Bayesian Estimation Under Informative Sampling

Based on work of T. D. Savitsky, Daniell Toth, Michail Sverchkov

T. D. Savitsky

Office of Survey Methods Research

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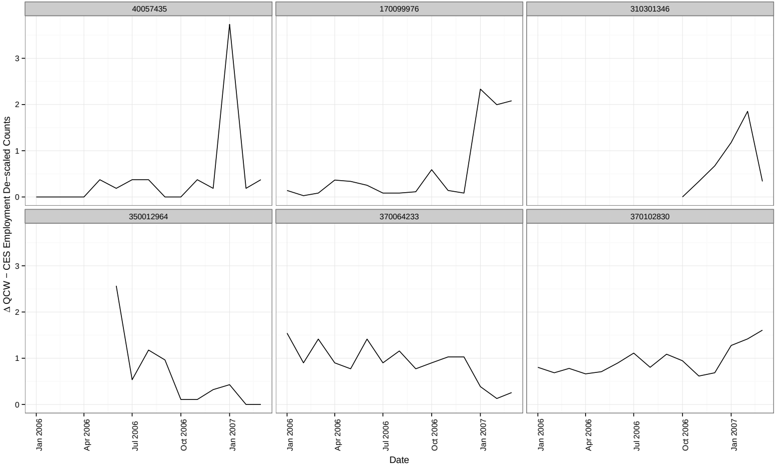
Employment Count Reporting Errors

- Make inference on the population distribution
- When the data are acquired under an informative sampling design
- Where $f\left(\mathbf{y}_o|\boldsymbol{\lambda}\right) \neq f\left(\mathbf{y}_p|\boldsymbol{\lambda}\right)$
- Analyst may not know the sampling design to parameterize it

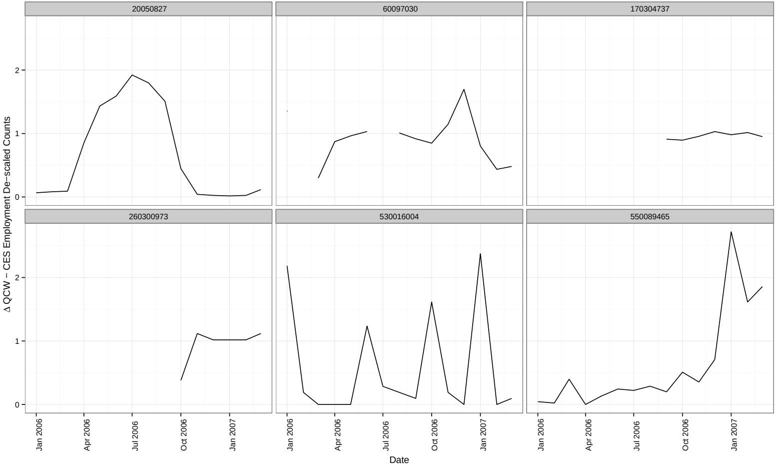
Employment Count Reporting Errors

- Establishments report number of employees
- On both Quarterly Census of Employment and Wages (QCEW)
- And Current Employment Survey (CES)
- Response Analysis Survey (RAS)
- Our data are $y = |y_{\text{QCEW}} - y_{\text{CES}}|$
- Collected on $N = 9777$ establishments, each for $T = 15$ months.
  - June, $2006-$ March, 2007
- Produces N, $T \times 1$ functions, $\{\mathbf{y}_i\}_{i=1,\ldots,N}$
- Functions each standardized to variance 1
- Desired inference about set of latent functional bases

Randomly-selected observed $\mathbf{y}_i$



More Randomly-selected observed $\mathbf{y}_i$



Stratified Sampling Design

- Sampled 2917 establishments - 2013 responded from $N = 9777$
- Stratified design with $H = 6$ strata
- Assigned to strata based on *a priori* “interesting” features
- Focused on selected portions of functions
- e.g. spike at year-end indicates counting checks, rather than employees

Bayesian Estimation Model: DP mixture over GP's

$$
\begin{aligned}
\overset{T\times 1}{\mathbf{y}_i} &\sim \mathcal{N}_T\left(\mathbf{f}_i, \tau_\epsilon^{-1}\mathbb{I}_T\right),\ i = 1,\ldots,N \\
\mathbf{f}_i &\sim \mathcal{N}_T\left(\mathbf{0}, \overset{T\times T}{\mathbf{C}}(\boldsymbol{\theta}_i)\right) \\
\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_N|G &\sim G \\
G &\sim \text{DP}(\alpha_0, G_0) \\
&\Updownarrow \\
\mathbf{f}|G &\sim \int \mathcal{N}_T\left(\mathbf{0}, \mathbf{C}(\boldsymbol{\theta})\right) G(d\boldsymbol{\theta})
\end{aligned}
$$

