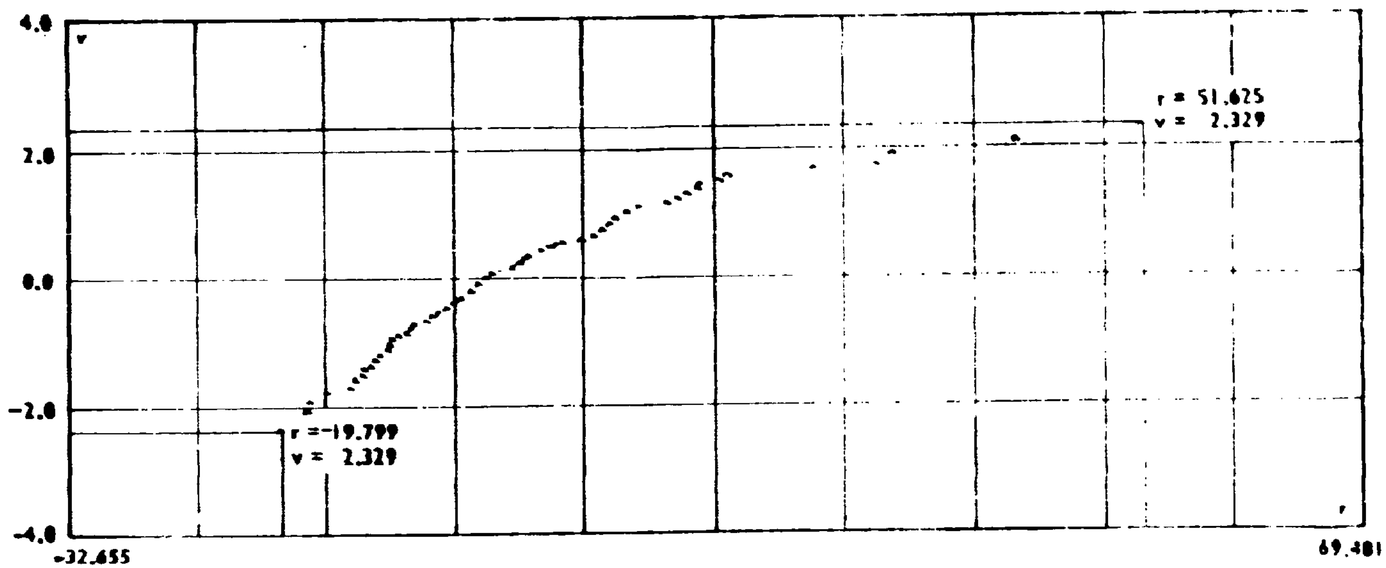

FIGURE 7. RESIDUALS OF EXAMPLE I PLOTTED ON NORMAL PROBABILITY PAPER

$$\sum_{i=1}^{n}\left(r_i - g_1 - \frac{v_i}{a_1}\right)^2,$$

or that of the vertical distances,

$$\sum_{i=1}^{n}\left[v_i - a_2\left(r_i - g_2\right)\right]^2 ,$$

is minimized. Here a_1 and a_2 represent the two estimates of δ, and g_1 and g_2 represent the two estimates of γ. If the partial derivatives of the first sum with respect to a_1 and g_1 are set to zero, the results are

$$-2\sum_{i=1}^{n}\left(r_i - g_1 - \frac{1}{a_1}v_i\right) = 0$$

and

$$\frac{2}{a_1^2}\sum_{i=1}^{n} v_i\left(r_i - g_1 - \frac{1}{a_1}v_i\right) = 0$$

or

$$\bar{r} - g_1 - \frac{1}{a_1}\bar{v} = 0 ,$$

$$\overline{rv} - g_1\bar{v} - \frac{1}{a_1}\overline{v^2} = 0 ,$$

where

$$\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i,\quad \bar{v} = \frac{1}{n}\sum_{i=1}^{n} v_i ,$$

$$\overline{rv} = \frac{1}{n}\sum_{i=1}^{n} r_i v_i,\quad \overline{v^2} = \frac{1}{n}\sum_{i=1}^{n} v_i^2 .$$

For a normal distribution, $\bar{v} = 0$. Thus, the solution is

$$\frac{1}{a_1} = \frac{\overline{rv}}{s_v^{\,2}}$$

$$g_1 = \overline{r} \; ,$$

where

$$s_v^{\,2} = \overline{v^2} - \overline{v}^2 = \overline{v^2} \; .$$

If the sum of squares of the vertical deviations is to be minimized, the partials with respect to g_2 and a_2 are set to 0,

$$-2 \sum_{i=1}^{n} \left[v_i - a_2 \left(r_i - g_2 \right) \right] = 0$$

and

$$-2 \sum_{i=1}^{n} \left(r_i - g_2 \right) \left[v_i - a_2 \left(r_i - g_2 \right) \right] = 0$$

or

$$\overline{v} - a_2 \, \overline{r} + a_2 \, g_2 = 0$$

$$\overline{rv} - a_2 \, \overline{r^2} + a_2 \, g_2 \, \overline{r} = 0 \; ,$$

where

$$\overline{r^2} = \frac{1}{n} \sum_{i=1}^{n} r_i^{\,2} \; .$$

With $v = 0$, the solution is

$$\frac{1}{a_2} = \frac{\overline{r^2} - \overline{r}^2}{\overline{rv}}$$

$$g_2 = \overline{r} \; .$$

The estimate $1/a_2$ is altered slightly to

$$\frac{1}{a_2} = \frac{s_r^{\,2}}{\overline{rv}} \; ,$$

where s_r^2 is the sample variance,

$$s_r^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(r_i - \bar{r} \right)^2 = \frac{(\overline{r^2} - \bar{r}^2)\, n}{n-1} .$$

In order to combine the two estimates of $1/\sigma$ and eliminate the cross product $\overline{rv}$, the geometric mean of $1/a_1$ and $1/a_2$ is found. Thus, the two combined estimates are

$$\frac{1}{a} = \sqrt{\frac{1}{a_1} \frac{1}{a_2}} = \frac{s_r}{s_v}$$

$$g = \bar{r} .$$

These estimates require the calculation of only $\bar{r}$ and s_r from the sample. The value of s_v depends only upon the number of points n. As the sample size n increases, s_v approaches 1 and the estimate $\bar{r}$ and s_r approach the true values μ and σ; thus, the equation of the straight line approaches $r = \mu + \sigma v$, discussed in Section 8.

The estimated straight line,

$$r = \bar{r} + \frac{s_r}{s_v} v ,$$

is plotted on the same paper as the points. If the points lie close to the estimated line, the distribution is considered to be approximately normal. If the scatter is too great with respect to the line, the distribution is considered non-normal. To determine whether the scatter is too great, control bands are placed around the line.

For Example I, the values range from -19.799 to 51.625. The end-points of the graph are -37.655 and 69.481. The average residual, $\bar{r} = 0.00286$; the estimate of $1/\sigma$ is $1/a = 12.353$. In Figure 8 are shown the points of Example I plotted along with the estimated straight line, $r = 0.00286 + 12.353\, v$.

