Chapter 1 Solutions

1.3 The probability density function for the gamma distribution is

$$f(t \mid \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \exp(-\lambda t).$$

What is the MTTF for the gamma distribution?

By definition,

$$\begin{aligned} MTTF &= \int_{-\infty}^{\infty} t f(t) \mathrm{d}t \\ &= \int_0^\infty t \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \exp(-\lambda t) \mathrm{d}t \\ &= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha} \exp(-\lambda t) \mathrm{d}t \end{aligned}$$

Recognizing that this is a $Gamma(\alpha+1, \lambda)$ distribution, we can write the integral as

$$\int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha} \exp(-\lambda t) \mathrm{d}t = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} = \frac{\alpha}{\lambda}.$$

1.4 The probability density function for the Weibull distribution is

$$f(t \mid \lambda, \beta, \theta) = \lambda\beta(t-\theta)^{\beta-1} \exp\left[-\lambda(t-\theta)^\beta\right], \qquad 0 \le \theta < t, \lambda > 0, \beta > 0.$$

What is the reliability function for the Weibull distribution?

By definition,

$$\begin{aligned} R(t) &= \int_t^\infty f(s) \mathrm{d}s \\ &= \int_t^\infty \lambda\beta(s-\theta)^{\beta-1} \exp[-\lambda(s-\theta)^\beta] \mathrm{d}s. \end{aligned}$$

Noticing that if $u = -\exp(-\lambda(s-\theta)^\beta)$, then $\mathrm{d}u = \lambda\beta(s-\theta)^{\beta-1}\exp(-\lambda(s-\theta)^\beta)\mathrm{d}s$, we have

$$\begin{aligned} R(t) &= -\exp(-\lambda(s-\theta)^\beta) \mid_t^\infty \\ &= \exp(-\lambda(t-\theta)^\beta). \end{aligned}$$

1.5 The probability density function for an exponential random variable is

$$f(t) = \lambda \mathrm{e}^{-\lambda t}, \qquad t > 0, \qquad \lambda > 0.$$

What is the average hazard rate for an exponential random variable?

In Example 1.1.3, it is shown that the cumulative hazard function is $H(t) = \lambda t$. Therefore, by definition, $AHR(t_1, t_2) = \frac{\lambda t_2 - \lambda t_1}{t_2 - t_1} = \frac{\lambda(t_2 - t_1)}{t_2 - t_1} = \lambda$.

1.7 Suppose that we are using the exponential distribution to model an item's lifetime.

(a) We observe that the item failed at 6 hours. What is the likelihood function for this observation?

Let $T \sim \mathit{Exponential}(\lambda)$. Then $f(t \,|\, \lambda) = \lambda e^{-\lambda t}$ and the likelihood function is $l(\lambda \,|\, t = 6) = \lambda e^{-6\lambda}$.

(b) We observe that the item failed at some time between 5 and 10 hours. What is the likelihood function for this observation?

From Table 1.6, we know that the likelihood function for an interval censored observation has the form $F(t_R) - F(t_L)$. The cumulative distribution function for an exponential random variable is $F(t \,|\, \lambda) = 1 - \exp(-\lambda t)$. Consequently, the likelihood function is $l(\lambda \,|\, 5 \leq t \leq 10) = \exp(-5\lambda) - \exp(-10\lambda)$.

(c) We observe the item for 20 hours, and it does not fail. What is the likelihood function for this observation?

From Table 1.6, we know that the likelihood function for a right-censored observation has the form $1 - F(t_R)$. The cumulative distribution function for an exponential random variable is $F(t \,|\, \lambda) = 1 - \exp(-\lambda t)$. Consequently, the likelihood function is $l(\lambda \,|\, t \geq 20) = 1 - \exp(-20\lambda)$.