GP covariance function - Rational Quadratic

$$\mathbf{f}_i \sim \mathcal{N}_T\left(\mathbf{0}, \overset{T\times T}{\mathbf{C}}(\boldsymbol{\theta}_i)\right)$$

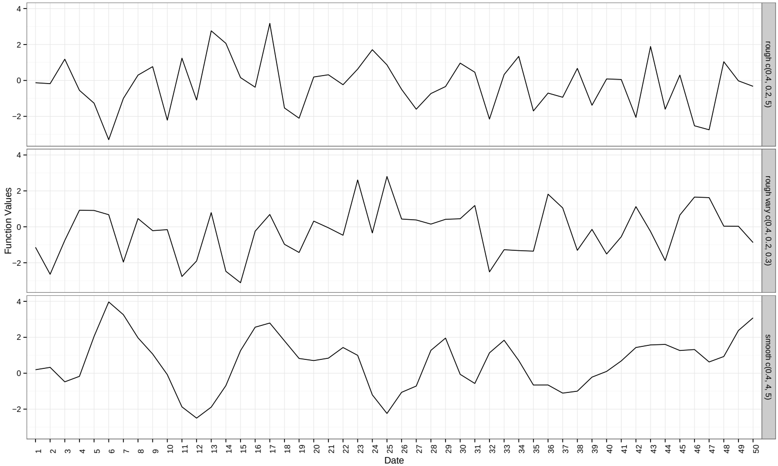
$$\mathbf{C}(\boldsymbol{\theta}_i) \equiv \mathbf{C}_i = \left(C_{f_{ij},f_{i\ell}}\right)_{j,\ell\in(1,\ldots,T)}$$

$$C_{f_{ij},f_{i\ell}} = \frac{1}{\theta_{i,1}}\left(1 + \frac{(t_{ij} - t_{il})^2}{\theta_{i,2}\theta_{i,3}}\right)^{-\theta_{i,3}}$$

- $\theta_{i,1}$ controls the vertical magnitude
- $\theta_{i,2}$ controls the base length-scale $\leftrightarrow$ wavelength
- $\theta_{i,3}$ controls smooth deviations across scales
  - As $\theta_{i,3} \uparrow \infty$, converge to single length-scale, $\theta_{i,2}$

GP Draws

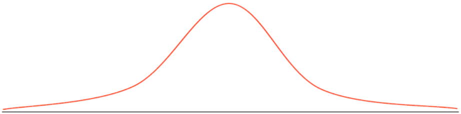
- Top - “Rough”: $(\theta_1, \theta_2, \theta_3) = (0.4, 0.2, 5.0)$
- Middle - “Rough with Variation”: $(\theta_1, \theta_2, \theta_3) = (0.4, 0.2, 0.3)$
- Bottom - “Smooth”: $(\theta_1, \theta_2, \theta_3) = (0.4, 4.0, 5.0)$



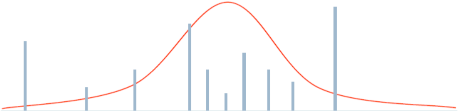
Draws from $G \sim$ DP are discrete, a.s.