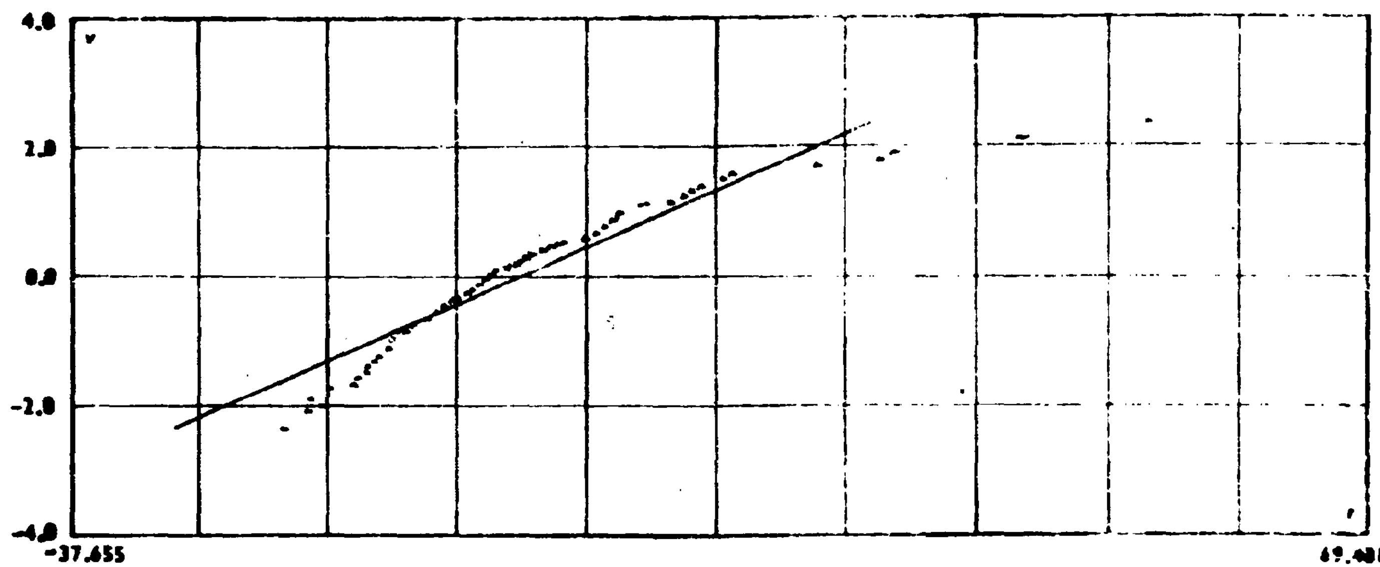

FIGURE 8. RESIDUALS AND BEST FITTING LINE OF EXAMPLE I

c. <u>Control Bands</u>

The j^{th} largest observation r_j in a sample of size n is called the j^{th} order statistic. Each observation was drawn from a population with density function f and distribution function F. In this case, the initial distribution is assumed to be normal. The j^{th} order statistic r_j has a derived density $f_n\left(r_j\right)$ that depends upon the initial distribution F and upon the values of j and n. It can be shown that this derived density function is

$$f_n\left(r_j\right) = \frac{n!}{(n-j)!\,(j-1)!}\, F^{j-1}\left(r_j\right)\left[1 - F\left(r_j\right)\right]^{n-j} f\left(r_j\right) .$$

Substituting the equations for a normal variate for $F\left(r_j\right)$ and $f\left(r_j\right)$, one obtains

$$f_n\left(r_j\right) = \frac{n!}{(n-j)!\,(j-1)!} \left\{\int_{-\infty}^{r_j} \frac{\exp\left[-\frac{1}{2\sigma^2}(t-\mu)^2\right]}{\sqrt{2\pi}\,\sigma}\,dt\right\}^{j-1}$$

$$\cdot \left\{1 - \int_{-\infty}^{r_j} \frac{\exp\left[-\frac{1}{2\sigma^2}(t-\mu)^2\right]}{\sqrt{2\pi}\,\sigma}\,dt\right\}^{n-j}$$

$$\cdot \left\{\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}\left(r_j-\mu\right)^2\right]\right\} .$$

This complicated form does not reduce to anything more reasonable and is difficult to manipulate and calculate. Thus, asymptotic distributions of the order statistics are used.

As n becomes larger, either j will increase with n so that j/n remains approximately constant or j will remain constant so that j/n decreases. In the former case, the j^{th} value r_j is called the j^{th} central value; in the second case, the values r_j and n-j + 1 are called extreme values.

It can be shown that as n increases, the distribution of the j^{th} central value, r_j, becomes asymptotically normal with mean and variance

$$E\, r_j = r_j^*$$

$$\operatorname{var}\left(r_j\right) = \sigma_j^2 = \frac{\left[F\left(r_j^*\right)\right]\left[1 - F\left(r_j^*\right)\right]}{n\, f^2\left(r_j^*\right)} \,,$$

where r_j^* is the solution of

$$F\left(r_j^*\right) = \frac{j}{n+1} \,.$$

The asymptotic distribution is used within the interval $0.15 \leq F \leq 0.85$, although its accuracy within this interval depends upon the sample size.

For the reduced variate $v_j = \alpha\left(r_j - \gamma\right)$, the variance is given by

$$\operatorname{var}\left(v_j\right) = \alpha^2 \operatorname{var}\left(r_j\right) \,,$$

and is independent of the parameters a and g. The product $\sqrt{n \cdot \operatorname{var}\left(v_j\right)}$ is independent of the sample size and is a function of the initial distribution only. A chart of this product for several values of F, assuming that the initial standard deviation $\sigma = 1$, is shown in Table VIII.

TABLE VIII. REDUCED STANDARD ERRORS FOR UNIT STANDARD DEVIATION

Probability, F	$\sqrt{n \cdot \operatorname{var}\left(v_j\right)}$
0.15	1.532
0.20	1.429
0.25	1.363
0.30	1.318
0.35	1.288
0.40	1.268
0.45	1.257
0.50	1.253
0.55	1.257
0.60	1.268
0.65	1.288
0.70	1.318
0.75	1.363
0.80	1.429
0.85	1.532

Because of the relationship $r = \frac{1}{\alpha} v + \gamma$, the standard deviations are related by s.d. $\left(r_j\right)$ = s.d. $\left(v_j\right)/\alpha$. This can be written in the form

$$\text{s.d.} \left(r_j\right) = \frac{\sqrt{n \cdot \text{var}\left(v_j\right)}}{\alpha \sqrt{n}} .$$

Thus, the standard error of the j^{th} value, r_j, can be estimated by multiplying the value $\sqrt{n \cdot \text{var}\left(v_j\right)}$ by $\left(\frac{1}{a} \times \frac{1}{\sqrt{n}}\right)$. For example, for the data given in Section 8.a, n = 100 and $\frac{1}{a} = 12.353$, so s.d. (r_{20}) = 1.429 (12.353)/10 = 1.756. This standard error is used in the construction of control bands.

Assume that a and g have been estimated and that the observations and the estimated straight line

$$r = g + \frac{1}{a} v$$

have been plotted. For each probability value listed in Table VIII, the estimate of r_j is found by intercepting the estimated line with a horizontal line from the probability value and reading the corresponding r value, $r_{j,\,est}$. The standard error, s.d. $\left(r_j\right)$, is added and subtracted from the value $r_{j,\,est}$. The points $r_{j,\,est}$ + s.d. $\left(r_j\right)$ are joined to form one curve; the points $r_{j,\,est}$ - s.d. $\left(r_j\right)$ are joined to form another curve.

To complete the curves, it is necessary to find the standard error of some of the extreme values. The asymptotic distribution of the extremes is not normal and is, in fact, very complicated. In Table IX are given some values of the reduced standard error for the largest reduced order statistic. By the symmetry of the normal distribution, this is also the standard error for the smallest reduced order statistic.

The values in Table IX are approximated in the TEN COB program by the formula,

$$\text{s.d.} \left(v_n\right) = \text{s.d.} \ (v_1) = 0.71 - 0.061\, \ln_e \ (n) \ ,$$

which is fairly accurate for samples of 20 to 500 points. The standard error of the reduced largest (or smallest) value is multiplied by the estimate 1/a to

TABLE IX. STANDARD ERRORS FOR REDUCED LARGEST AND SMALLEST VALUES

Sample Size	s.d. (r_n) = s.d. (r_1)
20	0.52
25	0.51
30	0.50
40	0.48
50	0.46
75	0.45
100	0.43
200	0.40
300	0.36

obtain s.d. (r_n) [or s.d. (r_1)]. This value is then added to and subtracted from $r_{n,\,est}$ or $(r_{1,\,est})$ to give the points $r_{n,\,est}$ + s.d. (r_n) and $r_{n,\,est}$ + s.d. (r_n) or $[r_{1,est}$ + s.d. (r_1) and $r_{1,\,est}$ - s.d. $(r_1)]$. These points are added to the proper curves to extend the control curves. There is a probability of 0.68 for each j^{th} observation to lie within the band formed by the two control curves. Figure 9 shows the 68-percent control band for Example I.

Multiplication of s.d. (r_j) by 0.6745, 1.960, 2.576, 2.807, and 3.290 leads to bands corresponding to the probabilities 0.50, 0.95, 0.99, 0.995, and 0.999. In Figure 10 the 95-percent control band has been added. This is the standard graph produced by the TEN COB program.

d. <u>Testing for Normality by Control Bands</u>

The method of control bands used by the TEN COB program gives a graphical criterion for the goodness of fit between the theoretic normal distribution and the observations.

If almost all of the observations fall within the 95-percent band, the underlying distribution can be assumed to be normal for most purposes and statistical tests and confidence intervals that depend upon an underlying normal distribution (such as the Student's t, the Chi-square, and the F tests) can be applied. If almost all of the observations fall within the 68-percent band, more confidence can be placed on the population's being normal. The term "almost all"

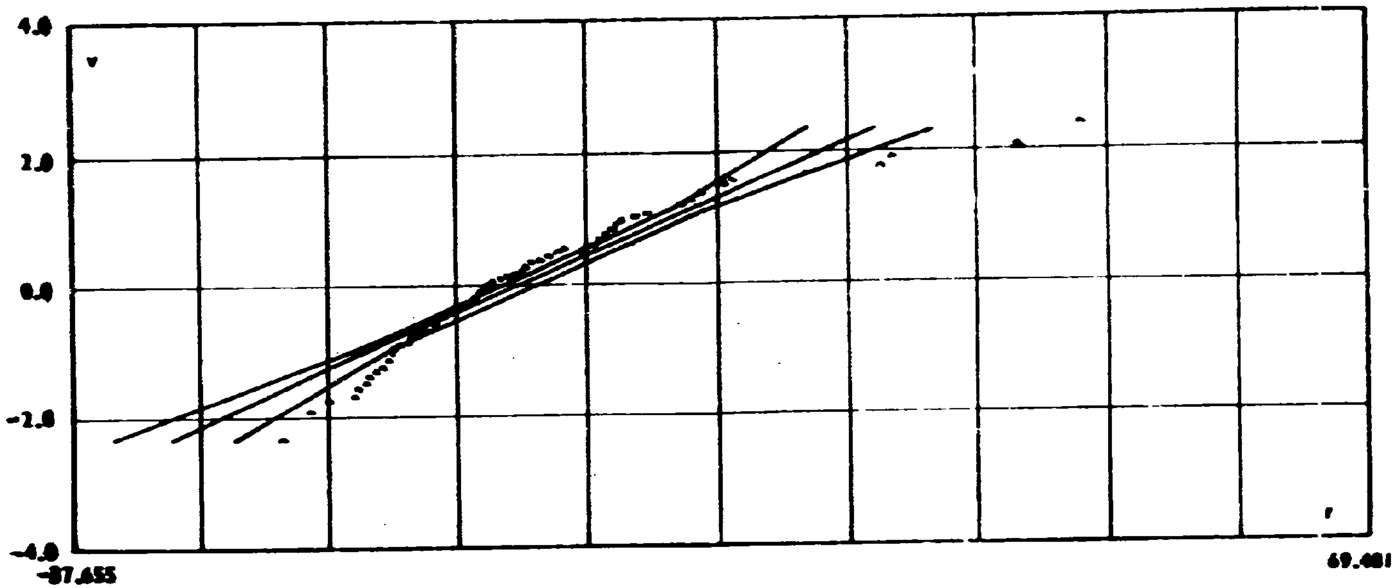

FIGURE 9. RESIDUALS, BEST FITTING LINE, AND 68-PERCENT CONTROL BAND FOR EXAMPLE I

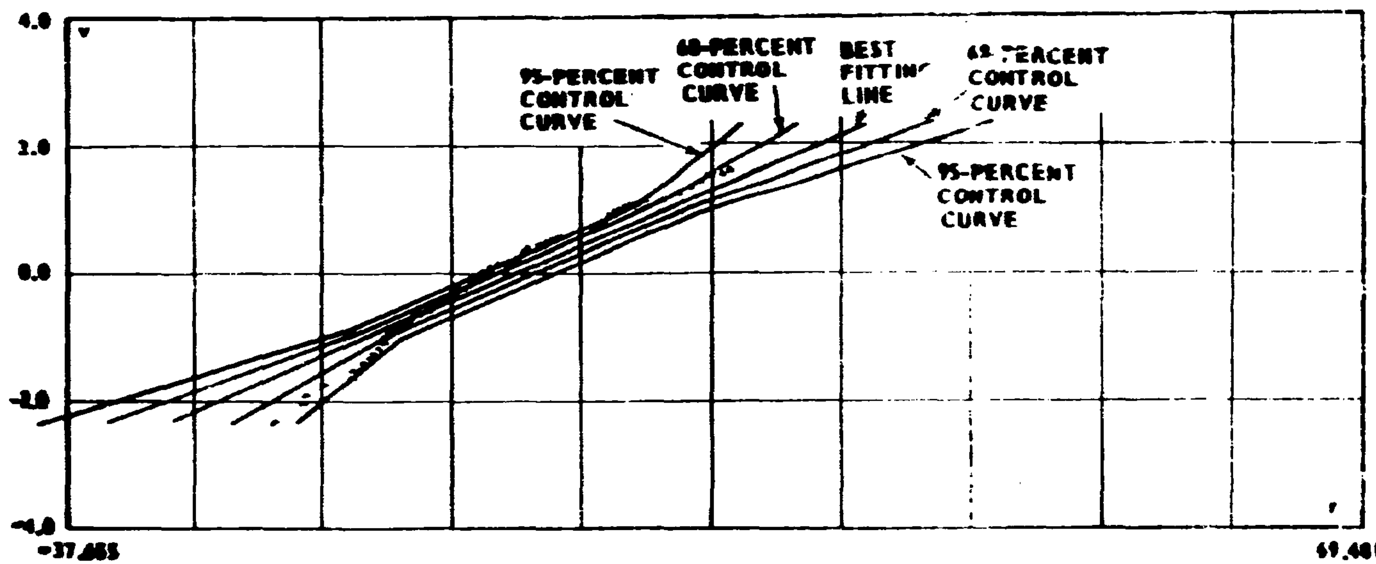

FIGURE 10. RESIDUALS, BEST FITTING LINE, 68-PERCENT CONTROL BAND, AND 95-PERCENT CONTROL BAND FOR EXAMPLE I

is necessarily vague since the degree to which the distribution must match a normal one varies with each set of observations.

In Example I, 67 percent of the residuals fell outside the 68-percent control curve and 18 percent fell outside the 95-percent control band. Thus, the data are probably not normally distributed.

When the observations are shown to be not normally distributed, one of two methods can be employed. The data can be tested against other types of distributions, such as the negative exponential, the log normal, etc. There are disadvantages to this approach. For one, the list of distributions to be tried is long. Further, even if a distribution is found that will approximate that of the observation, it would not have as many well-known associated tests and procedures as the normal distribution.

The other approach is to find a transformation of the observations that will result in normally distributed transformed observations. A list of suggested transformations can be found in Snedecor and Cochran [5].

9. REVISED MODEL FOR THE MISSILE DATA

Since the residuals for the radial miss distance of Example I were not normally distributed, it was decided to try transforming the miss distances to obtain normality of the residuals. The transformation $z = \ln_e y$ was made and the data were used in the GELSF program. The resulting calculation are shown in Tables X through XIII. The prediction equation was:

$$\begin{aligned}\ln y = {} & 2.758 - 0.00947\, x_1 - 0.0363\, x_2 + 0.000632\, x_3 + 0.0165\, x_4 \\ & - 0.000726\, x_5 + 0.00301\, x_6 + 0.0910\, x_7 + 0.250\, x_8 - 0.167\, x_9 \\ & + 0.134\, x_{10}.\end{aligned}$$

The residuals were plotted by the TEN COB program, as shown in Figure 11. These residuals appear to be normally distributed. Thus, this was the model upon which the conclusions were drawn.

10. INTERPRETING THE RESULTS

The probability distribution of the residuals when the logarithms of the radial miss distances are used in Example I is approximately normal. Thus, several methods of testing hypothesis and setting confidence regions are applicable. A few of these are discussed below.

TABLE X. THE ESTIMATED COEFFICIENTS USING LOGARITHMS, EXAMPLE I

I	BETA(I)	BASIC TYPE	IDENTIFICATION	NUMBER OF VECTORS
0	0.27579525E 01		CONSTANT	
1	-0.94697267E-02	1	TARGET ALT.	1
2	0.36300478E-01	1	RANGE,LAUNCH	2
3	0.63246976E-03			
4	0.16511960E-01	1	RANGE,INTER.	2
5	-0.72613758E-03			
6	0.30117154E-02	1	TAR. CL.VEL.	1
7	0.91036746E-01	2	MISSILE MOD.	2
8	0.25004798E 00			
9	-0.16691358E 00	2	TARGET TYPE	1
10	0.13372951E 00	2	RADAR POWER	1

THE SUM OF SQUARES FOR ERROR IS 0.35121311E 02

THE ESTIMATE OF THE VARIANCE IS 0.39462148E 00

THE ESTIMATE OF THE STANDARD DEVIATION IS 0.62818904E 00

THE NUMBER OF DEGREES OF FREEDOM IS 89.