Dirichlet Process

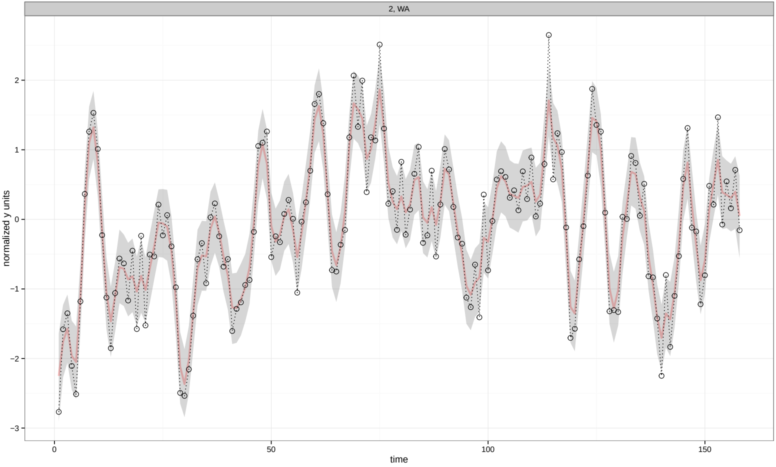
- Consider Gaussian G_0



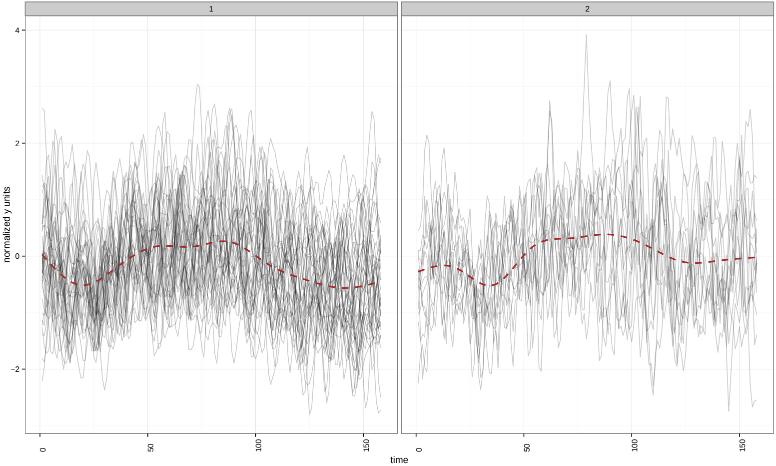
- $G \sim DP(\alpha, G_0)$



f_i: Monthly CPS data



f_*: Monthly CPS data



Estimating Population Density from Sampled Units

- Consider sequence of populations, $\{N_\gamma\}_{\gamma \in \mathbb{N}}$.
- Increasing in size as $\gamma \uparrow$.
- $m_\gamma\left(y\right) := \mathbb{E}\left(I_{\gamma i} | Y_i = y\right)$
- $c_\gamma\left(y_1, y_2\right) :=$ Cov $\left(I_{\gamma i}, I_{\gamma \ell} | Y_i = y_1, Y_\ell = y_2\right)$
- $w_\gamma(y) := \frac{1}{m_\gamma(y)}$
- Assume $m_\gamma\left(y\right) \stackrel{\text{a.s.}}{\rightarrow} m\left(y\right)$
- Assume $\forall y_1, y_2,\ c_\gamma\left(y_1, y_2\right) = o_\gamma(1)$

$$\frac{w_\gamma}{\mathbb{E}_o(w_\gamma(y))} f_o\left(y|\boldsymbol{\lambda}\right) \stackrel{\text{a.s.}}{\rightarrow} f_p\left(y|\boldsymbol{\lambda}\right)$$

- Bonnéry et al. (2013) extends Glivenko-Cantelli

Empirical Generation of Pseudo-populations

The Procedure

- Generate B pseudo-populations, each of size n, by weighted sampling of observed data with replacement using normalized inverse inclusion probabilities
- Estimate $(\boldsymbol{\lambda}_b)_{b=1,\ldots,B}$ under each pseudo-population
- Concatenate estimates, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_B)$

Empirical Generation of Pseudo-populations

The Idea

- Weighted empirical predictive distribution, $\hat{f}\left(\mathbf{y}_p|\mathbf{y}_o\right)$, generates "pseudo-populations" (Pfeffermann et al., 1998, Pfeffermann and Sverchkov, 2003, Sverchkov and Pfeffermann, 2004), applies result of Bonnéry et al. (2013)
- $\mathbf{y}_p$ are treated as parameters

$$\hat{f}\left(\boldsymbol{\lambda}|\mathbf{y}_o\right) = \int f\left(\mathbf{y}_p|\boldsymbol{\lambda}\right)\hat{f}\left(\mathbf{y}_p|\mathbf{y}_o\right)f\left(\boldsymbol{\lambda}\right)d\mathbf{y}_p = \int c(\mathbf{y}_p)f\left(\boldsymbol{\lambda}|\mathbf{y}_p\right)\hat{f}\left(\mathbf{y}_p|\mathbf{y}_o\right)d\mathbf{y}_p.$$

- Full uncertainty accounted for in estimation of $\boldsymbol{\lambda}$
  - population generation, $\hat{f}\left(\mathbf{y}_p|\mathbf{y}_o\right)$
  - $\boldsymbol{\lambda}$ estimation given the finite population, $\hat{f}\left(\boldsymbol{\lambda}|\mathbf{y}_p\right)$
  - conditioned on $\mathbf{y}_o$