TABLE XI. THE RESIDUALS USING LOGARITHMS, EXAMPLE I

THE RESIDUALS (ACTUAL OBSERVATIONS - PREDICTED)

1	0.13385597E 01	2	-0.11799058E 01	3	-0.10448189E 01
4	0.10502299E 01	5	-0.13319544E 01	6	-0.11663169E 01
7	0.79629727E 00	8	-0.10740552E 01	9	-0.94477469E 00
10	0.60914525E 00	11	-0.61321748E 00	12	-0.77579280E 00
13	0.46630567E 00	14	-0.62421568E 00	15	-0.39156256E 00
16	0.75759597E 00	17	-0.52438433E 00	18	-0.53046221E 00
19	0.59447633E 00	20	-0.30190491E 00	21	-0.51173764E 00
22	0.60510261E 00	23	-0.37542816E 00	24	-0.45460771E 00
25	0.44047211E 00	26	-0.55882984E 00	27	-0.25522122E 00
28	0.40321058E 00	29	-0.43049059E 00	30	-0.47737660E 00
31	0.13784327E 00	32	-0.19021748E 00	33	-0.30079470E 00
34	0.22922132E 00	35	-0.34795099E 00	36	-0.31082278E-01
37	0.13296481E 00	38	-0.14447162E 00	39	-0.11162561E 00
40	0.61089530E-02	41	-0.87367991E-01	42	-0.16238207E 00
43	0.39692866E-01	44	-0.10614727E 00	45	0.17858613E 00
46	0.12051401E 00	47	-0.21784411E 00	48	-0.28523697E-01
49	0.19604937E 00	50	0.73008554E-02	51	0.13371450E 00
52	0.57462015E-01	53	0.24589617E-01	54	0.62577974E-01
55	0.32117526E-01	56	0.11823651E 00	57	0.17930253E 00
58	0.12092239E 00	59	0.17378875E 00	60	0.25149329E 00
61	0.25372824E 00	62	0.29134430E-01	63	0.18998992E 00
64	0.16346773E 00	65	-0.21211904E 00	66	0.28135192E 00
67	0.26003527E 00	68	0.31340526E 00	69	0.29511288E 00
70	0.28351834E 00	71	0.48460059E 00	72	0.65685284E 00
73	0.47215514E 00	74	0.40259142E 00	75	0.39993109E 00
76	0.41434385E 00	77	0.38470396E 00	78	0.45086261E 00
79	0.42076650E 00	80	0.46927812E 00	81	0.45927099E 00
82	0.58228585E 00	83	0.50373919E 00	84	0.71316219E 00
85	0.72576783E 00	86	0.56370444E 00	87	0.59241320E 00
88	0.94417223E 00	89	0.72913634E 00	90	0.57049624E 00
91	0.86563863E 00	92	0.69634185E 00	93	0.73910604E 00
94	0.10282168E 01	95	0.31617459E 00	96	0.12577172E 01
97	0.11652044E 01	98	0.25843141E 00	99	0.12695047E 01
100	0.16652924E 01				

TABLE XII. THE ESTIMATED COVARIANCE MATRIX USING LOGARITHMS, EXAMPLE I

COLUMNS 1 THROUGH 4 OF THE COVARIANCE MATRIX

0.21074712E-02	-0.91714849E-04	-0.14291480E-05	-0.14266172E-03
0.91714849E-04	0.48811196E-02	-0.60794278E-04	-0.14337896E-02
0.14291480E-05	-0.60794278E-04	0.89987955E-06	0.24092065E-04
0.14266172E-03	-0.14337896E-02	0.24092065E-04	0.31801027E-02
0.70044566E-05	0.11421468E-04	-0.49218695E-06	-0.71649763E-04
0.21890713E-04	0.17066520E-04	-0.70210537E-06	-0.27066986E-04
0.45097491E-02	-0.36673246E-03	-0.10914733E-04	-0.21582248E-02
0.78418342E-03	-0.32173945E-04	-0.15469287E-04	-0.25695781E-02
0.42147766E-02	-0.25150089E-02	0.23552413E-04	0.13179417E-02
0.10207698E-02	0.55419948E-03	-0.12167119E-04	-0.15815845E-02

COLUMNS 5 THROUGH 8 OF THE COVARIANCE MATRIX

0.70044566E-05	0.21890713E-04	-0.45097491E-02	-0.78418342E-03
0.11421468E-04	0.17066520E-04	-0.36673246E-03	-0.32173945E-04
0.49218695E-06	-0.70210537E-06	-0.10914733E-04	-0.15469287E-04
0.71649763E-04	-0.27066986E-04	-0.21582248E-02	-0.25695781E-02
0.21228377E-05	0.13463233E-05	0.55997700E-04	0.82093212E-04
0.13463233E-05	0.54422052E-05	0.16359211E-03	0.12477445E-03
0.55997700E-04	0.16359211E-03	0.18043962E 00	0.76971968E-01
0.82093212E-04	0.12477445E-03	0.76971968E-01	0.73970126E-01
0.30289409E-04	-0.29778322E-03	0.27586871E-01	0.71816980E-02
0.36763153E-04	0.21602682E-04	0.57510257E-02	0.16732122E-01

TABLE XII. THE ESTIMATED COVARIANCE MATRIX USING LOGARITHMS, EXAMPLE I (Concluded)

COLUMNS 9 THROUGH 10 OF THE COVARIANCE MATRIX

0.42147086E-02	0.10207689E-02
0.25158089E-02	0.55419948E-03
0.23552413E-04	-0.12167119E-04
0.13179417E-02	-0.15815645E-02
0.30289409E-04	0.36763153E-04
0.29778322E-03	0.21602682E-04
0.27586671E-01	0.57510257E-02
0.71816580E-02	0.16732122E-01
0.65895710E-01	-0.36253922E-02
0.36253922E-02	0.22527393E-01

TABLE XIII. THE ESTIMATED CORRELATION MATRIX USING LOGARITHMS, EXAMPLE I

COLUMNS 1 THROUGH 4 OF THE CORRELATION MATRIX

0.10000000E 01	-0.28595596E-01	-0.33001279E-01	-0.55106910E-01
0.28595596E-01	0.10000000E 01	-0.92243847E 00	-0.36391936E 00
0.33001279E-01	-0.92243847E 00	0.10000000E 01	0.45288490E 00
0.55106910E-01	-0.36391936E 00	0.45288490E 00	0.10000000E 01
0.10472135E 00	0.11220284E 00	-0.35810149E 00	-0.87203876E 00
0.20440505E 00	0.10471233E 00	-0.31904300E 00	-0.20574622E 00
0.23126270E 00	-0.12357303E-01	-0.27238397E-01	-0.90096964E-01
0.62807087E-01	-0.16932320E-02	-0.60294333E-01	-0.16753790E 00
0.35764994E 00	-0.14027794E 00	0.97261560E-01	0.31043105E-01
0.14814639E 00	0.52850727E-01	-0.85934328E-01	-0.18686000E 00

COLUMNS 5 THROUGH 8 OF THE CORRELATION MATRIX

0.10472135E 00	0.20440505E 00	-0.23126270E 00	-0.62807087E-01
0.11220284E 00	0.10471233E 00	-0.12357303E-01	-0.16932320E-02
0.35810149E 00	-0.31904300E-00	-0.27238397E-01	-0.60294333E-01
0.87203876E 00	-0.20574622E 00	-0.90096964E-01	-0.16753790E 00
0.10000000E 01	0.39609879E 00	0.90478593E-01	0.20714181E 00
0.39609879E 00	0.10000000E 01	0.16508550E 00	0.19665727E 00
0.90478593E-01	0.16508550E 00	0.10000000E 01	0.66625189E 00
0.20714181E 00	0.19665727E 00	0.66625189E 00	0.10000000E 01
0.80984859E-01	-0.45726076E 00	0.25299265E 00	0.10286565E 00
0.16811207E 00	0.61697099E-01	0.90203603E-01	0.40989004E 00

TABLE XIII. THE ESTIMATED CORRELATION MATRIX USING LOGARITHMS. EXAMPLE I (Concluded)

COLUMNS 9 THROUGH 10 OF THE CORRELATION MATRIX

0.35764994E 00	0.14814639E 00
-0.14027794E 00	0.52850727E-01
0.97261560E-01	-0.85934328E-01
0.91043105E-01	-0.18686000E 00
0.80984859E-01	0.16811207E 00
0.49726076E 00	0.61697099E-01
0.25299265E 00	0.90203603E-01
0.10286565E 00	0.40989004E 00
0.10000000E 01	-0.94095956E-01
0.94095956E-01	0.10000000E 01

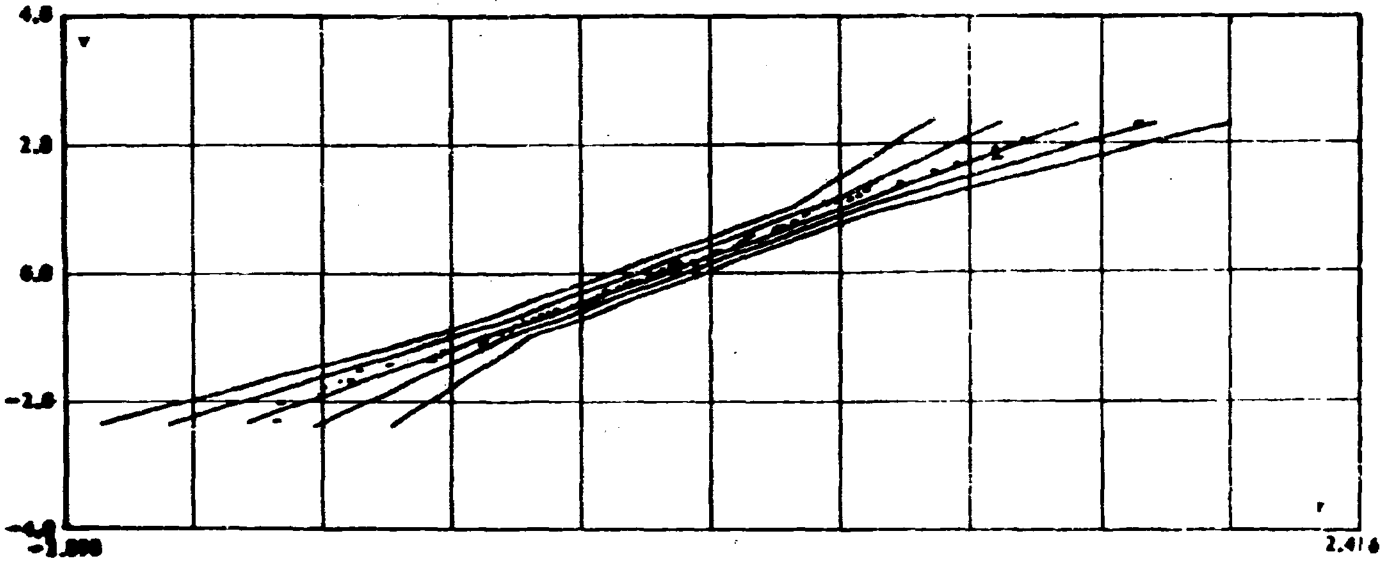

FIGURE 11. POINTS, BEST FITTING LINE, 68-PERCENT CONTROL BAND, AND 98-PERCENT CONTROL BAND FOR THE RESIDUALS USING LOGARITHMS IN EXAMPLE I

a. Testing $b_i = b_i^*$ Using a Student's t Random Table

The CELSP program prints the covariance matrix of the estimates $\hat{b}_1$ through $\hat{b}_k$ of the coefficients of the linear statistical model. The i^{th} diagonal element of the covariance matrix, c_{ii}, estimates the variance of $\hat{b}_i$. The ratio $\frac{\hat{b}_i - b_i}{\sqrt{c_{ii}}}$ has a Student's t distribution with n-k-1 degrees of freedom.

Suppose that a predetermined estimate b_i^*, of b_i, the true value, is available and the following hypotheses are postulated:

$$H_0:\ b_i = b_i^*;\quad H_1:\ b_i \neq b_i^* .$$

This is a two-sided test since there is no advantage if $b_i < b_i^*$. As before, the level of the test is α. The critical t value is found for the level of the test, the number of degrees of freedom and the fact that it is a two-sided test, $t\left(\frac{\alpha}{2}, \text{n-k-1}\right)$. Since the t distribution is symmetric, the probability that a t random variable is greater than $t\left(\frac{\alpha}{2}, \text{n-k-1}\right)$ or less than $-t\left(\frac{\alpha}{2}, \text{n-k-1}\right)$ is α, i.e.,

$$\text{Prob}\left[|t| > t\left(\frac{\alpha}{2}, \text{n-k-1}\right)\right] = \alpha$$

where $|t|$ is the absolute value of t. This defines the critical region of the test. If H_0 is true, then $\frac{\hat{b}_i - b_i^*}{\sqrt{c_{ii}}}$ is a t random variable and

$$\text{Prob}\left[\left|\frac{\hat{b}_i - b_i^*}{\sqrt{c_{ii}}}\right| > t\left(\frac{\alpha}{2}, \text{n-k-1}\right)\right] = \alpha .$$

Thus, the decision is

$$\text{Reject } H_0 \text{ if } \left|\frac{\hat{b}_i - b_i^*}{\sqrt{c_{ii}}}\right| > t\left(\frac{\alpha}{2}, \text{n-k-1}\right) .$$

Do not reject H_0 otherwise.

If it is desired to test whether the variable x_i has a "significant effect" upon y, then set $b_i^* = 0$ and perform the above test.

b. Testing the Coefficients for Missile Modifications

In the revised model for the missile, using $z = \ell n$ (radial miss distance), the coefficient b_7 represents the difference in z caused by a difference in Mod 1 and Mod 2 missile. That is, $b_7 = z_{\text{Mod } 2} - z_{\text{Mod } 1}$. From Table X, the estimate of this coefficient is

$$\hat{b}_7 = 0.0910 = \hat{z}_{\text{Mod } 2} - \hat{z}_{\text{Mod } 1} .$$

Since $z = \ell n$ (y), a positive increase in z corresponds to a positive increase in y. Thus, it would appear that the radial miss distance would be larger for Mod 2 missile than for Mod 1 missiles. To see whether this difference is statistically significant, a t test is performed, with $b_7^* = 0$.

It was decided to set $\alpha = 0.10$. The number of degrees of freedom is, from Table X, 89. The hypotheses are:

$$H_0: \ b_7 = 0, \qquad H_1: \ b_7 \neq 0 .$$

The critical t value is

$$t \ (0.05, \ 89) = 1.64 .$$

The estimate of the variance of $\hat{b}_7$ is, from Table XII,

$$c_{77} = 0.1804 .$$

The test statistic is

$$\left| \frac{b_7 - 0}{\sqrt{c_{77}}} \right| = \frac{0.0910}{\sqrt{0.1804}} = 0.21 .$$

The test statistic is less than the critical value of 1.64. Therefore, it cannot be concluded that there is a significant difference in the radial miss distance for Mod 2 and Mod 1 missiles.

A similar test is made for the difference in z of Mod 3 and Mod 2 missiles. Again, $\alpha = 0.10$, t (0.05, 89) = 1.64. The hypotheses are H_0: $b_8 = 0$. H_1: $b_8 \neq 0$. From Tables X and XII, $\hat{b}_8 = 0.250$, $c_{88} = 0.0740$. The test statistic is thus

$$\left| \frac{\hat{b}_8 - 0}{\sqrt{c_{88}}} \right| = \frac{0.250}{\sqrt{0.0740}} = 0.92 \; .$$

This is less than the critical value. It cannot be concluded that there is a significant difference in the effect of Mod 3 and Mod 2 on radial miss distance.

There is one more difference to be examined; that between Mod 3 and Mod 1. To estimate this difference, the following sum is used:

$$\begin{array}{l} \hat{b}_7 = \hat{z}_{\text{Mod 2}} - \hat{z}_{\text{Mod 1}} \\ \hat{b}_8 = \hat{z}_{\text{Mod 3}} - \hat{z}_{\text{Mod 2}} \\ \hline \hat{b}_7 + \hat{b}_8 = \hat{z}_{\text{Mod 3}} - \hat{z}_{\text{Mod 1}} \end{array}$$

In this case

$$\hat{b}_7 + \hat{b}_8 = 0.0910 + 0.250 = 0.341 \; .$$

Since $\hat{b}_7$ and $\hat{b}_8$ are normally distributed, then $\hat{b}_7 + \hat{b}_8$ is also normally distributed with mean $b_7 + b_8$ and variance

$$\text{var } (\hat{b}_7 + \hat{b}_8) = \text{var } (\hat{b}_7) + \text{var } (\hat{b}_8) + 2 \text{ cov } (\hat{b}_7, \hat{b}_8) \; .$$

This variance is estimated by

$$c_{77} + c_{88} + 2c_{78} \; .$$

From Table XII, $c_{78} = 0.0770$. Thus

$$c_{77} + c_{88} + 2c_{78} = 0.1804 + 0.0740 + 0.0770 = 0.3314 \; .$$

To test the hypotheses

$$H_0\text{: } b_7 + b_8 = 0; \qquad H_1\text{: } b_7 + b_8 \neq 0 \; ,$$

for $\alpha = 0.10$, the critical value is $t\ (0.05,\ 89) = 1.64$. The test statistic is

$$\left| \frac{\hat{b}_7 + \hat{b}_8 - 0}{\sqrt{c_{77} + c_{88} + 2c_{78}}} \right| = \frac{0.341}{\sqrt{0.3314}} = 0.59 \; .$$

Again, there is insufficient evidence of a difference in miss distance between Mod 3 and Mod 1.

c. Using the F Test in Analysis of Variance

Suppose that a model, called Model I or the complete model, has been fitted to a set of data and it is desired to reduce the model by dropping all terms which do not test as significant. Assume that the level of each test is 0.05 and that the tests are independent. If only one term is tested without positive results by a t test and dropped from the model, then the probability of falsely rejecting is 0.05. If two terms are tested separately with t tests and dropped, the probability of falsely rejecting at least one coefficient becomes $1 - (0.95)^2 = 0.0975$. If four terms are tested and dropped, the probability of falsely rejecting at least one coefficient is $1 - (0.95)^4 = 0.1855$. If this is extended to eight terms, it becomes $1 - (0.95)^8 = 0.3366$. Thus, the t test can be safely used only when one term alone is to be dropped. If more than one term is to be tested, the method of analysis of variance should be used.