Substitution "Plug-in" Method

- $f_o\left(\boldsymbol{\lambda}|H\left(\mathbf{Y}_o\right)\right) \equiv$ sample model
- Use $\left(\tilde{w}_i\right)$ to adjust $H\left(\cdot\right)$ from sample to population
- $\widehat{f_p}\left(\boldsymbol{\lambda}|H\left(\mathbf{Y}_o, \tilde{\mathbf{w}}\right)\right) \equiv$ population model
- Statistic, $\widehat{f_p}\left(\boldsymbol{\lambda}|\mathbf{H}\left(\mathbf{y}_o, \tilde{\mathbf{w}}\right)\right) = \displaystyle\prod_{j=1}^{p} \widehat{f_p}\left(\lambda_j|\mathbf{H}\left(\mathbf{y}_o, \tilde{\mathbf{w}}, \boldsymbol{\lambda}_{-j}\right)\right)$

$$\log\widehat{f_p}\left(\theta^{*}_{qm}|H\left(\{\mathbf{y}_i\}_{i\in s_m}, \{\tilde{w}_i\}_{i\in s_m}, \tau_{\epsilon}, \boldsymbol{\theta}^{*}_{m}\right)\right)$$
$$\propto -\frac{1}{2}n_m\log\left(|\mathbf{C}^{\tau}\left(\boldsymbol{\theta}^{*}_{m}\right)|\right) - \frac{1}{2}\sum_{i\in s_m}\mathbf{y}^{'}_{o,i}\mathbf{C}^{\tau}\left(\boldsymbol{\theta}^{*}_{m}\right)\mathbf{y}_{o,i}\tilde{w}_i + \log f(\theta^{*}_{qm})$$

- $\mathbf{C}^{\tau}\left(\boldsymbol{\theta}^{*}_{m}\right) = \mathbf{C}\left(\boldsymbol{\theta}^{*}_{m}\right) + \left(1/\tau_{\epsilon}\right)\mathbb{I}_T$
- s_m collects the units assigned to cluster m
- $n_m = \sum_{i\in s_m}\tilde{w}_i,\ \ \sum_{m=1}^{M} n_m = n$

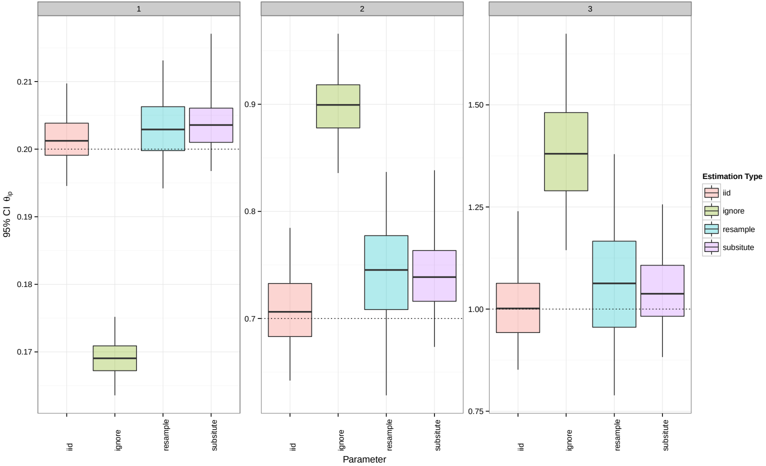
Comments on using Sampling Weights

- Both re-sampling and substitution don't require to re-parameterize
- This estimator employs harmonic averages for full conditionals
  - May break-down under high variation in the weights
- "Gold standard" to jointly model $(\mathbf{y}_i, \pi_i)$

A Warm Up

- Generate $N = 10000$ population units
- $\mathbf{f}_i \sim \mathcal{N}_t\left(\mathbf{0}, \mathbf{C}\left(\boldsymbol{\theta}\right)\right)$
- $\theta = (0.2, 0.7, 1.0)$ are global, not indexed by unit - No clustering
- Generate $y_{ij} = \mathcal{N}\left(f_{ij}, \tau_{\epsilon}^{-1}\right)$
- τ_{ϵ} chosen to achieve noise-to-signal of 0.1
- Employ $H = 4$ strata
- Assign units based on quantiles of variances for N, $T \times 1$ functions, $\mathbf{y}_i$
- Idea is that more interested in high magnitude (error) functions
- (STSI) Sample $n = 750$ from $H = 4$ strata with,
  - $\mathbf{n} = (82, 113, 211, 344)$, $\mathbf{p} = (0.03, 0.05, 0.08, 0.14)$
- Generate B = 25 pseudo-populations for re-weighted estimation