Let Model I (the complete model) be the following:

$$\text{Model I:}\quad y = b_0 + b_1 x_1 + b_2 x_2 + \ldots + b_k x_k + e\ .$$

Model I has been fitted to the data and the sum of squares of error, SSE_1, and the estimate of the variance, s_1^2, have been calculated.

Now suppose that it is desired to test the significance of k-g of these terms. For convenience, assume that the last k-g terms are to be tested. The null hypothesis is then

$$H_0:\ b_{g+1} = b_{g+2} = \ldots = b_k = 0\ .$$

If H_0 is true, then the system can be described by a shorter or deleted model, Model II:

$$\text{Model II:}\quad y = b_0 + b_1 x_1 + \ldots + b_g x_g + e\ .$$

Model II is then fitted to the data and the sum of squares of error, SSE_2, and the estimate of the variance, s_2^2, are calculated.

Even if H_0 is true, the estimates of b_{g+1} through b_k will not be zero because of the finite random sample of observation noise. Thus, fitting Model I instead of Model II will reduce the sum of squares for error, $SSE_1 \leq SSE_2$. In fact, SSE_2 can be partitioned into two positive quantities

$$SSE_2 = SSE_1 + (SSE_2 - SSE_1)\ .$$

It can be shown that if H_0 is true, $\frac{SSE_1}{n-k-1}$ and $\frac{SSE_2 - SSE_1}{k-g}$ provide unbiased estimates of σ^2 and that $\frac{SSE_1}{\sigma^2}$ and $\frac{SSE_2 - SSE_1}{\sigma^2}$ are independent Chi-square random variables with n-k-1 and k-g degrees of freedom, respectively. Thus, the ratio of these two independent Chi-square random variables divided by the respective degrees of freedom is an F random variable, if H_0 is true.

$$F = \frac{\dfrac{SSE_2 - SSE_1}{\sigma^2 \ (k-g)}}{\dfrac{SSE_1}{\sigma^2 \ (n-k-1)}} = \frac{SSE_2 - SSE_1}{(k-g) \ s_1^2} .$$

To test H_0, a one-sided test on this F ratio is used. The level of the test α must again be specified. The critical value of F depends upon α and upon the degrees of freedom, k-g and n-k-1; $F_{crit} = F\ (\alpha,\ k-g,\ n-k-1)$. This is found in statistical tables.

The decision is then based upon a comparison of F with the critical value:

If $F > F\ (\alpha,\ k-g,\ n-k-1)$, reject H_0.

If $F \leq F\ (\alpha,\ k-g,\ n-k-1)$, do not reject H_0.

If H_0 is rejected, the model cannot be shortened by dropping the entire group of k-g terms. Possibly, a subgroup of these k-g terms can safely be dropped, but other models must be postulated and tested to decide which can be dropped.

d. Testing the Effect of Missile Modification Upon Radial Miss Distance Using the F Statistic

A reduced model, without the terms for missile modification, was set up as:

$$z = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 + b_5 x_5 + b_6 x_6 + b_9 x_9$$
$$+ b_{10} x_{10} + e .$$

The estimated coefficients for this model are shown in Table XIV.

The null hypothesis for this test is H_0: $b_7 = b_8 = 0$. It was decided to set $\alpha = 0.10$. The critical F value is

TABLE XIV. THE ESTIMATED COEFFICIENTS FOR THE REDUCED MODEL USING LOGARITHMS, EXAMPLE I

I	BETA(I)	BASIC TYPE	IDENTIFICATION	NUMBER OF VECTORS
0	0.28229044E 01		CONSTANT	
1	-0.12045744E-01	1	TARGET ALT.	1
2	-0.36753511E-01	1	RANGE,LAUNCH	2
3	0.69349830E-03			
4	0.26067385E-01	1	RANGE,INTER.	2
5	-0.10532011E-02			
6	0.26468312E-02	1	TAR. CL.VEL.	1
9	-0.15728307E 00	2	TARGET TYPE	1
10	0.57511915E-01	2	RADAR POWER	1

THE SUM OF SQUARES FOR ERROR IS 0.35567401E 02

THE ESTIMATE OF THE VARIANCE IS 0.39085057E 00

THE ESTIMATE OF THE STANDARD DEVIATION IS 0.62518042E 00

THE NUMBER OF DEGREES OF FREEDOM IS 91.

$$F\ (0.10,\ 2,\ 89) = 2.77\ .$$

From Table X for the complete model,

$$SSE_1 = 35.1213$$

$$s_1^2 = 0.3946\ .$$

From Table XIV for the reduced model,

$$SSE_2 = 35.5674\ .$$

The test statistic is

$$\frac{SSE_2 - SSE_1}{(k-g)\ s_1^2} = \frac{35.5674 - 35.1213}{(2)\ (0.3946)} = 0.57\ .$$

This value is less than the critical value. It cannot be concluded that missile modification affects the radial miss distance.

11. CONCLUSIONS

Often observations or results of experiments can logically be represented by a linear statistical model relating the observation y to various known quantitative and qualitative variables $x_1, \ldots, x_k$, random noise e, and fixed but unknown coefficients $b_0, b_1 \ldots, b_k$:

$$y = b_0 + b_1 x_1 + b_2 x_2 + \ldots + b_k x_k + e\ .$$

The least squares estimates of the coefficients $b_0, b_1, \ldots, b_k$ have been developed theoretically and are calculated by the GELSF program.

If the noise e is Gaussian (normal) with zero mean and covariance matrix $\sigma^2 I$, where I is the identity matrix, then certain quantities have well known distributions. A graphical procedure for making tests for normality of the noise is described.

Crucial in the entire discussion is the model. The model must be of the correct form or the theory collapses. Statistical tests, in particular the analysis of variance, can suggest which terms should be retained in the model, but they cannot prescribe which new terms should be added. The model must be constructed from physical considerations and sound judgment.

Sound judgment should also be employed in the statistical evaluation of the data. In no case should statistical techniques described here or elsewhere be applied in a purely mechanical manner or divorced from the other aspects of the system under consideration.

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INFERENCE PROCEDURES BASED ON
CENSORED TIME DEPENDENT OBSERVATIONS

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1. Introduction. Frequently, experiments are conducted in such a way that they may be considered to consist of n trials, where in each trial, the time required to achieve some objective ("success") is observed. If an upper limit, T_0, is placed upon the possible duration of each trial, then the outcome on each trial is either the time until success, or T_0, whichever is smaller. Such observations are said to be censored at T_0. For example, it might be desired to determine whether a certain type of combat aid, such as a target detection device, is effective, or whether one type of device is better than another. In order to make such inferences about a single device, often an experiment of the following design is conducted: n players are selected, and each player uses the device in an attempt to detect a target. If a given player has not succeeded within 4 minutes (say), his trial terminates and the next player begins. The observed data then consists of the times of detection for those trials terminating before $T_0 = 4$ (together with the number of trials terminating at $T_0 = 4$). If it is desired to compare two devices, then samples on each device may be taken as described above.

If the distributions of (uncensored) time until success are identical exponentials, this situation falls under the body of results generally known as "life testing." In what follows, we shall discuss statistical procedures for making inferences about the mean rate λ of success (the reciprocal of mean time to success), based upon such censored time dependent observations. These inference techniques include point estimators, confidence intervals and tests of hypotheses for both the single population case, and that of comparing two populations. Since there is a strong parallel between making a confidence interval for a parameter and tests of hypotheses concerning that parameter, we shall discuss only one or the other of these in each of the approaches considered below. While it is assumed that the populations involved are exponentially distributed, attention is given to the robustness of the procedures under departures from exponential distributions.

In Section 2, we discuss procedures for making inferences about one population (one-sample inferences), including a review of several approaches in the literature and an easily applied approximate procedure that we have been investigating. A similar treatment for two-sample inferences is given in Section 3.

2. <u>One-Sample Procedures.</u> Suppose that $Z_1, \ldots, Z_n$ is a random sample of size n from an exponential population with mean rate λ. Let $X_i = \min\{Z_i, T_o\}$; $i = 1, 2, \ldots, n$. Based upon observations $x_1, \ldots, x_n$ on the X's, we wish to make inferences concerning the parameter λ. The (point) estimation of λ has been discussed by several writers (see, for example, Bartholomew [1]). Theyhave shown that the maximum likelihood estimate of λ is

$$\hat{\lambda} = k / \sum_{i=1}^{n} \left\{ a_i x_i + (1 - a_i) T_o \right\} , \tag{1}$$

where

$$a_i = \begin{cases} 1 & \text{if } x_i < T_o \\ 0 & \text{otherwise} \end{cases} ; \; i = 1, 2, \ldots, n,$$

and

$$k = \sum_{i=1}^{n} a_i .$$

While $\widehat{1/\lambda}$ is asymptotically normal with mean $1/\lambda$ and variance

$\left[n\lambda^2(1-e^{-\lambda T_0}\right]^{-1}$, the convergence is slow [1] and, for small samples, $\hat{\lambda}$ is seriously biased [6]. The exact distribution of $\widehat{1/\lambda}$ is given by Bartholomew [1] in a form useful with small samples, along with some approximations that are useful with moderate sized samples.

Several procedures for obtaining confidence intervals for λ and $1/\lambda$, based upon approximate distributions for certain functions of $1/\lambda$, have been suggested. Bartholomew [1] discusses two procedures based upon normal distributions, and a technique given in NAVORD O. D. 29304 [6] uses an approach based upon a Poisson distribution. The statistical properties of confidence intervals obtained by these approximations are apparently not fully known at the present time.

We have investigated a method for obtaining confidence intervals for λ, based upon a general approach given by Halperin [4], described as follows: Let

$$p = 1 - e^{-\lambda T_0} \tag{1}$$

denote the probability that the result in a given trial is not censored (i.e., $p = P[X_i < T_0]$). Then each experimental trial may be viewed as a Bernoulli trial, where "success" is associated with non-censoring and occurs on each trial with probability p. Based upon the observed number k of success in the n experimental trials, a $100(1-\alpha)$ percent upper confidence bound, say P_U, for p can be constructed using well-known methods. Using equation (1), this bound can be "inverted" to obtain a $100(1-\alpha)$ percent upper confidence bound for λ as

$$\lambda_U = \frac{-\ln(1-P_U)}{T_0} \tag{2}$$

intervals and bounds of this type are very easy to compute, and appear to perform nearly as well as those based upon approximating distributions. The results of a Monte Carlo study on the performances of the two procedures based upon the Poisson distribution [6] and the binomial distribution [4] are presented in Tables 1, 2, and 3 below. 1000 samples of size n ($n = 20, 30, 40, 50$) were generated from exponential distributions with mean rate λ ($\lambda = .1, .2, .5, 1, 2, 5, 10$). For different truncation times T_o the upper confidence limits λ_U were calculated using both procedures. Table 1 gives the average value of the upper confidence limit $\bar{\lambda}_U$. For each choice of λ and T_0 the first row contains $\bar{\lambda}_U$ for the O. D. 29304 [6] procedure (based on the poisson distribution); the numbers in the second row are those for the binomial procedure. Table 2 contains the sample variance of the upper bounds λ_U. In Table 3, the empirical coverage probability i.e., the proportion of the λ_U which actually exceed the true parameter value λ is given. The relative sensitivity of these procedures to departures from the exponential distribution are apparently not known at present.

Finally, we mention another approach to finding confidence intervals for λ, which seems to have received less attention in the literature than it deserves. Imagine that the time censored trials are conducted sequentially in time, and that we disregard the times between when each trial terminates and the next begins (see Figure 1). Then the times between successive successes is exponential with parameter λ, so that the "success arrival

		Sample Size n							
		20		30		40		50	
		Confidence Coefficient $1-\alpha$							
λ	T_o	.95	.99	.95	.99	.95	.99	.95	.99
.1	8.	.165 .178	.199 .209	.151 .158	.177 .183	.144 .149	.164 .168	.138 .142	.156 .159
.2	2.	.379 .397	.477 .478	.340 .352	.408 .408	.316 .324	.375 .376	.299 .305	.352 .352
.5	1.	.919 .969	1.12 1.13	.804 .836	.973 .985	.752 .772	.896 .904	.723 .738	.840 .846
1.	.7	1.68 1.78	2.05 2.13	1.53 1.60	1.82 1.86	1.44 1.49	1.68 1.71	1.40 1.43	1.60 1.62
2.	.3	3.50 3.72	4.28 4.39	3.10 3.23	3.74 3.82	2.97 3.07	3.47 3.52	2.84 2.92	3.26 3.30
5.	.05	10.6 11.2	14.5 14.3	9.46 9.80	12.0 11.9	8.63 8.76	10.7 10.6	8.14 8.33	9.95 9.88
10.	.06	17.0 18.0	21.6 22.2	15.6 16.2	18.8 19.2	14.6 15.1	17.2 17.3	14.1 14.5	16.3 16.5

TABLE 1. Average upper 100 (1 - α) percent confidence bounds $\bar{\lambda}_U$ for the O. D. and the binomial procedures.

		Sample Size n							
		20		30		40		50	
		Confidence Coefficient 1 - α							
λ	T_o	.95	.99	.95	.99	.95	.99	.95	.99
.1	8.	.002	.002	.001	.001	.001	.001	.000	.001
		.003	.004	.001	.002	.001	.001	.001	.001
.2	2.	.012	.015	.008	.009	.005	.006	.004	.005
		.014	.017	.008	.010	.006	.007	.004	.005
.5	1.	.067	.076	.036	.045	.028	.033	.020	.022
		.080	.094	.042	.053	.030	.038	.021	.024
1.	.7	.194	.267	.124	.142	.086	.096	.063	.074
		.265	.366	.146	.175	.102	.119	.073	.088
2.	.3	.874	1.08	.448	.630	.375	.485	.264	.353
		1.11	1.46	.578	.759	.431	.567	.306	.386
5.	.05	12.7	16.8	7.30	9.34	5.02	6.68	3.88	4.74
		14.0	18.5	7.80	10.2	5.24	6.96	4.05	4.96
10.	.06	19.8	29.4	13.7	16.8	9.15	11.0	6.78	7.96
		25.0	40.1	16.2	20.0	10.6	12.4	7.60	9.29

TABLE 2. Sample Variances of λ_U

λ	T_o	n = 20, .95	n = 20, .99	n = 30, .95	n = 30, .99	n = 40, .95	n = 40, .99	n = 50, .95	n = 50, .99
		Sample Size n / Confidence Coefficient $1-\alpha$							
.1	8.	.954	.995	.954	.994	.951	.993	.955	.996
		.977	.994	.963	.995	.959	.991	.955	.998
.2	2.	.950	.996	.967	.993	.960	.997	.950	.995
		.980	.996	.968	.993	.986	.993	.961	.995
.5	1.	.968	.997	.955	.993	.951	.995	.963	.992
		.985	.997	.984	.992	.952	.994	.969	.989
1.	.7	.951	.991	.958	.993	.956	.996	.950	.994
		.972	.991	.958	.993	.959	.993	.964	.996
2.	.3	.961	.995	.961	.993	.958	.996	.965	.988
		.985	.995	.972	.994	.961	.995	.966	.985
8.	.05	.949	1.00	.981	.997	.948	.995	.953	.995
		.949	.991	.981	.997	.948	.995	.975	.994
10.	.06	.951	.996	.954	.996	.948	.992	.954	.995
		.982	.995	.965	.997	.957	.991	.962	.991

TABLE 3. Proportion of the λ_U that exceed the true parameter value λ .

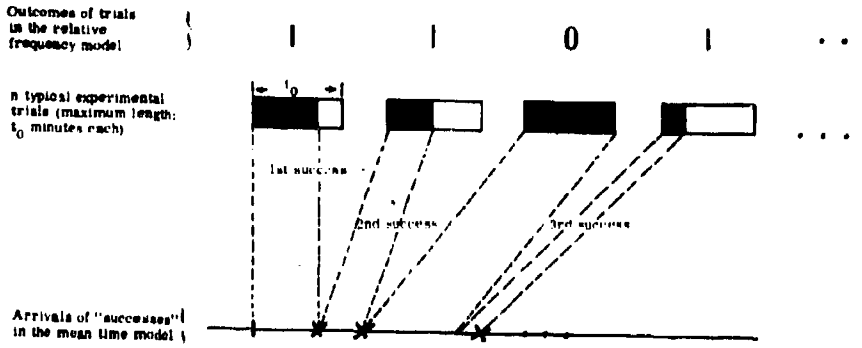

FIGURE 1.

process" is a poisson process with mean rate λ per unit time. Suppose that $k > 0$ successes are observed in n experimental trials. Then the conditional distribution of the waiting time to kth arrival, W_k, given k, is approximately gamma distributed. Thus $2\lambda W_k$ is chi-square distributed with $2k$ degrees of freedom. Using the tabulated χ^2 distribution, and given $k > 0$, one can easily find an interval (C_L, C_U) which will contain $2\lambda W_k$ with probability $1-\alpha$. Algebraic manipulation upon this interval yields

$$\left(\frac{C_L}{2W_k} , \frac{C_U}{2W_k} \right) \tag{3}$$

as a 100 $(1-\alpha)$ percent conditional confidence interval for λ.
The sensitivity of this approach to departures from exponential distributions depends upon T_0; with sufficiently small T_0, the interarrival times should be nearly exponential even with non-exponentially distributed times to success within the individual trials.