$\theta = (0.2, 0.7, 1.0)$ and $\mathbf{n} = (82, 113, 211, 344)$



2− stage Sampling Design

- Form $n_I = 10$ homogenous blocks by variance quantiles (of $\mathbf{y}_i$)
- 1^{st} stage: Sample 7 blocks proportional to variance
  - $p = (0.04, 0.06, 0.07, 0.08, 0.09, 0.10, 0.11, 0.12, 0.14, 0.19)$
- 2^{nd} stage: $H = 4$ strata based on variance quantiles within blocks

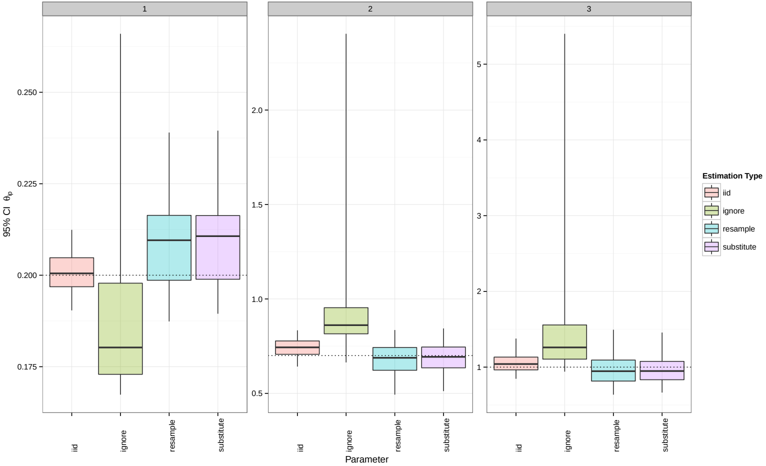
$p_I(\mathbf{s}_I) * p_i(\mathbf{s}_{Ii})' =$

block/stratum	H=1	2	3	4
1	0.001	0.002	0.004	0.006
2	0.002	0.003	0.006	0.009
3	0.002	0.004	0.007	0.010
4	0.003	0.004	0.008	0.012
5	0.003	0.005	0.009	0.013
6	0.004	0.005	0.010	0.015
7	0.004	0.006	0.011	0.017
8	0.004	0.006	0.012	0.019
9	0.005	0.007	0.014	0.022
10	0.007	0.010	0.019	0.029

- $n = 750$ units from $N = 10000$ population

$2-$ stage Posterior Distributions for $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$

- Monte Carlo draws of 10 samples, each selecting 7 of 10 blocks
- "Most" population elements have non-zero selection probabilities
- Conditional inclusion dependence is low



Generate Functions Under $M = 3$ Clusters

- Generate $N = 10000$ population units
- Here, $\{ \overset{P\times 1}{\boldsymbol{\theta}_i} \}_{i=1,\ldots,N}$
- 3 clusters with location values, $\boldsymbol{\Theta}^* = (\boldsymbol{\theta}_1^*, \ldots, \boldsymbol{\theta}_M^*)$:

$\boldsymbol{\Theta}^* =$

parameter/cluster	$m=1$	2	3
$p=1$	0.30	0.30	0.30
2	0.30	2.50	0.70
3	0.80	1.50	2.00
	rough-varied	smooth	rough

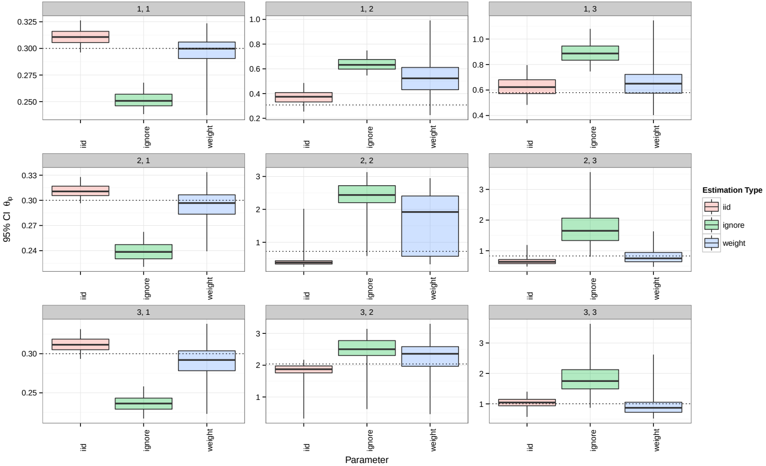
- $\mathbf{s} = s_1, \ldots, s_N$ indexes randomly-assigned cluster memberships
- $s_i \in (1, \ldots, M = 3)\,,\; i = 1, \ldots, N$
- $\boldsymbol{\theta}_i \equiv \boldsymbol{\theta}^*_{s_i}$

Generate Functions Under $M = 3$ Clusters

- $\mathbf{f}_i \sim \mathcal{N}_t\left(\mathbf{0}, \mathbf{C}\left(\boldsymbol{\theta}^*_{s_i}\right)\right)$
- Generate $y_{ij} = \mathcal{N}\left(f_{ij}, \tau_\epsilon^{-1}\right)$
- τ_ϵ chosen to achieve noise-to-signal of 0.1
- Employ $H = 4$ strata
- Assign units based on quantiles of variances for N, $T \times 1$ functions, $\mathbf{y}_i$
- (STSI) Sample $n = 800$ from $H = 4$ strata with unequal probabilities, $\mathbf{p} = (0.03, 0.06, 0.07, 0.16)$, $\mathbf{n} = (75, 155, 178, 392)$
- Generate B = 25 pseudo-populations for re-weighted estimation

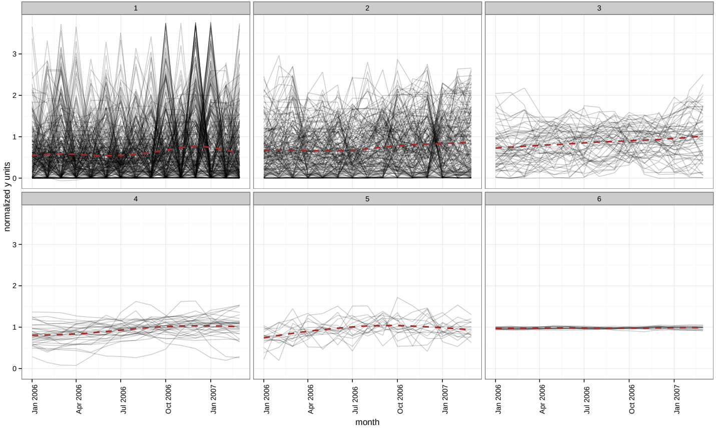
Posterior Distributions for $\boldsymbol{\theta}_i = (\theta_{i,1}, \theta_{i,2}, \theta_{i,3})$

- One unit randomly chosen within each cluster
- Non-informative performs worst on members of smooth cluster
- Does best on rough cluster - over-samples ↑ variance functions



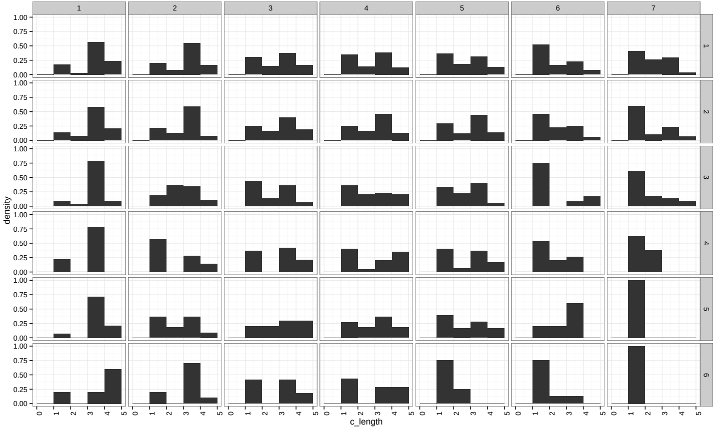
Clustering of $\{f_i\}$ for RAS Data

- 6 clusters contain $(950, 648, 204, 106, 57, 48)$ establishments
- Based on relative magnitude of errors committed than on shape



Closing Period by Size × Cluster

- 7 employee size categories, $(1-9,\ 10-19,\ 20-49,\ 50-99,\ 100-249,\ 250-499,\ \ >499)$
- BLS defines four time-indexed closing periods
- larger sized establishments in lower error clusters report sooner



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