3. Two-Sample Procedures. In this section, techniques for comparing two exponential populations based on censored observations, are discussed. Suppose independent samples of sizes n_1 and n_2 are drawn from two exponential populations with means $1/\lambda_1$ and $1/\lambda_2$ respectively. It will be assumed that both sets of sample observations are censored at the same time point T_0; that is, an upper limit T_0, is placed upon the possible duration of each trial. Let k_1 and k_2 denote the number of uncensored observations in the first and second sample respectively and let $p = k_1 + k_2$. Methods for testing the hypothesis $H_0 : \lambda_1 = \lambda_2$ and obtaining confidence intervals for $\rho = \lambda_1/\lambda_2$ and $\psi = \lambda_1 - \lambda_2$ are discussed below.

a. F test for $H_0 : \lambda_1 = \lambda_2$: Let W_{1,k_1} denote the total elapsed time time until k_1 uncensored observations are obtained from the first sample. Then, as mentioned earlier, $2\lambda_1 W_{1,k_1}$ has a chi-square distribution with $2k_1$ degrees of freedom. If W_{2,k_2} is similarly defined for the second sample, so that $2\lambda_2 W_{2,k_2}$ is distributed as a chi-square with $2k_2$ degrees of freedom, then, if H_0 is true, the ratio

$$W = \frac{2W_{1,k_1}/2k_1}{2W_{2,k_2}/2k_2} \tag{4}$$

has an F distribution with $(2k_1, 2k_2)$ degrees of freedom. The hypothesis H_0 will be rejected in favor of $H_1 : \lambda_1 < \lambda_2$, at significance level α, if the observed value W exceeds the 100αth percentile of the F distribution with $(2k_1, 2k_2)$ degrees of freedom. A confidence interval for $\rho = \lambda_1/\lambda_2$ can be obtained, based on the F distribution of $k_2\lambda_1 W_{1,k_1} / k_1\lambda_2 W_{2,k_2}$.

b. Cox's F' test : Let $n = n_1 + n_2$ and let the scores $t_{r,n}$ ($r = 1, 2, \ldots, n$) denote the expected values of the order statistics of a random sample of size n from an exponential distribution with mean equal to 1. It can be shown that

$$t_{r,n} = \sum_{s=0}^{r-1} \frac{1}{n-s} \qquad (r = 1, 2, \ldots, n)$$

Combine the $p = k_1 + k_2$ uncensored observations, defined in the beginning of the section and rank them. Replace the observation with rank r with the corresponding score t_{rn} ($r = 1, 2, \ldots, n$) . If two or more of the censored observations are equal, replace each one with the average of the corresponding scores t_{rn}. Let $\bar{t}_1$ denote the average of the scores assigned to the observations fron the first sample and $\bar{t}_2$ the average of the scores assigned to the observations from the second sample. Cox [2] has shown that the ratio

$$W' = \frac{\left[k_1 \bar{t}_1 + (n_1 - k_1)\, t_{p+1,n} \right] \Big/ k_1}{\left[k_2 \bar{t}_2 + (n_2 - k_2)\, t_{p+1,n} \right] \Big/ k_2} \tag{5}$$

is approximately distributed as an F with ($2k_1$, $2k_2$) degrees of freedom, when $H_o : \lambda_1 = \lambda_2$ is true. The rejection region for the hypothesis $H_o : \lambda_1 = \lambda_2$ can be determined from the F tables.
A confidence interval for $\rho = \lambda_1/\lambda_2$ can be obtained using this approximate distribution of W' , as follows: Multiply each of the uncensored observations from the second sample by a fixed number ρ_o and apply Cox's procedure. This will lead to a test of the hypothesis $H'_o : \lambda_1 = \rho_o \lambda_2$.

For a given significance level α, the set of all values of ρ_0 that will lead to rejection of H'_0 will from a confidence interval for ρ with confidence coefficient $1-\alpha$.

Recently, Gehan and Thomas [3] have reported a Monte Carlo study comparing the powers of the F and F' tests for small sample sizes. It was found that, when the assumption that the two samples are from exponential distributions is valid, these tests have comparable operating characteristics, (see Figure 2 below). However, if the samples are from Weibull distributions, the F test is not robust and the F' test is superior. It should be noted that the F' test requires that both sets of sample observations are censored at the same time point T_0; the F test is not constrained with this requirement.

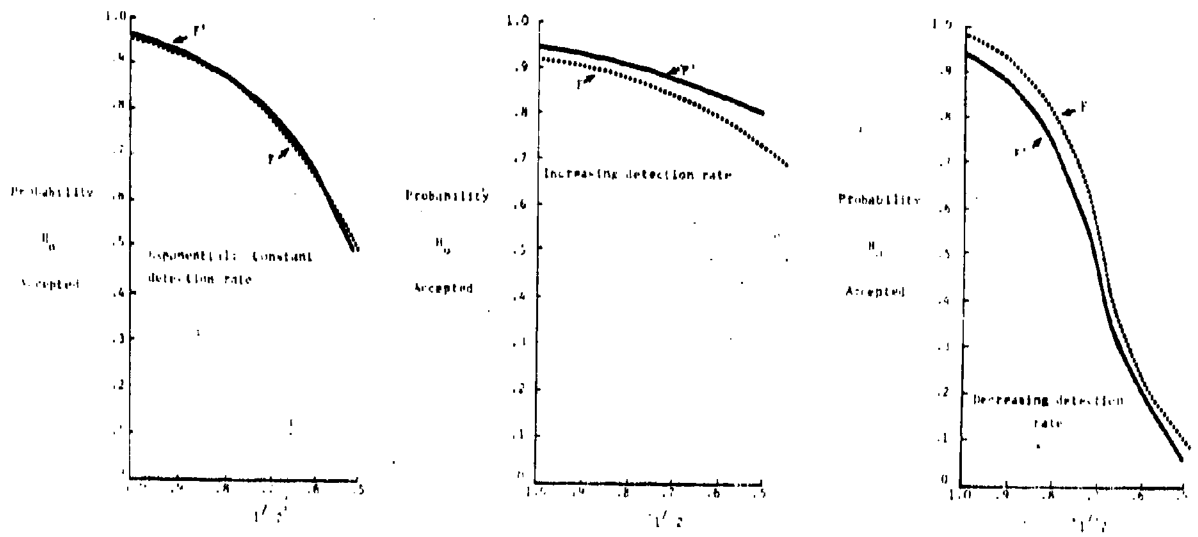

FIGURE 2. Operating Characteristic Curves for the F and F' Tests With Censored Observations for Exponential and Certain Weibull Distribution. ($n_1 = n_2 = 20$, $\alpha = .05$ and $\lambda_2 = 1.0$)

c. Confidence interval for $\psi = \lambda_1 - \lambda_2$: Suppose W_1 and W_2 are independent random variables with gamma distributions with parameters λ_1 and λ_2. Lenter and Buehler [5] obtained the conditional distribution of $H(u/v; \psi)$ of $U = W_1$ given $V = W_1 + W_2$. This conditional distribution function involves only the parameter $\psi = \lambda_1 - \lambda_2$. A confidence interval for ψ can be obtained as follows: Set $H(u/v; \psi)$ equal to $\alpha/2$ and $1-\alpha/2$ respectively and solve for ψ. The two solutions for ψ will be the lower and upper confidence limits for ψ with confidence coefficient $1-\alpha$.

The Lenter-Buehler technique can be used to derive a confidence interval for the difference $\lambda_1 - \lambda_2$ for two exponential distributions when the observations are censored. As was pointed out earlier in this paper, if the censored observations from an exponential distribution are treated as having been obtained sequentially, then $2\lambda W_k$, where W_k is the waiting time till kth arrival, given k, has a Chi-square distribution with 2k degrees of freedom. Thus W_k has a gamma distribution. We can now form two gamma distributed variables from the two sets of observations from the exponential distributions and apply Lenter-Buehler [5] technique to obtain a confidence interval for $\lambda_1 - \lambda_2$.

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